Exercises on holomorphic curves #3

Note Title 8/6/2009

(Easy) ① verify/correct

$$\Delta\xi := \partial_s^2\xi + \partial_t^2\xi = (\partial_s - J\partial_t)\,\underbrace{\bar\partial_J\xi}_{=\partial_s\xi + J(u)\partial_t\xi} - (\nabla_{\partial_s u}J)\,\partial_t\xi + (\nabla_{\partial_t u}J)\,\partial_s\xi$$

② for $\Omega \subset \mathbb{C}$ compact domain and $\xi \in \mathcal{C}^\infty(\Omega,\mathbb{R})$, $\xi|_{\partial\Omega} = 0$ show

$$\|d\xi\|_{L^2} \le \|\Delta\xi\|_{L^2} + \|\xi\|_{L^2}$$ by partial integration

②' same for $\xi \in \mathcal{C}^\infty(\Sigma,\mathbb{R})$ on closed Riemann surface Σ

with $\Delta\xi = *d*d\xi$ with Hodge $*$ given by a metric on Σ

(Medium) ②'' For $\xi \in \mathcal{C}^\infty(\Sigma,\mathbb{R})$ show $\|\xi\|^2_{W^{1,2}} \le \|\Delta\xi\|^2_{(W^{1,2})^*} + 2\|\xi\|^2_{L^2}$

where $\|f\|_{(W^{1,2})^*} = \sup_{\varphi \in W^{1,2}} \|\varphi\|^{-1}_{W^{1,2}} \left|\int_\Sigma f\cdot\varphi\right|$

③ bubbling – as in #2

consider the maps $u_\varepsilon[x:y] = [x^2 : \varepsilon y^2 : xy]$

$w_\varepsilon[x:y] = [\sqrt{\varepsilon}x^2 : \sqrt{\varepsilon}y^2 : xy]$, $v_\varepsilon[x:y] = [\varepsilon x^2 : y^2 : xy]$

describe their limits as $\varepsilon \to 0$. In which topology / on which subset does the convergence hold? How is that related to the behaviour of $|du_\varepsilon|$, $|dv_\varepsilon|$, $|dw_\varepsilon|$?

(Hard) ④ Show that $Aut(S^2, j_0, z_0)$ acts freely

$$\tilde{\mathcal{M}}(J) = \{u : S^2 \to S^2 \times T \mid \bar{\partial}_J u = 0,\ u_*[S^2] = [S^2 \times pt],\ u(z_0) = p_0\}$$

... and properly discontinuously

$\boxed{\text{for exercises ③-⑤ on board see lecture script}}$