# Exercises on holomorphic curves #2

Note Title 8/5/2009

① (J-hol. curves are minimal)

- Let $J \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ with induced inner product $g_J(\cdot,\cdot) = \omega_0(\cdot, J\cdot)$

  Show that $\omega_0(v,w) \leq g_J$-area of parallelogram spanned by $v, w$

  and equality holds iff $\boxed{w \in \mathrm{span}(v, Jv)}$. correction

- The area of $u : \Omega \to M$ ($\Omega \subset \mathbb{C} \ni s+it$) is

  $$\mathrm{Area}(u) = \int_\Omega \sqrt{\det \begin{pmatrix} g_J(\partial_s u, \partial_s u) & g_J(\partial_s u, \partial_t u) \\ g_J(\partial_t u, \partial_s u) & g_J(\partial_t u, \partial_t u) \end{pmatrix}}\, ds \wedge dt$$

  Check that $\bar\partial_J u = 0 \Rightarrow \mathrm{Area}(u) = \int_\Omega u^*\omega$

  and $v : \Omega \to M$, $\int v^*\omega = \int u^*\omega \Rightarrow \mathrm{Area}(v) \geq \mathrm{Area}(u)$.

  (i.e. J-holomorphic curves minimize the $g_J$-area in their homology class)

② (towards nonsqueezing)

Show that the only holomorphic maps $u : S^2 \to S^2 \times T$

with respect to $j_0$ on $S^2$, $j_0 \times J$ on $S^2 \times T$ (for any $J \in \mathcal{J}(T, \omega_0)$)

with $u_*[S^2] = [S^2 \times pt]$ are reparametrizations of

$u(z) = (z, p_0)$ for some $p_0 \in T$.

③ (Bubbling – Gromov compactness)

$C_\varepsilon := \{[z_0 : z_1 : z_2] \mid z_0 z_1 = \varepsilon z_2^2\} \subset \mathbb{CP}^2$

are $J_0$-holomorphic spheres for $\varepsilon > 0$. Check that by using

$u_\varepsilon : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^2 \,,\; [x : y] \longmapsto [x^2 : \varepsilon y^2 : xy]$ .

Compare $\lim\limits_{\varepsilon \to 0} C_\varepsilon$ with $\lim\limits_{\varepsilon \to 0} u_\varepsilon$ .

Explain the missing part of $C_\varepsilon$ by $u_\varepsilon \circ \varphi_\varepsilon$ for some $\varphi_\varepsilon \in Aut(\mathbb{CP}^1)$.