A NOTE ON HILBERT SPACE BUNDLES

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This note provides an example of an infinite rank Hilbert space bundle $E \to M$ and two finite rank subbundles $E_1, E_2 \subset E$ such that no proper subbundle of E contains E_1 and E_2. For that purpose let H be an infinite dimensional separable Hilbert space, let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H, and denote by $\mathcal{L}(H)$ the space of bounded linear operators from H to itself.

Theorem. *There exists a smooth map $x : \mathbb{R} \to H \setminus \{0\}$ such that the following holds. Assume that $\Pi : \mathbb{R} \to \mathcal{L}(H)$ is continuous in the norm topology and satisfies*

$$\Pi(s)^2 = \Pi(s), \qquad \Pi(s)e_1 = e_1, \qquad \Pi(s)x(s) = x(s), \tag{1}$$

for every $s \in \mathbb{R}$. Then $\Pi(s) = \mathbb{1}$ is the identity for every $s \in \mathbb{R}$.

Every topological subbundle of $\mathbb{R} \times H \to \mathbb{R}$ can be represented by a continuous family of projections Π as a family of images $\bigcup_{s\in\mathbb{R}}\{s\} \times \operatorname{im}\Pi(s)$. So the theorem shows that $E_1 = \mathbb{R} \times \mathbb{R}e_1$ and $E_2 = \bigcup_{s\in\mathbb{R}}\{s\} \times \mathbb{R}x(s)$ are smooth rank one subbundles of $\mathbb{R} \times H$ so that every topological subbundle that contains E_1 and E_2 is given by $\Pi(s) = \mathbb{1}$, and hence is equal to $\mathbb{R} \times H$.

Proof. **Step 1.** *Construction of the map $x : \mathbb{R} \to H \setminus \{0\}$.*

Define a sequence $(e_{k_n})_{n\geq 2}$ of unit vectors by setting $k_{2^i+\ell} := 2 + \ell$ for $i \in \mathbb{N}$ and $0 \leq \ell < 2^i$. Choose a smooth cutoff function $\beta : \mathbb{R} \to [0, 1]$ so that $\beta(0) = 1$ and $\operatorname{supp}\beta \subset (-\frac{1}{2}, \frac{1}{2})$. Then define

$$x(s) := e_1 + \sum_{n=2}^{\infty} \beta_n(s)e_{k_n} \qquad \text{with} \qquad \beta_n(s) := \begin{cases} 2^{-n}\beta\left(s^{-1} - n\right), & \text{for } s > 0, \\ 0, & \text{for } s \leq 0. \end{cases}$$

The functions $\beta_n : \mathbb{R} \to [0, 1]$ for $n \geq 2$ satisfy $0 \leq \beta_n \leq 2^{-n} = \beta_n(\frac{1}{n})$ and are supported in the pairwise disjoint intervals $\left(\frac{2}{2n+1}, \frac{2}{2n-1}\right)$. Hence $x(s) \in H \setminus \{0\}$ for all $s \in \mathbb{R}$ and $x(\frac{1}{n}) = e_1 + 2^{-n}e_{k_n}$.

Step 2. *The map $x : \mathbb{R} \to H$ in Step 1 is smooth.*

That x is smooth on $\mathbb{R} \setminus \{0\}$ is obvious from the definition. It remains to check that all right-sided derivatives of x at $s = 0$ vanish. Given $\frac{2}{3} \geq s > 0$, denote by $n(s) \geq 2$ the unique integer with $n(s) - \frac{1}{2} \leq \frac{1}{s} < n(s) + \frac{1}{2}$. Then

$$\|x(s) - x(0)\| \;=\; \beta_{n(s)}(s) \;\leq\; 2^{-n(s)} \;\leq\; \sqrt{2}\, 2^{-1/s} \qquad \text{for} \qquad 0 < s \leq \tfrac{2}{3}.$$

This shows that $x : \mathbb{R} \to H$ is differentiable with $x'(0) = 0$. Now let $k \geq 1$ and assume, by induction, that x is k times differentiable with $x^{(k)}(0) = 0$. Similar to the previous estimate, there is a constant $c_k > 0$, depending on the $\mathcal{C}^k$-norm of β, such that $\|x^{(k)}(s)\| \leq c_k s^{-k-1}2^{-1/s}$ for $0 < s \leq \frac{2}{3}$. This shows that x is $k+1$ times differentiable and $x^{(k+1)}(0) = 0$. Hence x is smooth.

Step 3. *Let $x : \mathbb{R} \to H$ be the map in Step 1 and assume $\Pi : \mathbb{R} \to \mathcal{L}(H)$ satisfies* (1) *and is strongly continuous at $s = 0$. Then $\Pi(0) = \mathbb{1}$.*

By construction $e_{k_n} = 2^n\left(x(\frac{1}{n}) - e_1\right)$ and hence, by (1), $\Pi(\frac{1}{n})e_{k_n} = e_{k_n}$ for every integer $n \geq 2$. Given any integer $\ell \geq 0$ we have $e_{2+\ell} = e_{k_{n_i}}$ for a sequence $n_i = 2^i + \ell$ that diverges to infinity. Thus strong continuity implies $\Pi(0)e_{2+\ell} = \lim_{i\to\infty} \Pi\left(\frac{1}{n_i}\right)e_{2+\ell} = \lim_{i\to\infty} \Pi\left(\frac{1}{n_i}\right)e_{k_{n_i}} = \lim_{i\to\infty} e_{k_{n_i}} = e_{2+\ell}$. This shows that $\Pi(0)e_n = e_n$ for all $n \geq 2$. Since $\Pi(0)e_1 = e_1$ it follows that $\Pi(0) = \mathbb{1}$.

Step 4. *We prove the theorem.*

Let $x : \mathbb{R} \to H$ be the map in Step 1 and assume $\Pi : \mathbb{R} \to \mathcal{L}(H)$ satisfies (1) and is continuous in the norm topology. Then the set $U := \{s \in \mathbb{R} \,|\, \Pi(s) \text{ is bijective}\}$ is open and $0 \in U$, by Step 3. Since $\Pi(s)$ is a projection, $\Pi(s) = \mathbb{1}$ for every $s \in U$. Hence U is closed and hence $U = \mathbb{R}$. $\square$

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