Note Title 8/5/2009

2 - Basic properties of pseudoholomorphic curves

(Σ, j) Riemann surface, (M,ω) symplectic, $J \in \mathcal{J}(M,\omega)$

Cauchy-Riemann operator

$u : \Sigma \to M \quad \rightsquigarrow \quad du : T\Sigma \to TM \qquad du(z) : T_z\Sigma \to T_{u(z)}M$ linear

$\Rightarrow du \in \Omega^1(\Sigma; u^*TM) = \{ \eta : T\Sigma \to u^*TM \mid \eta(T_z\Sigma) \subset T_{u(z)}M \}$ (with j acting on $T\Sigma$, $J(u)$ acting on u^*TM)

$\| \quad \Omega^{1,0} = \{\eta \circ j = J \circ \eta\} \oplus \Omega^{0,1} = \{\eta \circ j = -J \circ \eta\}$

- $\bar\partial_J u = \frac{1}{2}(du + J du\, j)$ is the projection of du on $\Omega^{0,1}$
- $\bar\partial_J$ is a nonlinear operator:

$$\bar\partial_J u\,(X \in T_z\Sigma) = \tfrac{1}{2}\left(du(X) + J(u(z))\, du(j(z)X)\right)$$

in local coordinates $u : (\mathbb{C}, i) \to (\mathbb{R}^{2n}, J : \mathbb{R}^{2n} \to \mathbb{R}^{2n \times 2n})$, $s+it \in \mathbb{C}$

$$\bar\partial_J u = 0 \iff \partial_s u(s,t) + J(u(s,t))\, \partial_t u(s,t) = 0$$

<u>moduli spaces</u> - reparametrization of J-hol. maps

$Aut(\Sigma, j) = \{\varphi : \Sigma \to \Sigma \ (j,j)\text{-hol diffeom}\} \qquad d\varphi \circ j = j \circ d\varphi$

If $u : \Sigma \to M$ is J-hol, then so is $u \circ \varphi$.

(E.g. $Aut(S^2, j_0) = \{$Möbius transformations $z \mapsto \frac{az+b}{cz+d}$; $a,b,c,d \in \mathbb{C}$, $ad-bc = 1\}$
$= PSL(2,\mathbb{C})$ 6-dim noncompact)

(after fixing a point $Aut(S^2, j_0, z_0) = \{ \varphi(z_0) = z_0 \}$ still 4-dim)

We are interested in the moduli space for $A \in H_2(M)$, $p_0 \in M$, $z_0 \in \Sigma$

$$\mathcal{M}(J, A) = \{ u : \Sigma \to M \mid \bar{\partial}_J u = 0,\ u_*[\Sigma] = A,\ u(z_0) = p_0 \} / Aut(\Sigma, j, z_0)$$

(weak) compactness properties of $\mathcal{M}(J,A)$ hinge on

<u>energy identity</u> for compatible ω, J

$$E(u) := \frac{1}{2}\int_\Sigma |du|^2 dvol_\Sigma = \int_\Sigma |\bar{\partial}_J u|^2 dvol_\Sigma + \int_\Sigma u^*\omega$$

in local coordinates $|\eta|^2 dvol_\Sigma = \left(|\eta(\partial_s)|^2 + |\eta(\partial_t)|^2 \right) ds \wedge dt$

$$= \langle \eta \wedge * \eta \rangle_{g_J} = - \langle \eta \wedge (\eta \circ j) \rangle_{g_J}$$

<u>Proof:</u> $4 |\bar{\partial}_J u|^2 dvol = | du + J du j |^2 dvol$ $\quad j\partial_s = \partial_t$, $j\partial_t = -\partial_s$

$$= \left(|\partial_s u + J\partial_t u|^2 + |\partial_t u - J\partial_s u|^2 \right) ds \wedge dt$$

$$= \left(|\partial_s u|^2 + |J\partial_s u|^2 + |J\partial_t u|^2 + |\partial_t u|^2 + 2g(\partial_s u, J\partial_t u) - 2g(\partial_t u, J\partial_s u) \right) ds \wedge dt$$

$$= 2|du|^2 + 4 \underbrace{\omega(\partial_s u, J^2 \partial_t u)\, ds \wedge dt}_{= -u^*\omega}$$

■

<u>Cor.:</u> • J-hol. curves of fixed homology $u_*[\Sigma] = A$ have fixed energy

$$E(u) = \int_\Sigma u^*\omega = \langle A, [\omega] \rangle$$

• null-homologous J-hol. curves are constant $\left(\int |du|^2 = 0 \right)$

- J-hol. curves minimize energy $\Rightarrow$ harmonic maps ("Δu = lower order")

(Ex) J-hol. curves minimize area

Pseudoholomorphic curves behave almost like holomorphic functions

locally: $\partial_s u + J(u)\partial_t u = 0$ vs $\partial_s u + J_0 \partial_t u = 0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{(\partial_s - J_0\partial_t)(\partial_s + J_0\partial_t)}_{\partial_s^2 + \partial_t^2 = \Delta} u = 0$$

$$\underset{\text{linear operator}}{\Delta u} = \underbrace{(\nabla_{\partial_t u} J)\partial_s u - (\nabla_{\partial_s u} J)\partial_t u}_{\text{nonlinear but lower order}}$$

Regularity: If $u \in \mathcal{C}^1, \bar{\partial}_J u = 0$ then $u \in \mathcal{C}^\infty$

"because" $u \in \mathcal{C}^k \Rightarrow \Delta u \in \mathcal{C}^{k-1} \not\Rightarrow u \in \mathcal{C}^{k+1}$

... true once we work with Sobolev spaces

Carleman similarity principle

$u : \mathbb{C} \to \mathbb{R}^{2n}$, $J, C : \mathbb{C} \to \mathbb{R}^{2n \times 2n}$, $J^2 = -\mathbb{1}$

$$\partial_s u + J \partial_t u + C u = 0 \quad , \quad u(0) = 0$$

$\Rightarrow$ $\exists \delta > 0$, $\Phi : B_\delta(0) \to \mathbb{R}^{2n \times 2n}$ invertible :

$$\Phi^{-1} J \Phi = J_0 \quad , \quad \Phi^{-1} u =: v \text{ is holomorphic} : \ \partial_s v + J_0 \partial_t v = 0$$

This does not prove regularity or estimates (like $\|u\|_{\mathcal{C}^2} \le \|\bar{\partial}_J u\|_{\mathcal{C}^1} + \|u\|_{\mathcal{C}^1}$)

since
- u must map to Darboux chart
- Φ is as regular as $J = J \circ u$

But it does prove other useful properties of J-holomorphic curves:

Unique continuation: Σ connected, $u, v : \Sigma \to M$ J-hol.

$B \subset \Sigma$ open, $u|_B = v|_B$ (or $u = v$ to ∞ order at a point)

$\Rightarrow$ $u \equiv v$

Cor.: $u : \Sigma \to M$ J-hol., not constant

- $\Rightarrow$ $u^{-1}(p)$ finite $\forall p \in M$
- $\Rightarrow$ $\text{Crit}\, u = \{ z \in \Sigma \mid du(z) = 0 \}$ finite

The following notion and theorem will be crucial for "transversality" – smoothness of moduli space for certain J

<u>Def^n</u>: $u : \Sigma \to M$ is <u>simple</u> if it is not multi-covered,

i.e. there is no $\varphi : \Sigma \to \Sigma$, $\deg \varphi > 1$ such that $u = v \circ \varphi$

$\varphi_*[\Sigma] = \deg \varphi \cdot [\Sigma]$
OR $\deg \varphi = \# \varphi^{-1}(z_0)$ (count with sign)

$$\begin{array}{ccc} \Sigma & \xrightarrow{u} & M \\ \varphi \downarrow & \nearrow_{v} & \\ \Sigma & & \end{array}$$

<u>Thm</u>: $u : \Sigma \to M$ J-holomorphic, simple

$\Rightarrow \{ z \in \Sigma \mid du(z) \neq 0,\ u^{-1}(u(z)) = \{z\} \} \subset \Sigma$ open, dense

(injective points)

<u>Example</u>: $u : S^2 \to S^2 \times T$, $u_*[S^2] = [S^2 \times pt]$

$u = v \circ \varphi$

$\Downarrow$

$\deg(pr_{S^2} \circ u) = 1$

$\deg(pr_{S^2} \circ v \circ \varphi) = \deg(pr_{S^2} \circ v) \cdot \deg(\varphi) = 1 \quad \Rightarrow \deg(\varphi) = 1$

$\Rightarrow$ u simple