Banff 2014

How to extend 2+1 "field theories" to 2+1+1 dimensions

inspired by joint work w. C. Woodward ; Gay-Kirby ; Banff 2010 - ...

Goal: Every "natural" 3-dim. "TFT" extends naturally to a 4-dim. "TFT".

functor $\widetilde{\mathrm{Bor}}_{2+1} \to \mathcal{C}$ 2-functor $\widetilde{\mathrm{Bor}}_{2+1+1} \to \mathcal{C}^{\#}$

what gauge theory gives:
- defined on presentation
- natural 

guiding examples: Donaldson-Yang-Mills / Seiberg-Witten invariants and their (conjectural) symplectic versions

disclosure: gauge theoretic "TFT"s are not monoidal

since $\mathrm{Sym}^{g}(\Sigma_1 \sqcup \Sigma_2) \neq \mathrm{Sym}^{g_1}(\Sigma_1) \times \mathrm{Sym}^{g_2}(\Sigma_2)$ SW

$\pi_1(\Sigma_1 \sqcup \Sigma_2 \setminus pt) \neq \pi_1(\Sigma_1 \setminus pt) * \pi_1(\Sigma_2 \setminus pt)$ YM

Def^{n}: A presentation of a category $\mathcal{C}$ consists of

- simple morphisms $SMor \subset Mor_{\mathcal{C}}$

  that generate: $\forall\, Y \in Mor_{\mathcal{C}}\ \exists\, Y_1 \ldots Y_k \in SMor\ :\ Y = Y_1 \circ \ldots \circ Y_k$

- moves $\underset{\omega}{Mv} \subset \bigcup_{m,n \geq 1} SMor^m \times SMor^n$

  $m=n=2:\quad (Y_1, Y_2 ; Y_1', Y_2') \qquad Y_1 \circ Y_2 = Y_1' \circ Y_2'$

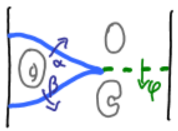
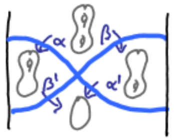
  s.t. any two decompositions $Y_1 \circ \ldots \circ Y_k = Y = Y_1' \circ \ldots \circ Y_{k'}'$ are

  related by a (not unique) sequence of moves

Ex.: $\mathcal{C} = Bor_{2+1}$ with $SMor$ = handle attachments

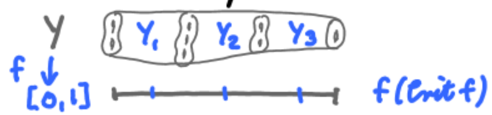
& diffeomorphisms

Mv = Cerf moves

$Y_\alpha \circ Y_\beta =$  $= Y_\varphi\ ,\quad Y_\alpha \circ Y_{\beta'} =$  $= Y_\beta \circ Y_{\alpha'}\ ,\quad \begin{matrix} Y_\alpha \circ Y_\beta \\ \wr\wr \\ Y_{\beta'} \circ Y_{\alpha'} \end{matrix}$

Note: A presentation $(SMor, Mv)$ of $\mathcal{C}$ induces a 2-category $\widetilde{\mathcal{C}}$: $\widetilde{Obj} = Obj_{\widetilde{\mathcal{C}}}$, $\widetilde{Mor}$ = sequences in $SMor$, ${}^2Mor$ = sequences in Mv with $\mathcal{C} \simeq \widetilde{\mathcal{C}}/_{{}^2Mor}$.

Ex B: $\widetilde{Bor}_{2+1}$ $\hat{=}$ 3-cobordisms with Cerf decomposition



Ex A: $\widetilde{Symp}$ given by Obj : symplectic manifolds defines $Symp := \widetilde{Symp}/_{{}^2Mor}$

$SMor(M,M')$: Lagrangian submanifolds $\subset M^- \times M'$

$\widetilde{Mor}(M,M')$: $M \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2 \dots \xrightarrow{L_{..}} M'$

Mv :
$$\begin{tikzcd} M_0 \arrow[r, "L_{01}"] \arrow[rr, bend right, "L_{01}\circ L_{12}"'] & M_1 \arrow[r, "L_{12}"] \arrow[d, phantom, "\Downarrow"] & M_2 \\ & {} & \end{tikzcd}$$

if $L_{01} \times L_{12} \pitchfork M_0 \times \Delta_{M_1} \times M_2$

$\downarrow pr_{M_0 \times M_2}$ injective

$L_{01} \circ L_{12} \subset M_0^- \times M_2$

is embedded

Thm [WW '07-..] To construct a "TFT" $\mathrm{Bor}_{2+1} \to \mathcal{C}at = \left(\begin{smallmatrix}\text{categories}\\ \text{functors}/\sim\end{smallmatrix}\right)$

that induces HF-valued 3-manifold invariants

it suffices to construct a functor $\mathrm{Bor}_{2+1} \to \mathrm{Symp} = \widetilde{\mathrm{Symp}}/_{2\mathrm{Mor}}$

on the Cerf presentation with values in (monotone...) $\widetilde{\mathrm{Symp}}$.

$\left(\mathrm{Obj}_{\mathrm{Bor}} \to \mathrm{Obj}_{\mathrm{Symp}},\ \mathrm{SMor}_{\mathrm{Bor}} \to \mathrm{SMor}_{\mathrm{Symp}} \text{ compatible with } \mathrm{Mv}_{\mathrm{Bor}} \to \mathrm{Mv}_{\mathrm{Symp}} \right)$

Ex: [Perutz] $\Sigma \mapsto \mathrm{Sym}^g \Sigma$, 

$\mapsto$ "$\left(\Sigma_g^g \times \alpha /_{\text{permute}} \hookrightarrow \mathrm{Sym}^g \Sigma_g \times \mathrm{Sym}^{g+1} \Sigma_{g+1} \right)$"

[WW] $\mathrm{Hom}(\pi_1(\Sigma \setminus pt), SU(n))/SU(n)$ with $\circledast \mapsto -\mathbb{1}$, $\quad \mathrm{Hom}(\pi_1(Y \setminus \mathit{line}), SU(n))/SU(n) \,\big|_{\partial Y}$ with $\mapsto -\mathbb{1}$

Proof: monotone $\widetilde{\mathrm{Symp}}$ extends to "the symplectic 2-category"

$\mathrm{Symp}^\#$: $\left(\begin{array}{l} \mathrm{Obj} = \mathrm{Obj}_{\mathrm{Symp}} \\ \mathrm{Mor} = \widetilde{\mathrm{Mor}} = \text{sequences in SMor} \\ {}^2\mathrm{Mor}(\underline{L}, \underline{L}') = \text{quilted } HF(M \overset{\underline{L}}{\underset{\underline{L}'}{\rightleftarrows}} M') \end{array} \right)$

$\Rightarrow$ $\exists$ functor $\mathrm{Symp} \longrightarrow \mathcal{C}at$

$M \longmapsto \mathrm{Mor}(pt, M)$

$\underline{L} \longmapsto$ "$\circ\, \underline{L}$" functor

$Y = \Sigma_{R,\ldots} \longmapsto M_\Sigma\ \underline{L}_Y \longmapsto {}^2\mathrm{Mor}(\Delta_{M_\Sigma}, \underline{L}_Y) = qHF(\underline{L}_Y)$

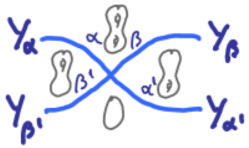
maybe Thm: To construct a "TFT" $\widetilde{Bor}_{2+1+1} \longrightarrow Cat = \left(\begin{array}{c} \text{categories} \\ \text{functors} \\ \text{nat.transf.} \end{array} \right)$

$\searrow Symp^{\#} \nearrow$

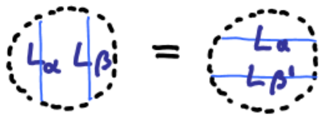
it suffices to construct a functor $Bor_{2+1} \to Symp$

on the Cerf presentation with values in (monotone ...) $\widetilde{Symp}$

that satisfies naturality:



$\Rightarrow$ "



"

Proof:

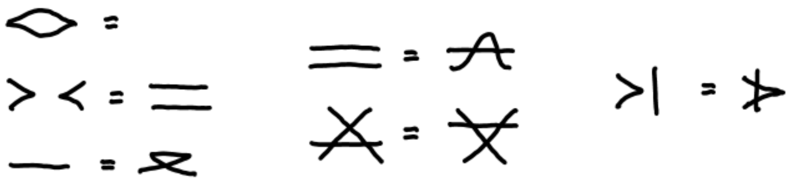
(i) [Whitney, ... Gay-Kirby]: Bor_{2+1+1} has a quilted presentation with

- generators $\widetilde{Bor}_{2+1}$, ${}^2SMor$ = 

  i.e. all 2-morphisms are represented by quilt diagrams tbd

  (Cerf diagrams of Morse 2-functions)

- moves



(ii) [WW] $\mathrm{Symp}^{\#}$ is a 2-category with cyclic ^{2}morphisms and

$$ {}^2\mathrm{Mor}\Big(\Sigma \overset{Y}{\underset{Y'}{\rightrightarrows}} P\Big) \simeq {}^2\mathrm{Mor}\left(\begin{smallmatrix} Y_1\ \Sigma_1\ Y_2 \\ \Sigma \qquad\quad P \\ Y_1'\ \Sigma_1'\ Y_2' \end{smallmatrix}\right) \simeq {}^2\mathrm{Mor}\left(\begin{smallmatrix} Y_i\ \Sigma_i\ Y_{i+1} \\ \vdots \\ Y_j\ \Sigma_j'\ Y_{j+1}' \end{smallmatrix}\right) $$

in Bor $\;\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$$ \{\partial X^4 = I\times\Sigma \,\square\, I\times P\} \qquad \{\partial X^4 = \bigcirc\} $$

quilts: every quilted surface
- S compact oriented 2-mfd
- $\Gamma \hookrightarrow S$ embedded graph
- $v_0 \in \Gamma$ "outgoing" vertex

with labels in $\mathrm{Symp}^{\#}$
- $\pi_0(S\setminus\Gamma) \to \mathrm{Obj}$
- edges $\to \mathrm{Mor}$
- vertices $\neq v_0 \to {}^2\mathrm{Mor}$

defines a 2-morphism $Q(S,\Gamma,\text{labels}) \in {}^2\mathrm{Mor}\left(\begin{smallmatrix} Y_1\ \Sigma_2\ Y_2 \\ \Sigma_1 \\ Y_k\ \Sigma_k \end{smallmatrix}\right)$ at $\begin{smallmatrix} Y_1\ \Sigma_2\ Y_2 \\ \Sigma_1\ v_0 \\ Y_k\ \Sigma_k \end{smallmatrix}$

satisfying "strip shrinking"

$$ (S,\Gamma) \supset v_1 \,(\mathbb{R}\times[0,1])\, v_2 \;\Rightarrow\; Q\left(X \begin{smallmatrix} \Sigma_2 \\ Y_{12} \\ \Sigma_1 \\ Y_{01} \\ \Sigma_0 \end{smallmatrix} X'\right) = Q\left(X \begin{smallmatrix} \Sigma_2 \\ Y_{01}\circ Y_{12} \\ \Sigma_0 \end{smallmatrix} X'\right) $$

(i') $\widetilde{\mathrm{Bor}}_{2+1+1}$ is a 2-category with cyclic morphisms and quilts (satisfying strip shrinking) whose 2-morphisms are represented by quilts with simple labels in $(\mathrm{Obj}, \mathrm{SMor}, {}^2\mathrm{SMor} = \ ,\ )$.

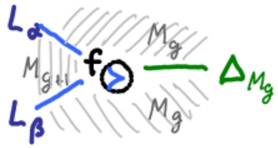
(ii) $\mathrm{Symp}^{\#}$ is a 2-category with cyc. mor., quilts, stripshrinking.

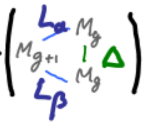
$\Rightarrow$ To extend
$$\begin{aligned} \widetilde{\mathrm{Bor}}_{2+1} &\longrightarrow \mathrm{Symp}^{\#} \\ \Sigma &\longmapsto M_\Sigma \\ Y=(Y_1,\dots Y_k) &\longmapsto (L_{Y_1},\dots L_{Y_k}) \end{aligned}$$
to $\widetilde{\mathrm{Bor}}_{2+1+1} = \begin{pmatrix} \text{surfaces} \\ \text{3-cobord. w. Cerf decomp.} \\ \text{4-cobordisms / diff rel } \partial \end{pmatrix}$ it suffices to

a) define $f_{\bigcirc}, f_{\bigotimes} \in {}^2\mathrm{Mor}_{\mathrm{Symp}^{\#}}$

b) check that moves between quilt representations in $\widetilde{\mathrm{Bor}}_{2+1+1}$ correspond to identities between 2-morphisms in $\mathrm{Symp}^{\#}$

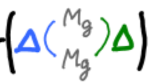
c) compose $\underbrace{\widetilde{\mathrm{Bor}}_{2+1+1}}_{\neq\ \mathrm{Bor}_{2+1+1}} \to \mathrm{Symp}^{\#} \to \mathcal{C}at$

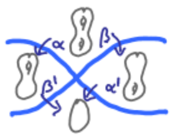
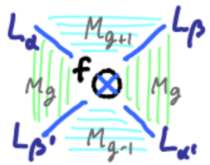
a)  

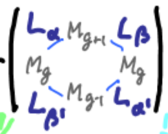
$^2\mathrm{Mor}\Big($  $\Big) \ni f_{>} := ss(1_{\Delta_{M_g}})$

$\alpha \pitchfork \beta = 1pt \Rightarrow L_\alpha \circ L_\beta = \Delta_{M_g} \Rightarrow$

$\Uparrow$ strip shrink

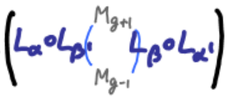
$^2\mathrm{Mor}\Big($  $\Big) \ni 1_{\Delta_{M_g}} =$ 

$^2\mathrm{Mor}\Big($  $\Big) \ni f_{\otimes} := ss(1)$

$ss \Uparrow \qquad \Uparrow ss$

$\alpha \cap \beta = \emptyset \Rightarrow$ $L_\alpha \circ L_\beta = L_{\beta'} \circ L_{\alpha'} \Rightarrow$ $^2\mathrm{Mor}\Big($  $\Big)$

$L_\alpha \circ L_{\beta'} = L_\beta \circ L_{\alpha'} \Rightarrow$ $^2\mathrm{Mor}\Big($  $\Big)$

$\Downarrow$ $1_{L_\alpha \circ L_\beta = L_{\beta'} \circ L_{\alpha'}}$ $=$

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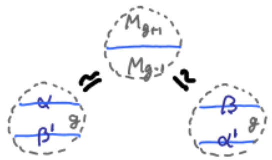
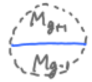


Conj.: The [Perutz]/[WW] functors $\mathrm{Bor}_{2+1} \to \mathrm{Symp}$ are natural, i.e.

$ss\Big($ 

$\Big) = ss\Big($ 

$\Big) \in {}^2\mathrm{Mor}\begin{pmatrix} L_\alpha & L_\beta \\ L_{\beta'} & L_{\alpha'} \end{pmatrix}$

Sketch of Proof: via SS-compatible degenerations of PDE's

[Salamon-W]

pseudoholomorphic quilts $\xleftrightarrow{\text{adiabatic limit}}$ Yang-Mills quilts $\xleftrightarrow{[SW14]}$ Yang Mills

[Dostoglou-S]+[W03-05]
+ work in progress

geometric morphism with explicit pert^{n} of Morse-Bott end

$\Downarrow$

3+1 Atiyah-Floer conjecture: The [WW] TFT extends Donaldson-Yang-Mills $\mathrm{Bor}_{3+1} \to \begin{pmatrix}\text{groups}\\ \text{homs}\end{pmatrix}$

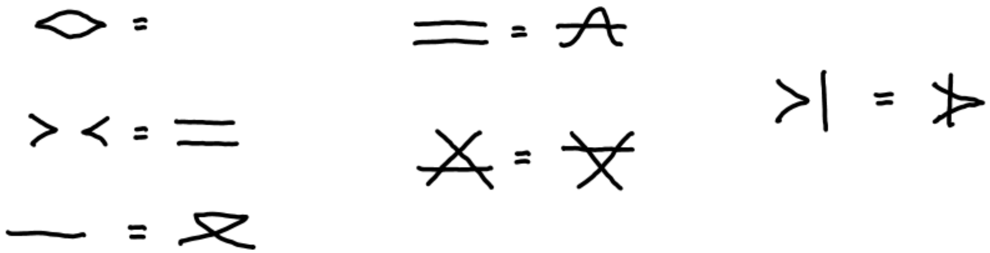
Consequences of Thm (maybe):

* To prove invariance of Perutz' matching invariants it suffices to show naturality  $\simeq$  and  $=$ 

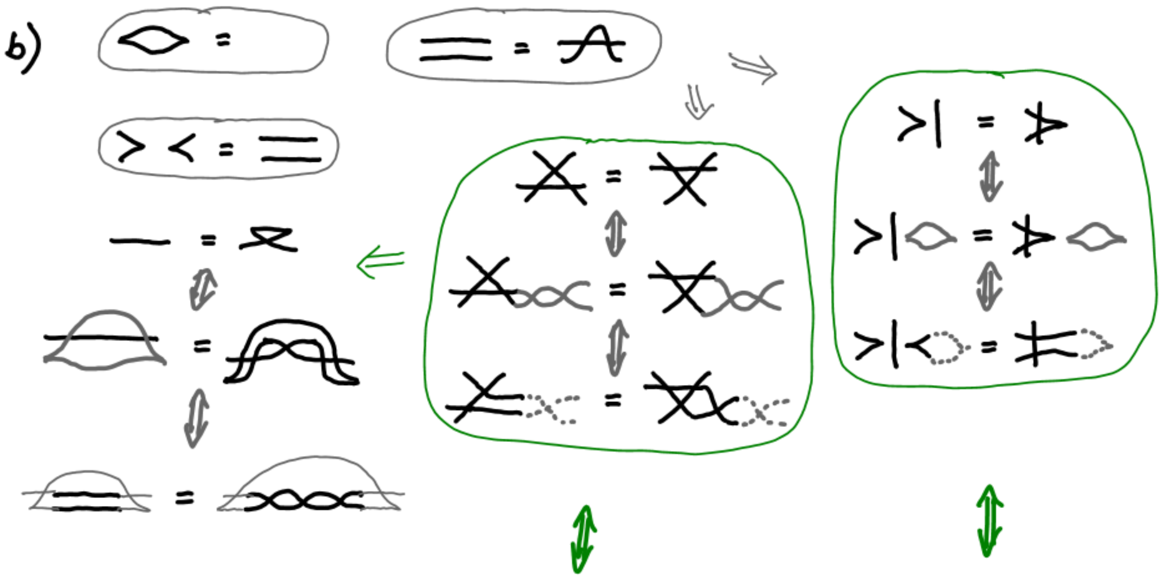
.

* To prove that the [Perutz] TFT extends Seiberg-Witten $\mathrm{Bor}_{3+1} \to \dots$ it suffices to show that [Taubes-...]-style isomorphisms of quilted 3-mfd invariants intertwine Q and Q_{SW}.

b)



b)



<u>TODO</u>: requiring naturality