Quilted Floer homology - news flash

joint w. Chris Woodward

David Gay for connected Cerf theory

Idea: Symplectic version of $U(r)$-instanton Floer homology

Fix $d \in \mathbb{N}$ coprime to r.

Σ closed, oriented $\rightsquigarrow \left(\begin{array}{ll} P & U(r) \text{ bundle} \\ \downarrow & \\ \Sigma & \text{of degree } d \end{array} \right) \rightsquigarrow \boxed{M(\Sigma)}$:= moduli space of central curvature, fixed determinant connections on P

Thm: $M(\Sigma)$ is smooth, symplectic, monotone, min $c_1 \geq 2$ for Σ connected.

$\Sigma_0 \xrightarrow{Y} \Sigma_1$ cobordism $\rightsquigarrow \left(\begin{array}{ll} Q & U(r) \text{ bundle} \\ \downarrow & \\ Y & Q|_{\partial Y} = P_0 \cup P_1 \end{array} \right) \rightsquigarrow \boxed{L(Y)} := M(Y)|_{\partial Y} \subset \boxed{M(\Sigma_0) \times M(\Sigma_1)}$

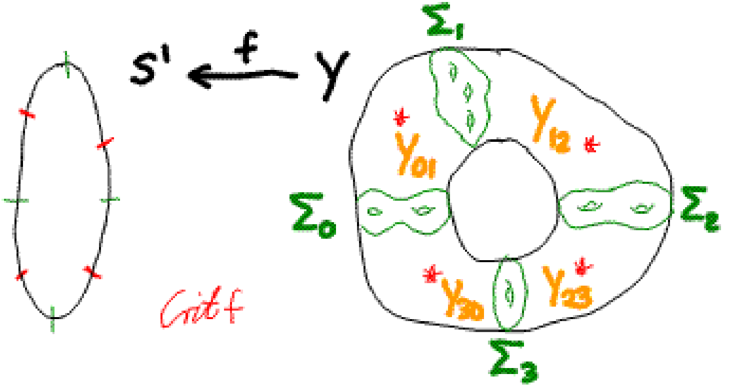
Thm: $L(Y)$ is smooth, Lagrangian for Y compression body
—"— spin, simply connected —"— elementary cobordism

- trivial
- handle attachment

To construct an invariant for 3-manifolds Y from this, need

a) decomposition of Y into elementary cobordisms w. connected dividing surfaces of genus ≥ 1 (since $M(S^2)=\emptyset$)

e.g. from S^1-valued Morse function



$\rightsquigarrow$

$$\begin{tikzcd}
 & M(\Sigma_1) \arrow[dr, "L(Y_{12})"] & \\
M(\Sigma_0) \arrow[ur, "L(Y_{01})"] & & M(\Sigma_2) \arrow[dl, "L(Y_{23})"] \\
 & M(\Sigma_3) \arrow[ul, "L(Y_{30})"] &
\end{tikzcd}$$

a') Floer homology for cyclic sequences of Lagrangian correspondences

b) invariance of decompositions e.g. up to Cerf moves :

- critical point switch
- critical point cancellation
- cylinder cancellation
- diffeomorphism equivalence

} can all be viewed as (several) compositions of elem. cobordisms

$$\begin{tikzcd}\Sigma_0 \arrow[r, "Y_{01}"] \arrow[rr, bend right, "Y_{01} \cup Y_{12}"'] & \Sigma_1 \arrow[r, "Y_{12}"] & \Sigma_2\end{tikzcd}$$

Thm [Cerf,..., Gay-Kirby,..] Cerf moves with connected levels suffice.

Thm:
$$\begin{tikzcd}M(\Sigma_0) \arrow[r, "L(Y_{01})"] \arrow[rr, bend right, "\boxed{L(Y_{01} \cup Y_{12}) = L(Y_{01}) \circ L(Y_{12})}"'] & M(\Sigma_1) \arrow[r, "L(Y_{12})"] & M(\Sigma_2)\end{tikzcd}$$

elementary cobordism compositions correspond to embedded composition of Lagrangian correspondences

b') isomorphism of Floer homology under embedded composition

quilted Floer homology for cyclic generalized Lagrangian correspondences

$$\underline{L} = \left(L_{i(i+1)} \subset (M_i \times M_{i+1}, (-\omega_i)\oplus\omega_{i+1}) \right)_{i\in\mathbb{Z}_r}$$

$$\begin{tikzcd}
& M_1 \arrow[r, "L_{12}"] & M_2 \arrow[d] \\
M_0 \arrow[ur, "L_{01}"] & & \cdots \\
& M_{r-1} \arrow[ul, "L_{(r-1)r}"] &
\end{tikzcd}$$

$$\boxed{HF(\underline{L}) := HF(\Delta_{M_0}\times\Delta_{M_1}\cdots\times\Delta_{M_{r-1}},\ L_{01}\times L_{12}\times\cdots\times L_{(r-1)r})}$$

$$\downarrow \qquad\qquad \cap$$

$$M_0\times M_1\times M_1\times\cdots\times M_{r-1}\times M_0$$

- generated by $L_{01}\times L_{12}\times\cdots\times L_{(r-1)r} \cap (\Delta_{M_0}\times\Delta_{M_1}\cdots\times\Delta_{M_{r-1}})^T \cong \{(p_i\in M_i)_{i\in\mathbb{Z}_r} \mid (p_i,p_{i+1})\in L_{i(i+1)}\}$
- ∂ counts $\{v:\mathbb{R}\times[0,1]\to M_0\times\cdots\times M_0 \mid \bar\partial_J v=0,\ v(\cdot,0)\in(\Delta\times\cdots\Delta)^T,\ v(\cdot,1)\in L\times\cdots\times L\}/_{\mathbb{R}}$

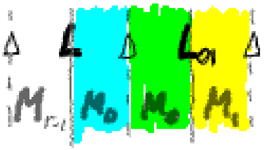
quilted setup: specialize & generalize to $(u_i:\mathbb{R}\times[0,\delta_i]\to M_i,\ \bar\partial_{J_i}u_i=0)_{i\in\mathbb{Z}_r}$

quilted Floer trajectories:

$$\left\{ v : \mathbb{R}\times[0,1] \to M_0\times\ldots\times M_0 \;\middle|\; \bar\partial_J v = 0 \,,\, v(\cdot,0)\in\in(\Delta\times\ldots\Delta)^T ,\, v(\cdot,1)\in L\times\ldots\times L \right\}/_{\mathbb{R}}$$

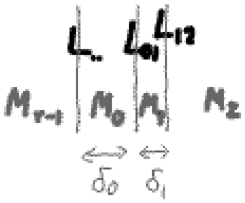
specialize: for "split" $J = (J_0, -J_1, J_1, \ldots, J_{r-1}, -J_0)$

$$\cong \left\{ (u_i : \mathbb{R}\times[-1,1]\to M_i)_{i\in\mathbb{Z}_r} \;\middle|\; \bar\partial_{J_i} u_i = 0 \,,\, (u_i(s,1), u_{i+1}(s,-1))\in L_{i(i+1)} \right\}/_{\mathbb{R}}$$



generalize to widths $\underline{\delta} = (\delta_i > 0)_{i\in\mathbb{Z}_r}$

$$\boxed{\begin{array}{l} \left\{ (u_i : \mathbb{R}\times[0,\delta_i]\to M_i)_{i\in\mathbb{Z}_r} \;\middle|\; \bar\partial_{J_i} u_i = 0 \,,\, \begin{array}{l} (u_i(s,\delta_i), u_{i+1}(s,0))\in L_{i(i+1)} \\ \lim\limits_{s\to\pm\infty} (u_i(s,\cdot)) = \underline{p}^{\pm} \end{array} \right\}/\mathbb{R} \\ \quad =: \mathcal{M}(\underline{p}^-, \underline{p}^+; \underline{\delta}) \end{array}}$$



Lemma: After split Hamiltonian perturbation, $\tilde{L}_{i(i+1)} := (\mathrm{id}_{M_i} \times \phi_{H_{i+1}}^{\delta_{i+1}})(L_{i(i+1)})$

all partial compositions are transverse (⇒ immersions)

$$\tilde{L}_{i(i+1)} \times \dots \times \tilde{L}_{(j-1)j} \pitchfork M_i \times \Delta_{i+1} \times \dots \times \Delta_{j-1} \times M_j \looparrowright \tilde{L}_{i(i+1)} \circ \dots \circ \tilde{L}_{(j-1)j} \subset M_i \times M_j$$

Thm: Fix widths $\underline{\delta}$. $\exists \mathcal{H}_{reg} \subset \{ (H_i : M_i \to \mathbb{R})_{i \in \mathbb{Z}_r} \}$ comeagre : $\forall \underline{H} \in \mathcal{H}_{reg}$

$\exists \mathcal{J}_{reg} \subset \{ (J_i : [0, \delta_i] \to \mathcal{J}(M_i, \omega_i))_{i \in \mathbb{Z}_r} \}$: $\mathcal{M}(\underline{p}^-, \underline{p}^+; \underline{\delta})$ cut out transversely.

Proof: 2006: • all u_i nonconstant
• all u_i constant

2011: • some u_i constant ⇝ using Lemma & a lot of Sard-Smale

problem: u_i constant but linearized seam cond$^{\underline{n}}$ nonconstant $\Big\}$ ⟶ does not happen generically

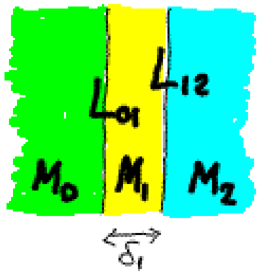
Thm: Differentials for varying widths are chain homotopy equivalent (modulo disk bubbling).

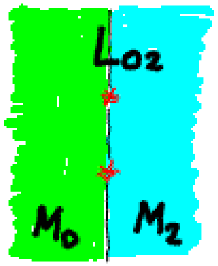
strip shrinking analysis for width $\delta_1 \to 0$

$$\underline{L} = (\dots, L_{01}, L_{12}, \dots) , \quad L_{01}\times L_{12} \pitchfork M_0\times\Delta_1\times M_2 =: \tilde{L}_{02} \xrightarrow[\text{immersion}]{\pi_0\times\pi_2} L_{02}\subset M_0\times M_2$$

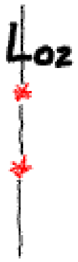
Conjecture: For $\delta_1>0$ sufficiently small, there is a bijection between

$\mathcal{M}(\underline{p}^-, \underline{p}^+, \underline{\delta})$ and "punctured quilted Floer trajectories for $(\dots, L_{02}, \dots)$"





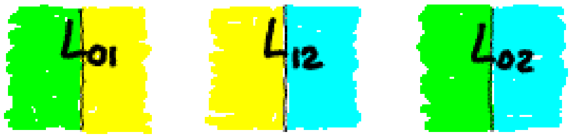
+ (trees of) bubbles at punctures (some of which switch sheets) + lift of



to $\tilde{L}_{02}$

bubbles

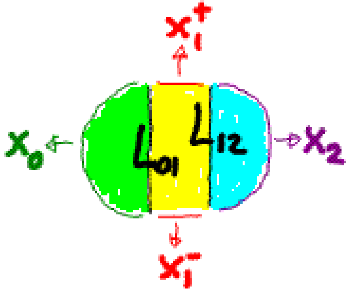
$\hat{=}$ $S^2 \to M_i$ should be codim ≥ 2



$\hat{=}$ $D^2 \to M_i \times M_j$ disk bubbles
$\partial D \to L_{ij}$

figure eight bubbles → switching if $x_1^- \neq x_1^+$
→ non-switching if $x_1^- = x_1^+$

know: convergence for a sequence of radii



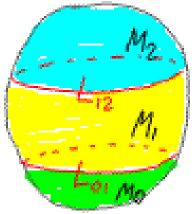
to $(x_0, x_1^{\pm}, x_2) \in \tilde{L}_{02}$
$\downarrow$
$(x_0, x_2) \in L_{02}$

don't know: removal of singularity, Fredholm theory, gluing, transversality

Cor.: Energy of figure eight bubbles with $x_1^- = x_1^+$ (e.g. for $\tilde{L}_{02} \xrightarrow{1-1} L_{02}$)

is given by $\langle [\omega_0, \omega_1, \omega_2], \text{"}\pi_2(M_0, M_1, M_2; L_{01}, L_{12})\text{"} \rangle$

$$\parallel \quad \left\{ \sum_{i=0}^{2} \int v_i^* \omega_i \;\middle|\; \right.$$



$$\left. \begin{array}{ll} v_2 : D^2 \to M_2 & (v_1 \times v_2)(S^1) \subset L_{12} \\ v_1 : S^1 \times [-1,+1] \to M_1 & \\ v_0 : D^2 \to M_0 & (v_0 \times v_1)(S^1) \subset L_{01} \end{array} \right\}$$

Theorem: If $\tilde{L}_{02} \to L_{02}$ injective and bubbling excluded (e.g. by monotonicity or energy control)

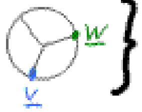
then $\boxed{HF(\ldots, L_{01}, L_{12}, \ldots) \simeq HF(\ldots, L_{02}, \ldots)}$.

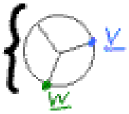
Example / Application: Chekanov-Polterovich torus

$$K = \Big\{ (\underline{v},\underline{w}) \in \underset{\overset{\|}{\{|\underline{v}|=1\} \subset \mathbb{R}^3}}{S^2 \times S^2} \;\Big|\; v_3 + w_3 = 0 ,\ \measuredangle(\underline{v},\underline{w}) = \tfrac{2\pi}{3} \Big\}$$

As correspondence $(S^2)^- \xrightarrow{K} S^2$ it "maps"

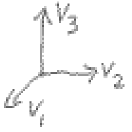
- $T_3' = \{\underline{v} \in S^2 \mid v_3 = 0\}$ to $T_3' \circ K : \{(\underline{v},\underline{w}) \in K \mid \underline{v} \in T_3'\} \xrightarrow{pr_2} S^2$

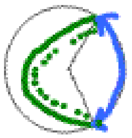
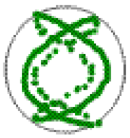
$\| \ \{$ 

$\} \cup \{$ 

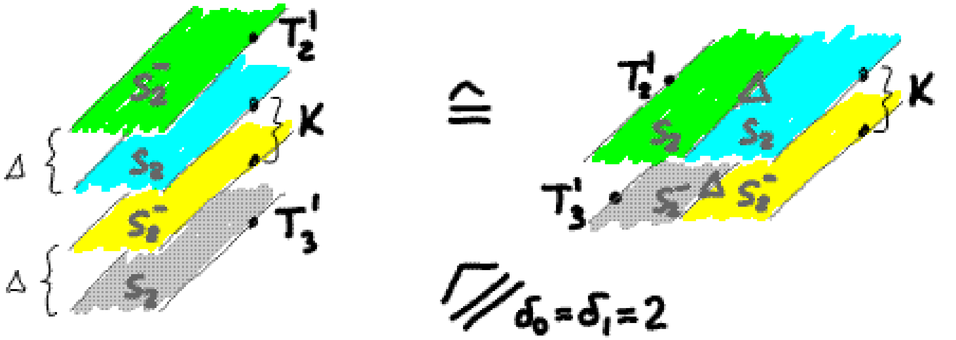
$\} \xrightarrow[(\underline{v},\underline{w}) \mapsto \underline{w}]{\text{triv. double cover}} T_3'$

- $T_2' = \{\underline{v} \in S^2 \mid v_2 = 0\}$ to $T_2' \circ K : \{(\underline{v},\underline{w}) \in K \mid \underline{v} \in T_2'\} \xrightarrow{pr_2} S^2$

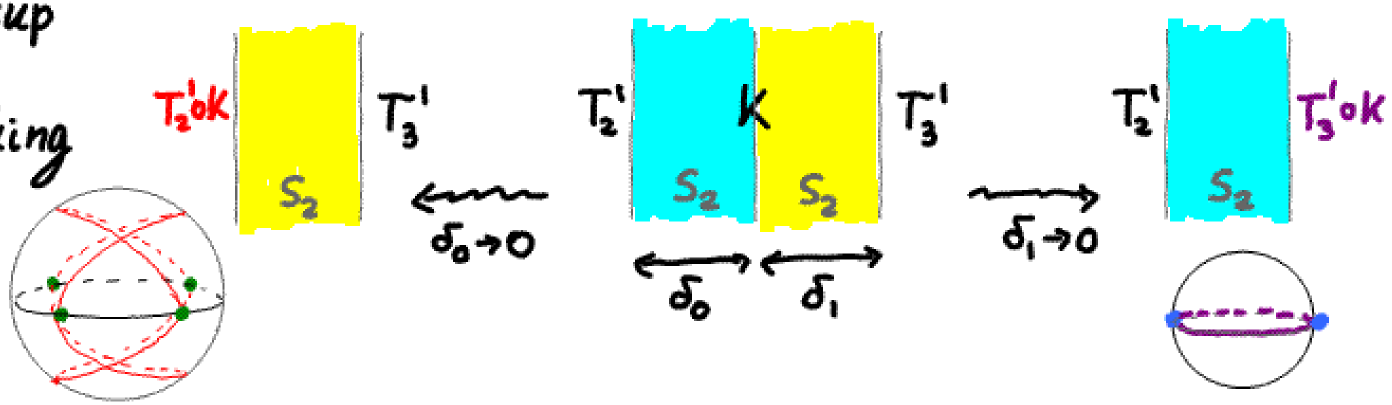


$\{ \underline{v}$  $\underline{w} \} \cup \{ \underline{w}$  $\underline{v} \} \longrightarrow$ 

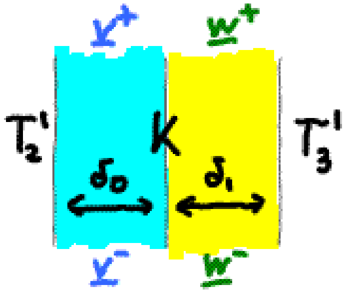
$$HF\left(\begin{tikzcd} pt \arrow[r, "T_2'"] & (S^2)^- \arrow[d, "K"] \\ & S^2 \arrow[ul, "T_3'"] \end{tikzcd} \right) = HF(\{(\underline{v},\underline{v},\underline{w},\underline{w})\}, T_2' \times K \times T_3') \cong HF(T_2' \times T_3', K)$$

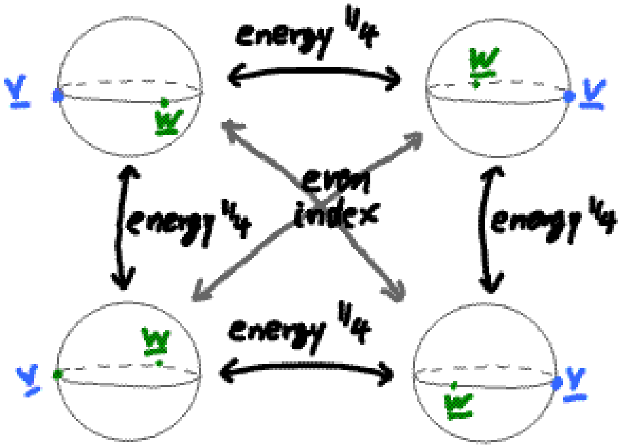


quilted setup
&
strip shrinking



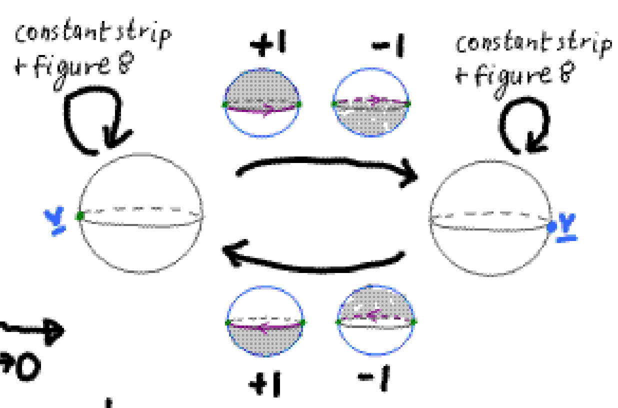
$CF(T_2^1 \times T_3^1, K)$



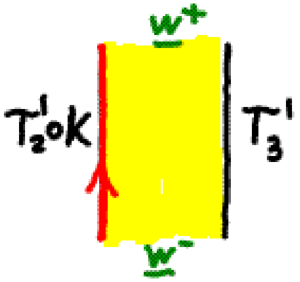


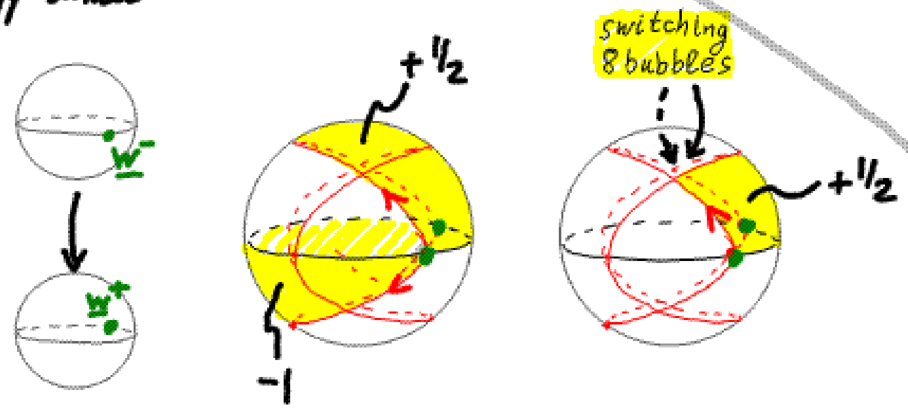
$\delta_1 \to 0$

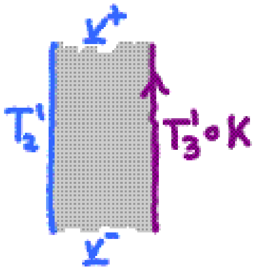
& energy $\in \frac{1}{4}\mathbb{N}$



$\delta_0 \to 0$ & energy small

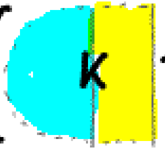
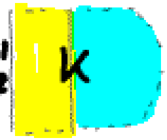






<u>possible bubbling</u>: • energy of trajectories : $1/4$; $\mathrm{Vol}\, S^2 = 1$

• energy of disk/sphere bubbles $\in \frac{1}{2}\mathbb{N}$ $\Rightarrow$ ∂ unchanged by varying widths

• energy of figure eight bubbles $\frac{1}{4}\mathbb{N} = \Big\{$  $T_3^1 \Big\}$, $\Big\{ T_2^1$  $\Big\}$ = smaller $\sim \frac{1}{16}$

<u>Thm</u>: With twisted coefficients s.t. $\partial^2 = 0$ get $\partial = 0$ in this presentation

$$\Rightarrow HF(T^1 \times T^1, K) \simeq H_*(T^2)$$

<u>Proof</u>: $\overset{\longleftrightarrow}{\longleftrightarrow}$ $\partial = 0$ from $\delta_1 \to 0$

$\updownarrow$ $\partial = \lambda$ from symmetry

$\partial^2 = 0$ from Chekanov-Schlenk classification of disks on K