Exercises on Holomorphic Curves # 1

Problem 1 (Integrability)

(i) Given a complex structure on a vector space ($J : V \to V,\ J^2 = -\mathbb{1}$), show that for any nonzero $v \in V$ the vectors v, Jv are linearly independent.

(ii) The Nijenhuis tensor N_J for an almost complex structure on a manifold M is given by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

for all vector fields $X, Y : M \to TM$. Show that it automatically vanishes for $\dim M = 2$.

Hint: $N_J(X,X) = 0$, $N_J(X,JX) = 0$, use (i).

Problem 2 (Cauchy-Riemann equation in local coordinates) Let (Σ, j) be a Riemann surface and (M, J) an almost complex manifold. Show that the following are equivalent (and find the formula for $J_{\mathcal{U}}$):

(i) u is (J,j)-holomorphic, i.e. $\overline{\partial}_J u := \frac{1}{2}(\mathrm{d}u + J \circ \mathrm{d}u \circ j) = 0$,

(ii) In local coordinates $(\Sigma, j) \supset \mathcal{U} \cong \Omega \subset (\mathbb{C}, i)$ the map $u|_{\mathcal{U}} \cong v : \Omega \to M$ satisfies

$$\partial_s v(s,t) + J_{\mathcal{U}}(v(s,t))\partial_t v(s,t) = 0 \qquad \forall s + it \in \Omega.$$

Problem 3 (Automorphisms of the sphere) Consider the sphere $(S^2, j_0) = (\mathbb{C}P^1, i)$ with its standard complex structure. Identify its automorphism group $\mathrm{Aut}(S^2, j_0)$ (the group of diffeomorphisms preserving the complex structure) with the (j_0, j_0)-holomorphic maps of degree 1,

$$\{\phi : S^2 \to S^2 \,\big|\, \deg\phi = 1, \phi^* j_0 = j_0\},$$

and with the group $PSL(2,\mathbb{C})$ of Möbius transformations

$$\phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \qquad \phi(z) = \frac{az+b}{cz+d}; \qquad ad - bc = 1.$$

Hints/Facts:

- $\deg\phi$ *is the total multiplicity of* $\phi^{-1}(z_0)$ *for any fixed* $z_0 \in S^2$.
- *The* (j_0, j_0) *holomorphic maps are rational functions* $u : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, $u(z) = p(z)/q(z)$.
- *If* p, q *are coprime polynomials then* $\deg(p/q) = \max\{\deg p, \deg q\}$.

Problem 4 (Moduli spaces) Let (M, J) be an almost complex manifold and fix a point $p_0 \in M$. Let (S^2, j_0) denote the standard complex structure on the sphere and fix a point $z_0 \in S^2$. Identify the following moduli spaces:

$$\left\{C \subset M \;\middle|\; J\text{-holomorphic submanifold}, C \overset{\text{diff}}{\cong} S^2, p_0 \in C\right\},$$

$$\left\{u : S^2 \to M \middle| (J, j_0)\text{-holomorphic embedding}, \exists z \in S^2 : u(z) = p_0\right\}/\mathrm{Aut}(S^2, j_0),$$

$$\left\{u : S^2 \to M \;\middle|\; (J, j_0)\text{-holomorphic embedding}, u(z_0) = p_0\right\}/\mathrm{Aut}(S^2, j_0, z_0),$$

where $\mathrm{Aut}(S^2, j_0, z_0) = \{\phi : S^2 \to S^2 \mid \text{diffeomorphism}, \phi^* j_0 = j_0, \phi(z_0) = z_0\}$.

Ponder about how to include homotopy or homology classes, and how to replace "embedding" by "map" for the case $M = S^2 \times T^{2n-2}$ that we will need for Gromov nonsqueezing.

Problem 5 Let $u : S^2 \to M$ be a smooth map to an almost complex manifold (M, J). Convince yourself that the vector-valued 1-forms $\Omega^1(S^2, u^*TM)$ form a complex vector bundle over S^2, which splits

$$\Omega^1(S^2, u^*TM) = \Omega^{1,0}(S^2, u^*TM) \oplus \Omega^{0,1}(S^2, u^*TM)$$

into bundles of complex linear and complex antilinear vectors (with respect to $j_0 : TS^2 \to TS^2$ and $J : TM \to TM$).

Compare the projection formulas with the Cauchy-Riemann operator $\overline{\partial}_J$.