Introduction and Gromov nonsqueezing

Note Title 8/4/2009

Invariants of symplectic manifolds (M,ω)

- ω nondegenerate $\rightsquigarrow$ volume $\int_M \omega\wedge\ldots\wedge\omega \neq 0$
- $d\omega = 0$ $\rightsquigarrow$ $[\omega] \neq 0 \in H^2(M;\mathbb{R})$
- $\mathcal{J}(M,\omega)$ contractible space of compatible almost complex structures

$\rightsquigarrow$ Chern classes $c_i(TM) \in H^{2i}(M)$

(independent of choice of J)

pseudoholomorphic curves $\longrightarrow$ Gromov-Witten invariants
(depend on choice of J)

"algebra to quotient out J-dependence"

quantum cohomology
Floer homology
symplectic field theory
Fukaya category
⋮

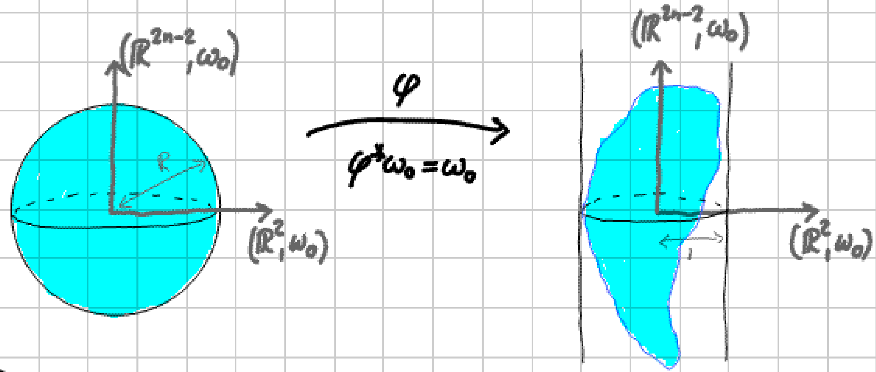
e.g. Gromov width

$$W_{Gr}(M,\omega) := \sup\{\pi r^2 \mid B^{2n}(r) \xhookrightarrow{\text{sympl.}} M\}$$

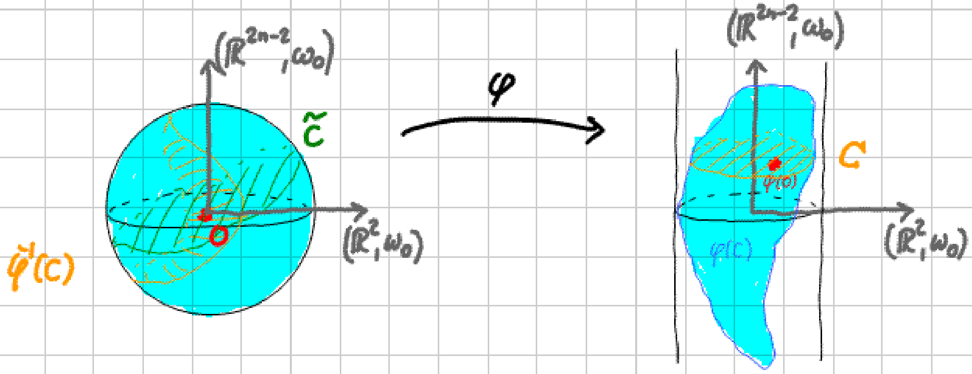
radius r ball in $(\mathbb{R}^{2n},\omega_0)$

	$B^2(1)\times\mathbb{R}^2$	Volume	W_{Gr}	
diffeom. volume preserv. **not** symplectom.	$B^2(1)\times B^2(4)$	$16\pi^2$	π	by Gromov nonsqueezing
	$B^2(2)\times B^2(2)$	$16\pi^2$	4π	
	$B^2(2)\times\mathbb{R}^2$			

Gromov nonsqueezing: If $B^{2n}(R) \xhookrightarrow{\text{sympl.}} B^2(1)\times\mathbb{R}^{2n-2}$ then $R \le 1$



Plan of Proof



$\varphi^{-1}(C)$ has $\varphi^* g_0$-area = area(C)
through 0
is a $\varphi^* g_0$-minimal surface

$C = \mathbb{R}^2 \times pt \cap \varphi(C)$ symplectic cross-section
through $\varphi(0)$ has g_0-area $\le \pi$
and is a g_0-minimal surface

If $\tilde C$ is a g_0-minimal surface through 0 with g_0-area $\le \pi$,

then $\quad \pi R^2 \le g_0\text{-area} \le \pi \quad \Rightarrow \quad R \le 1$ ∎

monotonicity lemma for minimal surfaces

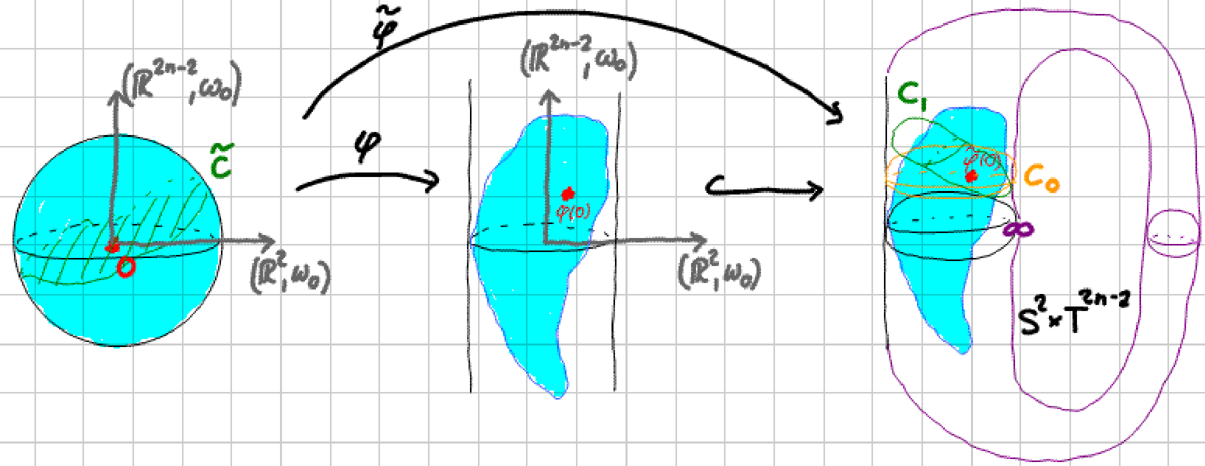
We will find $\tilde C = \tilde\varphi^{-1}(C_1)$ by embedding $B^2(1-\varepsilon) \times pr(\varphi(B^{2n}(R))) \hookrightarrow S^2 \times T^{2n-2}$

$\varepsilon > 0$ $\quad \cap \; \mathbb{R}^{2n-2}$ $\quad \omega_0 \times K\omega_0$

w.l.o.g. $B^{2n}(R) \xrightarrow{\varphi}$ $\quad \tilde\varphi$ $\quad$ area π $\quad K \gg 1$

(then $\varepsilon \to 0$)

and finding a $\tilde{\varphi}_* g_0$-minimal surface in $S^2 \times T^{2n-2}$ through $\tilde{\varphi}(0)$



<u>Note</u>: $C_0 = S^2 \times pt \underset{\ni \tilde{\varphi}(0)}{\subset} S^2 \times T^{2n-2}$ is J_0-holomorphic: $J_0 TC_0 = TC_0$

<u>Fact</u>: J-hol. $\Rightarrow$ g_J-minimal

<u>Idea</u>: Pick homotopy $(J_t) \subset \mathcal{J}(S^2 \times T^{2n-2})$ to $J_1 = \tilde{\varphi}_* J_0$ extended to $S^2 \times T^{2n-2}$

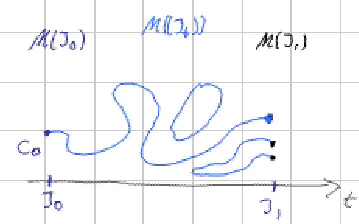
<u>Claim</u>: There exists $C_1 \subset S^2 \times T$ J_1-hol., $\tilde{\varphi}(0) \in C_1$, homotopic to C_0

Proof: (a) there is a regular (TBD) homotopy $(J_t)_{t \in [0,1]} \subset \mathcal{J}(S^2 \times T^{2n-2})$

(b) $\mathcal{M}(J_0) := \{ C \subset S^2 \times T \mid J_0 TC = TC, \tilde{\varphi}(0) \in C, [C] = [S^2 \times pt] \} = \{ C_0 \}$

(c) $\mathcal{M}((J_t)) := \{ (t \in [0,1], C \subset S^2 \times T) \mid J_t TC = TC, \text{—"—} \}$ is

- a manifold "transversality"
- dimension 1 "Fredholm theory"
- compact "Gromov compactness"
- has boundary $\partial \mathcal{M}((J_t)) = \mathcal{M}(J_0) \cup \mathcal{M}(J_1)$



$\Rightarrow$ $\mathcal{M}(J_1) \neq \emptyset$ since compact 1-manifolds have even number of boundary points

Pseudoholomorphic curves

(M,ω) symplectic

$J: TM \to TM$ compatible almost complex structure $\left.\begin{matrix} d\omega = 0 \\ J^2 = -\mathbb{1} \\ \omega(\cdot, J\cdot) = g_J(\cdot,\cdot) \end{matrix}\right.$ metric

$\searrow_M \swarrow$

- Darboux: locally $(M,\omega) \cong (\underset{(x_j + iy_j)}{\mathbb{C}^n}, \omega_0 = \sum dx_j \wedge dy_j)$

Integrability thm $\quad (M,J) \underset{\text{locally}}{\cong} (\mathbb{C}^n, i = J_0)$

$\Leftrightarrow$ Nijenhuis $N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] = 0$

$\forall X,Y : M \to TM$ vector field

Ex $\dim M = 2,\ J^2 = -\mathbb{1} \Rightarrow N_J = 0$

Cor: Every almost $\underbrace{\text{complex 2-manifold } (\Sigma, j)}$ is complex $\left(\underset{loc}{\cong} (\mathbb{C}, i)\right)$

They are called Riemann surface.

Uniformization thm: Every complex structure j on S is equivalent to $j_0 = i$ on $\mathbb{CP}^1$ (i.e. $\exists \varphi : S^2 \to \mathbb{CP}^1$ diffeom.: $\varphi^* j_0 = j$)

this is a holomorphic map

$\varphi^* j_0 = d\varphi^{-1} \circ j_0 \circ d\varphi = j$

$\Leftrightarrow \quad j_0 \circ d\varphi = d\varphi \circ i$

$\Leftrightarrow \quad -d\varphi = j_0 \circ d\varphi \circ i$

<u>Def$^{\underline{n}}$</u>: (N,j), (M,J) almost complex

- A map $u: N \to M$ is (J,j)-holomorphic if $J \circ du = du \circ j$

  $\Leftrightarrow \quad \bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0$ "Cauchy-Riemann operator"

- A submanifold $C \subset M$ is J-holomorphic if $J(TC) = TC$.

  (Then $Id_C : C \to M$ is a $(J, J|_C)$-holomorphic map.)

<u>Remark</u>: J-holomorphic maps/submanifolds are "very rare"

(an overdetermined PDE) unless

⊛ N and M are complex $\rightsquigarrow$ algebraic geometry:

(holomorphic functions $F: M \to \mathbb{C}^r$ cut out
holomorphic submanifolds $F^{-1}(0) \subset M$ of dimension $\dim M - 2r$)

OR ⊛ $\dim_{\mathbb{R}} N = 2$ (so N is complex)

then $u: N \to M$ or $u(N) = C \subset M$ is called

J-holomorphic curve because $\dim_{\mathbb{R}} u(N) = 2$

$\dim_{\mathbb{C}} u(N) = 1$