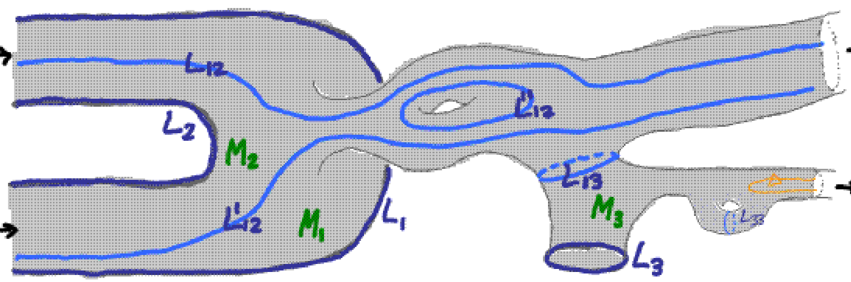
Stanford 2012

① From 4-manifolds to pseudoholomorphic quilts

② Transversality and strip shrinking

③ Construction recipe for 3-/4-manifold invariants



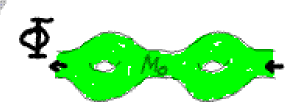
Recall: A quilted surface Q with symplectic labels (monotone/exact/or the like) defines a relative invariant $\Phi_Q : \bigotimes_{e \in E_{in}} HF(\underline{L}_e) \longrightarrow \bigotimes_{e \in E_{out}} HF(\underline{L}_e)$ satisfying

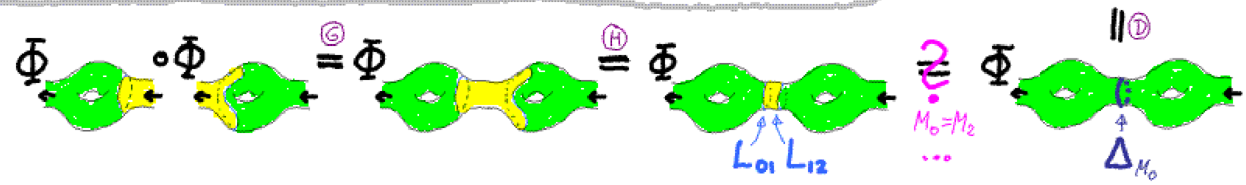
Ⓗ homotopy $(Q_t, j_t, \underline{J}_t) \Rightarrow \Phi_{Q_0} = \Phi_{Q_1}$

Ⓖ composition = gluing $\Phi_{Q_1} \circ_{E^1_{in} \cong E^0_{out}} \Phi_{Q_0} = \Phi_{Q_1 \# Q_0}$

Ⓓ insertion of diagonal $\Phi_{(Q,\mathcal{S})} \simeq \Phi_{(Q, \mathcal{S} \sqcup S')}$, $L_{S'} = \Delta_{M_p}$

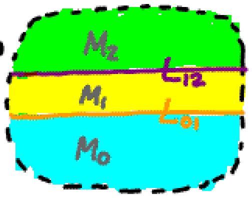
Ex.:





Thm (strip/annulus shrinking) [W.-Woodward] **for embedded composition:** If $L_{01}\times L_{12} \pitchfork M_0\times\Delta_{M_1}\times M_2 \subset M_0^-\times M_1\times M_1^-\times M_2$, with $\downarrow$ (embedded) onto $L_{01}\circ L_{12} \subset M_0^-\times M_2$

and "bubbling is a priori excluded" then

$Q \supset$ 

$Q' \supset$ 

$$HF(\ldots L_{01}, L_{12}\ldots) \xrightarrow{\iota_{SS} = id_{CF} \text{ for suitable data}} HF(\ldots L_{01}\circ L_{12}\ldots)$$

$$\begin{array}{ccc} \bigotimes_{e\ in} HF(\underline{L}_e) & \xrightarrow{\Phi_Q} & \bigotimes_{e\ out} HF(\underline{L}_e) \\ \iota_{SS}\downarrow\wr & & \wr\downarrow\iota_{SS} \\ \bigotimes_{e\ in} HF(\underline{L}'_e) & \xrightarrow{\Phi_{Q'}} & \bigotimes_{e\ out} HF(\underline{L}'_e) \end{array}$$

commutes.

Proof by adiabatic limit:

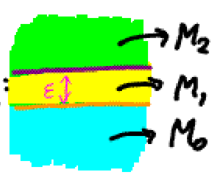
- in strip case $\exists$ Hamiltonian perturbations in $M_{i\neq 1}$ giving transversality in both

$$\left\{(\ldots p_0,p_1,p_2\ldots) \,\middle|\, \ldots \begin{array}{l}(p_0,p_1)\in L_{01}\\(p_1,p_2)\in L_{12}\end{array}\ldots\right\} \underset{\text{by injectivity of composition}}{\simeq} \left\{(\ldots p_0,p_2\ldots) \,\middle|\, \ldots (p_0,p_2)\in L_{01}\circ L_{12}\right\}$$

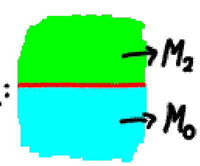
$\quad \| \qquad\qquad\qquad\qquad\qquad\qquad \|$

$CF(\ldots L_{01}, L_{12}\ldots)$ generators $\qquad$ $CF(\ldots L_{01}\circ L_{12}\ldots)$ generators

- for "regular $\underline{J}$" and strip/annulus width $\varepsilon > 0$ suff. small there is a bijection

$\left\{ \underline{u}: \right.$ 

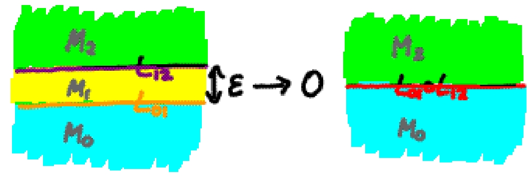
$$\left.\begin{array}{l} \bar\partial_{J_i} u_i = 0 \quad \forall i = \ldots 0,1,2\ldots \\ \ldots, (u_0,u_1)(S_{01}) \subset L_{01}, \\ (u_1,u_2)(S_{12}) \subset L_{12}, \ldots \\ \ldots + \int|du_0|^2 + \int|du_1|^2 + \int|du_2|^2 + \ldots = E \\ \text{ind}[\underline{u}] = 0 \ (+1) \end{array}\right\} / \mathbb{R} \;\overset{*}{\simeq}\; \left\{ \underline{v}: \right.$$



$$\left.\begin{array}{l} \bar\partial_{J_i} v_i = 0 \quad \forall i = \ldots 0,2\ldots \\ \ldots \\ (v_0,v_2)(S_{02}) \subset L_{01}\circ L_{12} \\ \ldots \\ \ldots + \int|dv_0|^2 + \int|dv_2|^2 + \ldots = E \\ \text{ind}[\underline{v}] = 0 \ (+1) \end{array}\right\} / \mathbb{R}$$

(+1) for Floer trajectories $\to$ $/\mathbb{R}$

* if we can a priori exclude all
possible bubbling in strip shrinking



$S^2 \to M_0$ $S^2 \to M_1$ $S^2 \to M_2$

$D^2 \to M_0^- \times M_1$ $D^2 \to M_1^- \times M_2$

"figure eight"
removal of singularity ~~unclear~~
and "bubbles off some energy"

∃ "homotopic" quilt map (not holom) with same energy

([Bottman-W]: proven!
...but not a Fredholm problem...yet...)

$D^2 \to M_0^- \times M_2$

Examples of a priori bubble exclusion

- "all is exact": $\omega_M = d\lambda_M$ for all M on patches
  $(\tau = 0)$ $(-\lambda_M) \times \lambda_N |_L = df_L$ for all $L \subset M^- \times N$ on seams

OR

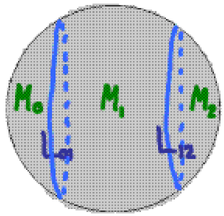
- "all is monotone": $[\omega_M] = 2\tau \cdot c_1(TM)$ for all M on patches
  $\tau > 0$ $[-\omega_M \times \omega_N] = \tau \cdot \mu_{Maslov}$ on $\pi_2(M^- \times N, L)$ for all $L \subset M^- \times N$ on seams
  similar ... on other relevant π_2's
  (e.g. guaranteed by $\pi_1(L) \hookrightarrow \pi_1(M^- \times N)$ torsion $\forall L$)

$\Downarrow$

energy $= E_0$(limits at ends) $+ \tau \cdot$ Fredholm index

OR

- "quilted aspherical" : $\underline{u} = (u_0, u_1, u_2)$ smooth, satisfies seam conditions (but not necc. holomorphic)



$$E_{S^2}(\underline{u}) := \int u_0^* \omega_{M_0} + \int u_1^* \omega_{M_1} + \int u_2^* \omega_{M_2} = 0$$

$$\left(\Rightarrow [\omega_i]|_{\pi_2(M_i)} = 0 \ , \ [-\omega_i \times \omega_j]|_{\pi_2(M_i^- \times M_j, L_{ij})} = 0 \right)$$

OR

- "energy control" : Fix energy E_Q and limit points of hol. quilts

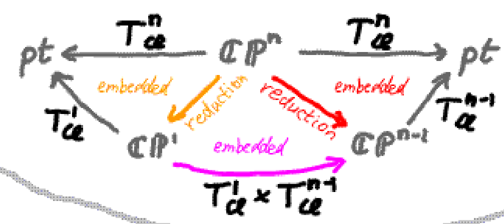
s.t. $E_Q < \min_{E(\underline{u})>0} E_{S^2}(\underline{u}) + E_{Q'}^{min}$

minimal bubble energy — minimal energy of Q' quilt with corresp. limit points

Ex: $HF(T^n_{cl}, T^n_{cl}) \cong HF(T^1_{cl}, T^1_{cl}) \otimes HF(T^{n-1}_{cl}, T^{n-1}_{cl})$ in $\mathbb{CP}^n$

Corollary: [c.f. Cho] in standard spin structure $HF(T^n_{cl}, T^n_{cl}) \cong H_*(T^n)$

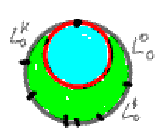
because of embedded compositions and monotonicity

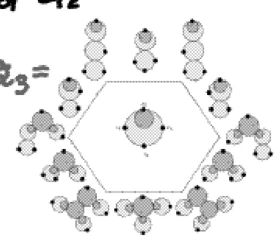


Ex: $L_{01} \subset M_0^- \times M_1$ induces a functor

$$\Phi_{L_{01}} : \mathcal{F}uk^{\#}(M_0) \to \mathcal{F}uk^{\#}(M_1) \ \text{ s.t. } \ \Phi_{L_{12}} \circ \Phi_{L_{01}} = \Phi_{L_{01} \circ L_{12}} \ \text{ if embedded}$$

"counts" $\left\{ \begin{array}{l} \underline{u} : Q \to (M_0, M_1, L_0^0 .. L_0^k, L_{01}) \\ (j, J_0, J_1)\text{-holomorphic} \end{array} \middle| \ j \in \mathcal{Q}_k \right\}$





[Ma'u - W - Woodward ...still in progress]

... but... how do we "count" $\{\underline{u} : Q \to (\underline{M}, \underline{L}) \mid \bar{\partial}\underline{u} = 0, \dots\} / \underline{u}\circ\varphi \sim \underline{u}$

Finite dimensional regularization

$E \downarrow \circlearrowleft s$, B — smooth section in finite dimensional vector bundle with $s^{-1}(0)$ compact

$\Rightarrow \exists \nu \approx 0 : (s+\nu)^{-1}(0)$ smooth manifold

$\forall \nu' \approx 0 : (s+\nu')^{-1}(0)$ cobordant

$\mathcal{E} \downarrow \circlearrowleft s$, $\bar{\partial}\underline{u} \uparrow$

$\mathcal{B} = \{\underline{u} : Q \to (\underline{M}, \underline{L}) \mid \dots\}$

We need $\{\varphi\}$-equivariant transverse perturbation ν although $\bar{s}^{-1}(0)$ noncompact.

Geometric magic: get ν from perturbing $\underline{J}$ if "solutions $\underline{u}$ are injective on parts of domain that $\underline{J}$ cannot depend on" (e.g. bubbles; $\mathbb{R}$ in HF).

Abstract machinery with $\bar{s}^{-1}(0)$ = compactified moduli space

To construct quilt invariants and prove strip shrinking, we assume monotonicity & orientability to exclude all bubbling for index 0,1.
Then e.g. for $\mathcal{M}(\underline{p}^-, \underline{p}^+; \underline{\delta}) = \{$quilted Floer cylinders of widths $\underline{\delta}\} / \mathbb{R}$

Thm: $\exists \mathcal{H}_{reg} \subset \{(H_i : M_i \to \mathbb{R})_{i \in \mathbb{Z}_r}\}$ comeagre : $\forall \underline{H} \in \mathcal{H}_{reg}$

$\exists \mathcal{J}_{reg} \subset \{(J_i : [0, \delta_i] \to \mathcal{J}(M_i, \omega_i))_{i \in \mathbb{Z}_r}\}$ comeagre : $\forall \underline{J} \in \mathcal{J}_{reg}$: $\bar{\partial}_{(\underline{J}, \underline{H})} \pitchfork 0$

$\Rightarrow \mathcal{M}(\underline{p}^-, \underline{p}^+; \underline{\delta}, \underline{H}, \underline{J})$ smooth manifold

Proof: 2006: • all u_i nonconstant
• all u_i constant

problem: u_i constant but linearized seam cond^n nonconstant } → does not happen for $\underline{H} \in \mathcal{H}_{reg}$, $\underline{J} \in \mathcal{J}_{reg}$

2011: • some u_i constant ⇝ use 7 times Sard-Smale

ABSTRACT OPTIONS

Polyfold regularization

[Hofer-Wysocki-Zehnder]

$E \downarrow B$, s: scale-smooth section in strong M-polyfold bundle with $s^{-1}(0)=\overline{\mathcal{M}}$ compact

$\exists \nu \approx 0$: $(s+\nu)^{-1}(0)$ smooth compact manifold

$\forall \nu' \approx 0$: $(s+\nu')^{-1}(0)$ cobordant

TODO for quilts: Fredholm setup for eg. figure 8 bubbling

virtual regularization

[Li/Liu-Tian; Siebert; Fukaya-Ono; Joyce]

McDuff-W. version for simplified trivial isotropy

$E \downarrow B$, s: smooth section functor in category bundle with finite dimensional object spaces

s.t. $|s^{-1}(0)| = s^{-1}(0)/_{\text{morph.}} \simeq \overline{\mathcal{M}}$ compact

$\exists \nu : B \to E$ admissible functor : $|(s+\nu)^{-1}(0)|$ smooth compact manifold

$\forall \nu'$ admiss. : $|(s+\nu')^{-1}(0)|$ cobordant

TODO for quilts: finite dim. reduction for eg. figure 8 bubbling

[Fukaya-Ono '96 $\pm \varepsilon$]

Def: A (smooth; trivial isotropy) Kuranishi chart / finite dimensional reduction for $\overline{\mathcal{M}}$ is

- trivial finite dimensional vector bundle $E \downarrow B$
- smooth section $s : B \to E$
- homeomorphism $\psi : s^{-1}(0) \xrightarrow{\sim} F \subset_{\text{open}} \overline{\mathcal{M}}$