3 - Implicit function theorem and Fredholm theory

Note Title 8/6/2009

(Σ, j) Riemann surface, (M,ω) symplectic mfd, $J \in \mathcal{J}(M,\omega)$, $A \in H_2(M)$

$\tilde{\mathcal{M}}(J,A) = \{ u : \Sigma \to J \mid \bar\partial_J u = 0 \,,\, u_*[\Sigma] = A \}$ is the zero set of a section:

$$\begin{array}{lll}
\mathcal{E} = \{ \eta \mid u \in \mathcal{B},\ \eta \in \Omega^{0,1}(\Sigma, u^*TM) \} & \ni \bar\partial_J u & \in \text{fiber } \mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM) \\
\downarrow \quad \nearrow s & \uparrow & \\
\mathcal{B} = \{ u : \Sigma \to M \mid u_*[\Sigma] = A \} & \ni u &
\end{array}$$

To apply the Implicit function theorem we need Sobolev completions:
fix $p > 2$

- $\mathcal{B}$ Banach manifold : $\mathcal{B} = \{ u \in W^{1,p}(\Sigma, M) \mid \ldots \}$

  near u is modelled on Banach space $W^{1,p}(\Sigma, u^*TM) \ni \xi$

$$\begin{array}{rcl}
\text{chart} & \downarrow & \downarrow \\
& \mathcal{B} & \ni \exp_u(\xi)
\end{array}$$

- $\mathcal{E}$ Banach vector bundle : $\mathcal{E} = \{ (u \in W^{1,p} , \eta \in L^p(\Sigma, \Lambda^{0,1}T\Sigma \otimes u^*TM)) \}$

  L^p-completion of $\Omega^{0,1}(\Sigma, u^*TM)$

- $s : \mathcal{B} \to \mathcal{E}$ smooth section transverse to 0-section ($b \longmapsto 0 \in E_b$)

  i.e. $\forall b \in s^{-1}(0)$ linearization $D_b s : T_b\mathcal{B} \longrightarrow \mathcal{E}_b$ is surjective

$$\begin{array}{ccc}
T_b\mathcal{B} & \xrightarrow{D_b s} & \mathcal{E}_b \\
{\scriptstyle ds}\downarrow & & \uparrow{\scriptstyle pr} \\
{\scriptstyle Id \times D_b s} \ T_{(b,0)}\mathcal{E} & \cong & T_b\mathcal{B} \times \mathcal{E}_b
\end{array}$$

  $s(u) = \bar\partial_J u \in L^p(\Sigma, \Lambda^{0,1}T\Sigma \otimes u^*TM)$, $u \in W^{1,p}(\Sigma, M)$ • smooth in u

  ⊛ call $J \in \mathcal{J}(M,\omega)$ regular if $D_u \bar\partial_J =: D_u$ is surjective $\forall \bar\partial_J u = 0$

$\Rightarrow$ $s^{-1}(0) \subset \mathcal{B}$ is a smooth submanifold ; $T_b s^{-1}(0) = \ker D_b s$

$= \{ u \in \mathcal{B} \mid \bar\partial_J u = 0 \in \mathcal{E}_u \} = \tilde{\mathcal{M}}(J,A)$ for J regular

<u>Note</u>: We expect $\widetilde{\mathcal{M}}(J,A)$ to be finite dimensional, i.e.

$$\underbrace{\mathrm{im}\, D_u = \mathcal{E}_u \quad , \ker D_u \text{ finite dim.}}_{\Rightarrow D_u \text{ Fredholm}} \quad , \dim \widetilde{\mathcal{M}}(J,A) = \underset{\substack{\| \\ \mathrm{index}\, D_u}}{\dim \ker D_u}$$

It is a good start to prove this (we'll need it to apply Sard-Smale and find regular J)

<u>Thm</u> For $\bar\partial_J u = 0$ $\quad D_u \xi = \frac{d}{dt}\big|_{t=0} \bar\partial_J \exp_u(t\xi)$

$$D_u : T_u\mathcal{B} = W^{1,p}(\Sigma, u^*TM) \longrightarrow \mathcal{E}_u = L^p(\Sigma, \Lambda^{0,1}T\Sigma \otimes u^*TM)$$

$$\xi \longmapsto \frac{1}{2}\left(\nabla\xi + J(u)\circ\nabla\xi\circ j\right) + \frac{1}{2}(\nabla_\xi J)(u)\circ\partial_J u\circ j$$

locally $\|$

$$\eta\, ds - J(u)\eta\, dt \;;\; \eta = \frac{1}{2}\left(\partial_s\xi + J(u)\partial_t\xi + \partial_\xi J(u)\partial_t u\right)$$

is Fredholm ($\mathrm{im}\, D_u \subset \mathcal{E}_u$ closed, finite)

with $\mathrm{index}\, D_u = \dim\ker D_u - \dim \mathcal{E}_u/\mathrm{im}\, D_u = n(2-2g(\Sigma)) + 2c_1(u^*TM)$

$= 2n + 4$

for $\Sigma = S^2$, $A = [S^2\times pt]$

<u>Cor</u>: For regular $J \in \mathcal{J}(S^2\times T)$ resp. $(J_t)_{t\in[0,1]}$

- $\{u \in \widetilde{\mathcal{M}}(J,[S^2\times pt]) \mid u(z_0) = p_0\}$ is a manifold of dimension $2n+4-2n = 4$
- ——— $/\mathrm{Aut}(S^2, j_0, z_0)$ ——— $4-4=0$
- $\{(t,u) \mid u \in \widetilde{\mathcal{M}}(J_t,[S^2\times pt]), u(z_0)=p_0\} / \mathrm{Aut}(S^2, j_0, z_0)$ ——— $2n+4+1-2n-4 = 1$

Proof of Cor:

• work with sections $\quad \begin{aligned} \mathcal{B} &\to \mathcal{E}\times M \\ u &\mapsto (\bar\partial_J u, u(z_0)) \end{aligned}$ $\quad$ resp. $\quad \begin{aligned} [0,1]\times\mathcal{B} &\to \mathcal{E} \\ (t,u) &\mapsto \bar\partial_{J_t} u \end{aligned}$

linearized operators are compact perturbation of $D_u \times 0 : \mathcal{B} \to \mathcal{E}_u \times T_{u(z_0)}M$ $\quad$ homotopic in $\left\{ \begin{matrix}\text{bounded}\\ \text{operators}\end{matrix} \right\}$ to $\underset{\text{w.r.t. } J_0}{D_u} : [0,1]\times\mathcal{B} \to \mathcal{E}_u$

hence by stability $\quad \begin{aligned} \text{index} &= \operatorname{ind}(D_u \times 0) \\ &= \operatorname{ind} D_u - \dim M \end{aligned}$ $\quad \begin{aligned} \text{index} &= \operatorname{ind}(\mathcal{S}) \\ &= \operatorname{ind} D_u + 1 \end{aligned}$

⊛ $\operatorname{Aut}(S^2, j_0, z_0) \hookrightarrow \tilde{\mathcal{M}}(\dots)$ acts freely (~ by unique continuation, finite energy, simpleness) and p

Proof of Thm:

step 1: $\underset{\|\nabla\xi\|_{L^p}+\|\xi\|_{L^p}}{\|\xi\|_{W^{1,p}}} \le C\big(\|D_u\xi\|_{L^p} + \underset{K\xi}{\|\xi\|_{L^p}} \big)$ $\quad ; K : W^{1,p} \hookrightarrow L^p$ compact

and hence $\dim \ker D_u < \infty$, $\operatorname{im} D_u$ closed

Step 2: $\dim \overset{\text{Banach space}}{\mathcal{E}_u/\operatorname{im} D_u} = \dim (\mathcal{E}_u/\operatorname{im} D_u)^* = \dim(\operatorname{im} D_u^\perp \subset \mathcal{E}_u^*) = \dim(\ker D_u^*)$

$D_u^* : \mathcal{E}_u^* \longrightarrow T_u\mathcal{B}^*$ $\quad$ adjoint operator $\quad (D_u^*\eta)(\xi) = \eta(D_u\xi)$

$\mathcal{E}_u^* = L^p(\dots)^* \cong L^{p'}(\dots)$ $\quad T_u\mathcal{B}^* = W^{1,p}(\dots)^* \cong W^{-1,p'}(\dots)$ $\quad \frac{1}{p}+\frac{1}{p'}=1$ $\quad \eta \in \mathcal{E}_u^*,\ \xi \in T_u\mathcal{B}$

as distributions $\quad \int_\Sigma \langle D_u^*\eta, \xi\rangle = \int_\Sigma \langle \eta, D_u\xi\rangle$

locally $\quad D_u^* = -\partial_s + J\partial_t + 0\text{th order} \quad D_u = \partial_s + J\partial_t + 0\text{th order}$

similar estimates as in Step 1 $\Rightarrow \dim \ker D_u^* < \infty$

$\Rightarrow D_u$ is Fredholm

<u>Step 3</u> D_u is a compact perturbation of $\bar\partial_{u^*TM} : \xi \mapsto \frac{1}{2}(\nabla\xi + J(u)\nabla\xi\, j)$, the Dolbeault operator of the complex vector bundle u^*TM.

Riemann-Roch thm $\Rightarrow$

$$\underbrace{\operatorname{index}\bar\partial_{u^*TM}}_{\overset{\text{stability}}{=}\ \operatorname{index} D_u} = \underbrace{\chi(\Sigma)}_{(2-2g)}\ \underbrace{\mathrm{rk}_{\mathbb{C}}\, u^*TM}_{n} + 2c_1(u^*TM)$$

Indeed: $D_u - \bar\partial_{u^*TM} : W^{1,p} \xrightarrow{\text{bounded}} W^{1,p} \xhookrightarrow{\text{compact}} L^p$

$$\xi \longmapsto \tfrac{1}{2}(\nabla_\xi J)(u)\, \partial_J u \circ j \qquad \blacksquare$$

<u>Estimates for Cauchy-Riemann</u>

local coordinates: $\underset{\text{open}}{\Omega \subset \mathbb{C}}$ $\quad u, \xi : \Omega \to \mathbb{C}^n \quad \bar\partial_J \xi = \partial_s\xi + J(u)\partial_t\xi$

- Step 1 requires ($K \subset \Omega$ compact)

$$\underbrace{\|\nabla\xi\|_{L^p(K)}}_{\|\xi\|_{W^{1,p}} - \|\xi\|_{L^p}} \le C\big(\underbrace{\|\bar\partial_J\xi\|_{L^p(\Omega)}}_{\le\ \|D_u\xi\|_{L^p} + \|D_u - \bar\partial_J\|\,\|\xi\|_{L^p}} + \|\xi\|_{L^p(\Omega)}\big)$$

(then cover by local charts $\Sigma = \bigcup_i K_i$; $K_i \underset{\text{compact}}{\subset} \Omega_i \subset \Sigma$)

Caldéron-Zygmund ($1<p<\infty$): $\Delta\xi = (\text{first order operator})f \Rightarrow \|\nabla\xi\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}$

$$(\partial_s - J\partial_t)\,\bar\partial_J\xi = \Delta\xi + (\nabla_{\partial_s u}J)\,\partial_t\xi - (\nabla_{\partial_t u}J)\,\partial_s\xi \qquad f = (\bar\partial_J\xi, \xi)$$

(multiply by cutoff function $h \in C^\infty(\Sigma)$, $\operatorname{supp} h \subset \Omega$, $h|_K = 1$ and extend to $\mathbb{R}^2 \supset \Omega$)

- More generally $\|\xi\|_{W^{k+1,p}(K)} \le C\big(\|\Delta\xi\|_{W^{k-1,p}(\Omega)} + \|\xi\|_{L^p(\Omega)}\big)$

$$\Rightarrow \|\xi\|_{W^{k+1,p}} \le C\big(\|\bar\partial_J\xi\|_{W^{k,p}} + \|(\nabla_{\partial_s u}J)\partial_t\xi - (\nabla_{\partial_t u}J)\partial_s\xi\|_{W^{k-1,p}} + \|\xi\|_{L^p}\big)$$

from John Etnyre's Chern explorations

Def$^{\underline{n}}$:

The <u>k-th Chern class</u> of $\mathbb{C}^n \hookrightarrow E \downarrow B$ is the obstruction $\in H^{2k}(B;\mathbb{Z})$

to the existence of a $\underbrace{(n-k+1)}_{N}$ - frame of E over $2k$-skeleton of B.

N sections, everywhere linearly independent

Fact: $\pi_j(\text{k-frames in } \mathbb{C}^n) = \begin{cases} 0 & ; \ j \leq 2(n-k) \\ \mathbb{Z} & ; \ j = 2(n-k)+1 \end{cases}$

Really define c_k by building sections of k-frame bundle over skeleton