PRELIMINARY DRAFT – A_∞-STRUCTURES FROM MORSE TREES WITH PSEUDOHOLOMORPHIC DISKS

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Abstract. For a Lagrangian submanifold, we define a moduli space of trees of holomorphic disk maps with Morse flow lines as edges, and construct an ambient space around it which we call the quotient space of disk trees. We show that this ambient space is an M-polyfold with boundary and corners by combining the infinite dimensional analysis in sc-Banach space with the finite dimensional analysis in Deligne-Mumford space. We then show that the Cauchy-Riemann section is sc-Fredholm, and by applying the polyfold perturbation we construct an A_∞ algebra over $\mathbb{Z}_2$ coefficients. Under certain assumptions, we prove the invariance of this algebra with respect to choices of almost-complex structures.

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1. Introduction

This introduction gives an informal description of the main objects and a quick survey of the main results. Let (M, ω) be a symplectic manifold of dimension $2n$ with $\omega|_{\pi_2(M)} = 0$, and L a Lagrangian submanifold of M. In addition, we choose a compatible almost complex structure J on M, and a Morse-Smale pair (f, g) on the Lagrangian L. We denote by D the closed unit disk in $\mathbb{C}$.

Let $p_k, \ldots, p_1, q$ be Morse critical points of f, μ an integer, and ν a non-negative real number. The **moduli space of holomorphic disk trees** $\mathfrak{M}([p_k \otimes \cdots \otimes p_1, q; \mu], \nu)$ consists of equivalence classes of the form $[\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}]$. Here T is an ordered tree with edges E and vertices V. Since the tree is ordered, each edge has a direction. Hence we view E as a subset of $(\mathrm{V} \times \mathrm{V}) \backslash \Delta$, and each edge is of the form $\mathrm{e} = (\mathrm{v}^-, \mathrm{v}^+)$. Furthermore, the set of vertices V is partitioned into **main vertices** and **critical vertices** $\mathrm{V} = \mathrm{V}^{\mathrm{m}} \sqcup \mathrm{V}^{\mathrm{c}}$. The set of critical vertices V^{c} contains k leaf vertices $\mathrm{w}_k, \ldots, \mathrm{w}_1$, and the root vertex $\mathrm{rt}(\mathrm{T})$, which correspond to the critical points $p_k, \ldots, p_1$, and q, respectively. The tuple $\underline{\gamma}$ consists of generalized Morse trajectories $\underline{\gamma}_{\mathrm{e}}$ for each edge $\mathrm{e} \in \mathrm{E}$. (A generalized Morse trajectory can be a series of broken Morse flow lines. See [20] for a thorough exposition and Section 2.2 for a brief description.) The tuple $(\underline{x}, \underline{u})$ consists of boundary marked points and J-holomorphic disk maps $(\underline{x}_{\mathrm{v}}, u_{\mathrm{v}})$ for each main vertex $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$. More precisely, $\underline{x}_{\mathrm{v}} = \{x_{\mathrm{v},\mathrm{e}} \in \partial D \,|\, \mathrm{e} \text{ adjacent to } \mathrm{v}\}$ is an *ordered* set of boundary marked points, aligned counter-clockwise on ∂D, and we require the map u_{v} to satisfy the Lagrangian boundary condition $u_{\mathrm{v}}(\partial D) \subset L$.

Furthermore, the tree of disk maps and generalized Morse trajectories satisfies the following additional conditions. We call $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$ a **ghost vertex** if $\langle [u_{\mathrm{v}}]_{H_2}, [\omega]_{H^2} \rangle = 0$. (Since u_{v} is J-holomorphic, $\langle [u_{\mathrm{v}}]_{H_2}, [\omega]_{H^2} \rangle$ is the same as the energy of u_{v}, and 0 energy implies u_{v} is constant, hence the name ghost vertex.) We require the **coincidence condition** (i.e., the value of the disk map at each marked point matches the value of the corresponding generalized Morse trajectory as in equation (2.9)), the **stability condition** (i.e., a ghost vertex is associated with at least three edges as in equation (2.8)), and the **topology condition** (i.e., the Maslov indices of disk maps sum up to μ, and the pairings of disk maps with the symplectic form $\langle [u_{\mathrm{v}}]_{H_2}, [\omega]_{H^2} \rangle$ sum up to ν as in equation (2.10)).

Lastly, the aforementioned equivalence relation is given as follows. We say $(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$ is equivalent to $(\mathrm{T}', \underline{\gamma}', \underline{x}', \underline{u}')$ if there is an ordered tree isomorphism $\zeta : \mathrm{T} \to \mathrm{T}'$ and a tuple of conformal disk automorphisms $\underline{\psi} = (\psi_{\mathrm{v}})_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}}$ such that

- ζ relabels the Morse trajectories $\underline{\gamma}_{\mathrm{e}} = \underline{\gamma}'_{\mathrm{e}'}$, and
- $\underline{\psi}$ push forward the marked points and map to the marked points and map relabeled by ζ,

$$(\psi_{\mathrm{v}}(\underline{x}_{\mathrm{v}}), u_{\mathrm{v}} \circ \psi_{\mathrm{v}}^{-1}) = (\underline{x}'_{\mathrm{v}'}, u'_{\mathrm{v}'}).$$

In this case, we write $(\zeta, \underline{\psi})(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}) = (\mathrm{T}', \underline{\gamma}', \underline{x}', \underline{u}')$.

One hopes to derive an algebra by counting the virtual dimension 0 part of such a moduli space, but it is not necessarily a smooth manifold without perturbations. Similar spaces and various perturbation schemes have been considered by Cornea and Lalonde [3], Charest [1], [2], Fukaya, Oh, and Ono [5]. Here we tackle the problem by constructing an ambient space around the moduli space $\mathfrak{M}$ and give the ambient space an M-polyfold structure, which is a powerful analytical tool invented by Hofer, Wysocki, and Zehnder [7] to deal with transversality problems in J-holomorphic curves. They have applied the polyfold theory to the construction of Gromov-Witten invariants of arbitrary genus [8] and are dealing with the transversality problem in symplectic field theory [6].

In our setting, the moduli space of disk trees $\mathfrak{M}([p_k \otimes \cdots \otimes p_1, q; \mu], \nu)$ lies naturally in an ambient space $\mathfrak{X}([p_k \otimes \cdots \otimes p_1, q; \mu], \nu)$, which we call the **quotient space of disk trees**. It is the same as the moduli space $\mathfrak{M}$ except that each disk map u_{v} is not necessarily J-holomorphic, but has $H^{3,\delta}$ regularity (Definition 3.12), which roughly speaking, has Sobolev H^3-regularity away from the marked points $\underline{x}_{\mathrm{v}}$, and H^3-regularity with δ-exponential decay towards the marked points. We denote $\mathfrak{X}$ to be the union over all $p_k \otimes \cdots \otimes p_1$, q, μ, and ν

$$\mathfrak{X} := \bigsqcup \mathfrak{X}([p_k \otimes \cdots \otimes p_1, q; \mu], \nu).$$

One of the main results of this thesis is that the quotient space of disk trees can be given the structure of an M-polyfold with boundary and corners (Theorem 6.13).

Theorem 1.1. *The quotient space of disk trees $\mathfrak{X}$ is an M-polyfold with boundary and corners.*

One can think of an M-polyfold as an infinite dimensional manifold with certain new local models (see [7] and Definition 6.3). In other problems, e.g., symplectic field theory, the ambient space is given a structure of a polyfold, which is a generalization of an infinite dimensional orbifold. The reason why we can have an M-polyfold instead of polyfold is because of the following proposition (Proposition 2.5).

Proposition 1.2. *Suppose we have $(\zeta, \underline{\psi})(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}) = (\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$. Then ζ is the identity tree isomorphism, and $\underline{\psi}$ are identity disk automorphisms.*

The goal of the thesis is to construct an M-polyfold strong bundle (see [7]) $\pi : \mathfrak{Y} \to \mathfrak{X}$, where each fiber consists of complex anti-linear sections, and show that the Cauchy-Riemann section $\bar{\partial}_J : \mathfrak{X} \to \mathfrak{Y}$ is sc-Fredholm (see [7]). Note that the moduli space of disk trees $\mathfrak{M}$ lies in the ambient space $\mathfrak{X}$ as the solution set of the $\bar{\partial}_J$ section. By general polyfold theory ([9]), there exists a perturbation section s so that the solution set $(\bar{\partial}_J + s)^{-1}(0)$ is a compact finite dimensional smooth manifold with boundary and corners. Using these perturbed moduli spaces, we will construct a $\mathbb{Z}_2$-coefficient A_∞ algebra, as opposed to one with rational coefficients, due to the trivial isotropy group as in Proposition 1.2.

We now elaborate on the topology defined on $\mathfrak{X}$ in Theorem 1.1, as it sheds some light on the process of deriving an A_∞ algebra. Let $\hat{\tau} := (\hat{\mathrm{T}}, \underline{\hat{\gamma}}, \underline{\hat{x}}, \underline{\hat{u}})$

be a disk tree representative. Denote by $\hat{\mathrm{E}}^{\mathrm{nd}}$ the set of **nodal edges**, whose Morse trajectories $\underline{\hat{\gamma}}_{\hat{\mathrm{e}}}$ have length 0; we introduce a **gluing parameter** $r_{\hat{\mathrm{e}}} \in (-\varepsilon, \varepsilon)$ to each nodal edge. Note that for each nodal edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \in \hat{\mathrm{E}}^{\mathrm{nd}}$, the coincidence condition implies that $\hat{u}_{\hat{\mathrm{v}}^-}(\hat{x}_{\hat{\mathrm{v}}^-,\hat{\mathrm{e}}}) = \hat{u}_{\hat{\mathrm{v}}^+}(\hat{x}_{\hat{\mathrm{v}}^+,\hat{\mathrm{e}}})$, hence giving rise to a nodal map. We define the ε-neighborhood $\mathcal{U}_\varepsilon(\hat{\tau})$ to consist of $(\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})$, where each $\underline{\gamma}_{\hat{\mathrm{e}}}$ is ε-close to $\underline{\hat{\gamma}}_{\hat{\mathrm{e}}}$ (in the metric given by (2.3)) while keeping the length of $\underline{\gamma}_{\hat{\mathrm{e}}}$ zero for nodal edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, and each $(\underline{x}_{\hat{\mathrm{v}}}, u_{\hat{\mathrm{v}}})$ is ε-close to $(\underline{\hat{x}}_{\hat{\mathrm{v}}}, \hat{u}_{\hat{\mathrm{v}}})$ (in the sense that $(\underline{x}_{\hat{\mathrm{v}}}, u_{\hat{\mathrm{v}}}) \in \mathcal{U}_\varepsilon(\underline{\hat{x}}_{\hat{\mathrm{v}}}, \hat{u}_{\hat{\mathrm{v}}})$ as in Definition 4.1), with coincidence condition (2.9) satisfied. Then we define the **gluing map** as follows

$$
\begin{aligned}
(1.1) \qquad [\#] : (-\varepsilon, \varepsilon)^{\hat{\mathrm{E}}^{\mathrm{nd}}} \times \mathcal{U}_\varepsilon(\hat{\tau}) &\to \mathfrak{X} \\
(\underline{r}, (\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})) &\mapsto [\#_{\underline{r}}(\hat{\mathrm{T}}), \#_{\underline{r}}(\underline{\gamma}), \#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{u})].
\end{aligned}
$$

For each nodal edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \in \hat{\mathrm{E}}^{\mathrm{nd}}$ with gluing parameter $r_{\hat{\mathrm{e}}} > 0$, we identify $\hat{\mathrm{v}}^-$ and $\hat{\mathrm{v}}^+$ to a single vertex in the glued tree $\#_{\underline{r}}(\hat{\mathrm{T}})$. Take an r_{e}-dependent strip near $x_{\hat{\mathrm{v}}^-,\hat{\mathrm{e}}}$ in the disk $D_{\hat{\mathrm{v}}^-}$ and an r_{e}-dependent strip near $x_{\hat{\mathrm{v}}^+,\hat{\mathrm{e}}}$ in the disk $D_{\hat{\mathrm{v}}^+}$, glue the strips together to get a glued disk, and then discard $x_{\hat{\mathrm{v}}^-,\hat{\mathrm{e}}}$ and $x_{\hat{\mathrm{v}}^+,\hat{\mathrm{e}}}$ in the glued boundary marked points $\#_{\underline{r}}(\underline{x})$; moreover, we interpolate the nodal map $(u_{\hat{\mathrm{v}}^+}, u_{\hat{\mathrm{v}}^-})$ on the glued strip to define a map on the glued disk (see (4.14)). On the other hand, for each edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \in \hat{\mathrm{E}}$ with $r_{\hat{\mathrm{e}}} \leq 0$, we displace the $u_{\hat{\mathrm{v}}^-}$ part of the nodal map $(u_{\hat{\mathrm{v}}^+}, u_{\hat{\mathrm{v}}^-})$ along the Morse flow so that $u_{\hat{\mathrm{v}}^+}$ and $u_{\hat{\mathrm{v}}^-}$ become separated (see (4.15)). This gives rise to the tuple of maps $\#_{\underline{x},\underline{r}}(\underline{u})$, and the new Morse flow between the pair is the displaced Morse trajectory $\#_{\underline{r}}(\gamma)$ (see (4.8)).

Now imagine the gluing parameter $r_{\hat{\mathrm{e}}}$ decreases from positive to negative, this curve in $\mathfrak{X}$ does not stop at the bubbling at $r_{\hat{\mathrm{e}}} = 0$ when a map becomes nodal, but it continues displacing along the Morse flow when $r_{\hat{\mathrm{e}}}$ becomes negative. In this way, disk bubbling is an *interior point* in $\mathfrak{X}$; the boundary and corners structure of $\mathfrak{X}$ comes from that of the Morse trajectory space. Later on we shall use this fact to prove that the A_∞ algebra we construct indeed satisfies the A_∞ equation.

Let κ be the equivalence class $[\hat{\tau}]$. We denote the image of $[\#]$ in (1.1) by $\mathfrak{U}_\varepsilon(\kappa; \hat{\tau})$. This collection of neighborhoods arsing from gluing defines a Hausdorff topology (Theorem 4.6).

Theorem 1.3. *The collection $\{\mathfrak{U}_\varepsilon(\kappa; \hat{\tau}) \,|\, \kappa \in \mathfrak{X}\}$ forms a basis in $\mathfrak{X}$ and defines a Hausdorff topology.*

We now briefly describe the M-polyfold with boundary and corners atlas (see [7]) for $\mathfrak{X}$ the quotient space of disk trees. We shall construct a chart around a disk tree κ by using a variation of the gluing map $[\#] : (-\varepsilon, \varepsilon)^{\hat{\mathrm{E}}^{\mathrm{nd}}} \times \mathcal{U}_\varepsilon(\hat{\tau}) \to \mathfrak{X}$ in (1.1). Note that the gluing map $[\#]$ itself is not a chart. It is *not* injective due to (1) the conformal disk automorphism action on each disk, and (2) the interpolation on the glued strip used in the gluing. We deal with problem (1) by restricting the domain to a "slice" of the neighborhood $\mathcal{U}_\varepsilon(\hat{\tau})$, and handle problem (2) by further restricting to a subset of maps called the "splicing core". Hofer, Wysocki, and Zehnder built M-polyfold theory with problem (2) in mind, adopting splicing cores as local models of the M-polyfold (see [7]). This collection of charts forms an atlas (Theorem 6.12).

Theorem 1.4. *The quotient space of disk trees $\mathfrak{X}$ has an M-polyfold with boundary and corners atlas.*

With some additional minor topological properties of $\mathfrak{X}$, Theorem 1.3 and Theorem 1.4 essentially constitutes Theorem 1.1.

In order to prove the above results on the topology and the atlas of the quotient space of disk trees, we study the Deligne-Mumford space $\mathfrak{DM}$, a finite dimensional analog of $\mathfrak{X}$. We refer the readers to [19], [18], [16], [14], and [10] for studies of other cases of the Deligne-Mumford space.

The **Deligne-Mumford space** with k incoming critical vertices $\mathfrak{DM}(k)$ consists of equivalence classes of the form $[\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O}]$. Here T is an ordered tree with a partition into main vertices and critical vertices $\mathrm{V} = \mathrm{V}^{\mathrm{m}} \sqcup \mathrm{V}^{\mathrm{c}}$, with critical vertices consisting of k leaves and the root $\mathrm{rt}(\mathrm{T})$. The tuple $\underline{\ell}$ consists of edge lengths $\ell_{\mathrm{e}} \in [0,1]$ for each edge $\mathrm{e} \in \mathrm{E}$. The tuple $(\underline{x}, \underline{O})$ consists of boundary marked points and interior marked points $(\underline{x}_{\mathrm{v}}, O_{\mathrm{v}})$ for each main vertex $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$, where each O_{v} is unordered. Furthermore, the tree of marked points and edge lengths satisfies the **stability condition** (i.e., each main vertex v with no interior marked points $O_{\mathrm{v}} = \emptyset$ is associated with at least three edges as in equation (5.1)).

Lastly, the aforementioned equivalence relation is given as follows. We say $(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{o})$ is equivalent to $(\mathrm{T}', \underline{\ell}', \underline{x}', \underline{o}')$ if there is an ordered tree isomorphism $\zeta : \mathrm{T} \to \mathrm{T}'$ and a tuple of conformal disk automorphisms $\underline{\psi} = (\psi_{\mathrm{v}})_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}}$ such that

- ζ relabels the edge lengths $\underline{\ell}_{\mathrm{e}} = \underline{\ell}'_{\mathrm{e}'}$, and
- $\underline{\psi}$ push forward the marked points to the marked points relabeled by ζ,
$$(\psi_{\mathrm{v}}(\underline{x}_{\mathrm{v}}), \psi_{\mathrm{v}}(O_{\mathrm{v}})) = (\underline{x}'_{\mathrm{v}'}, O'_{\mathrm{v}'}).$$

In this case, we write $(\zeta, \underline{\psi})(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}) = (\mathrm{T}', \underline{\gamma}', \underline{x}', \underline{u}')$.

We use gluing to construct a topology and an atlas on the Deligne-Mumford space $\mathfrak{DM}$ in an analogous way to $\mathfrak{X}$. Let $\hat{\mu} := (\hat{\mathrm{T}}, \hat{\underline{\ell}}, \hat{\underline{x}}, \hat{\underline{O}})$ be a representative. Denote by $\hat{\mathrm{E}}^{\mathrm{nd}}$ the set of nodal edges, whose edge lengths $\hat{\underline{\ell}}_{\hat{\mathrm{e}}}$ are 0; we introduce a gluing parameter $r_{\hat{\mathrm{e}}} \in (-\varepsilon, \varepsilon)$ to each nodal edge. We define the ε-neighborhood $\mathcal{U}_\varepsilon(\hat{\mu})$ and the gluing map in a similar way as in the quotient space of disk trees

$$\begin{aligned}(1.2) \qquad [\#] : (-\varepsilon, \varepsilon)^{\hat{\mathrm{E}}^{\mathrm{nd}}} \times \mathcal{U}_\varepsilon(\hat{\mu}) &\to \mathfrak{DM} \\ (\underline{r}, (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})) &\mapsto [\#_{\underline{r}}(\hat{\mathrm{T}}), \#_{\underline{r}}(\underline{\ell}), \#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{O})].\end{aligned}$$

We displace the edge lengths $\#_{\underline{r}}(\underline{\ell})$ in an analogous way as we displace Morse trajectories $\#_{\underline{r}}(\underline{\gamma})$, keep the glued boundary marked points $\#_{\underline{r}}(\underline{x})$ as before, and define the glued interior marked points $\#_{\underline{r},\underline{x}}(\underline{O})$ to be the union of interior marked points on the glued disk.

The Deligne-Mumford space has the structure of a manifold with boundary and corners (Theorem 5.18). Much like the boundary of quotient space of disk trees is given by elements with broken Morse flow lines, the boundary of the Deligne-Mumford space is given by elements with unit edge lengths.

Theorem 1.5. *The Deligne-Mumford space $\mathfrak{DM}$ is a manifold with boundary and corners.*

While the above result does not directly prove Theorem 1.3 and 1.4, the tools we develop along the way are instrumental to proving the M-polyfold structure of $\mathfrak{X}$. Here we roughly sketch this idea. For a disk representative $(\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})$, we can construct a set of codimension 2 sub-manifolds of M that are transversal to the disk maps $\underline{u}$. We call them **transversal constraints**. We then locally define a **stabilization map** which takes $(\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})$ to $(\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})$, where each $\ell_{\hat{\mathrm{e}}}$ is the renormalized length of the Morse trajectories $\underline{\gamma}_{\hat{\mathrm{e}}}$ (see (2.2)), and each $O_{\hat{\mathrm{v}}}$ is the pre-image of the transversal constraints under the map $u_{\hat{\mathrm{v}}}$. The stabilization $(\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})$ is a representative of an element in the Deligne-Mumford space. Suppose we have equivalence of gluing

$$(\zeta, \underline{\psi})(\#(\underline{r}, (\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u}))) = \#(\underline{r}', (\hat{\mathrm{T}}', \underline{\gamma}', \underline{x}', \underline{u}')),$$

where $\#$ is the gluing in (1.1) before taking the equivalence class. Then the same $(\zeta, \underline{\psi})$ gives equivalence to their stabilizations

$$(\zeta, \underline{\psi})(\#(\underline{r}, (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O}))) = \#(\underline{r}', (\hat{\mathrm{T}}', \underline{\ell}', \underline{x}', \underline{O}')).$$

Thus understanding how $(\zeta, \underline{\psi})$ depends on the gluing parameters $\underline{r}$ and the stabilization $(\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})$ help us understand the gluing map (11.19), which help us prove the topology and the atlas of $\mathfrak{X}$. We refer the readers to Proposition 5.23 and 8.11 for further details.

We now work towards constructing an A_∞ algebra. Firstly, let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be the **bundle of complex anti-linear sections**, where $\mathfrak{Y}$ consists of equivalence classes of the form $[\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}, \underline{\lambda}]$. The fiber consists of tuples $\underline{\lambda} = (\lambda_{\mathrm{v}})_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}}$ such that for each main vertex $\mathrm{v} \in \mathrm{V}^m$ and point $z \in D$,

$$\lambda_{\mathrm{v}}(z) : (T_z D, i) \to (T_{u_{\mathrm{v}}(z)} M, J(u_{\mathrm{v}}(z)))$$

is a complex anti-linear map. Moreover, we require each λ_{v} to have $H^{2,\delta}$ regularity. We define the **Cauchy-Riemann section** $\bar{\partial}_J : \mathfrak{X} \to \mathfrak{Y}$ by

$$\bar{\partial}_J(u_{\mathrm{v}}) = \frac{1}{2}(\partial_s u_{\mathrm{v}} + J(u_{\mathrm{v}})\, \partial_t u_{\mathrm{v}}).$$

We give the bundle the structure of an M-polyfold strong bundle (Proposition 9.5), and show that the $\bar{\partial}_J$ section is sc-Fredholm (Theorem 9.10).

Theorem 1.6. *The bundle of complex anti-linear sections $\pi : \mathfrak{Y} \to \mathfrak{X}$ is an M-polyfold strong bundle, and the section $\bar{\partial}_J : \mathfrak{X} \to \mathfrak{Y}$ is sc-smooth and sc-Fredholm.*

We are now in the right position to construct an A_∞ algebra from the quotient space of disk trees. For each disk tree $z = [\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}] \in \mathfrak{X}([p_k \otimes \cdots \otimes p_1, q; \mu], \nu)$, we call $[p_k \otimes \cdots \otimes p_1, q; \mu]$ the **type** of z. Suppose two disk trees $z^{\mathrm{v}} \in \mathfrak{X}([p_k \otimes \cdots \otimes p_1, q; \mu], \nu)$ and $z^{\mathrm{w}} \in \mathfrak{X}([r_l \otimes \cdots \otimes r_1, s; \delta], \sigma)$ are such that $q = r_i$. Then we can concatenate the Morse trajectory in z^{v} which flows into q with the Morse trajectory in z^{w} which flows out of r_i and obtain a concatenated disk tree. In general, for a tree of disk trees such that all edges $\mathrm{e} = (\mathrm{v}, \mathrm{w})$ satisfy the above coincidence condition at critical points, then we define their **concatenation** similarly as before (see Definition 11.9 for details). We denote the concatenation operation by $\circ$, and note that a concatenated disk tree lies on the boundary of $\mathfrak{X}$ because it has at least one broken Morse flow line.

One can find a sc^+ perturbation (see Section 1.4 of [7]) of $\bar{\partial}_J$ which is compatible with the concatenation $\circ$, and the perturbed solution set has a compact manifold structure (Theorem 11.15 and Theorem 10.9).

Theorem 1.7. *There exists a $\circ$-compatible sc^+ section $s : \mathfrak{X} \to \mathfrak{Y}$ such that for all types Z and $\nu \geq 0$, the solution set of the perturbed section $(\bar{\partial}_J + s)^{-1}(0) \cap \mathfrak{X}(Z, \nu)$ is a compact manifold with boundary and corners.*

In order to define the desired A_∞, we shall focus on those disk trees with Fredholm index 0. By an index calculation, all disk trees z of a fixed type $Z = [p_k \otimes \cdots \otimes p_1, q; \mu]$ have the same $\bar{\partial}_J$ Fredholm index

$$\operatorname{ind}_{\bar{\partial}_J}(z) = \operatorname{ind}_{\bar{\partial}_J}(Z) = \sum_{i=1}^{k} |p_i| - |q| + \mu - (k-1)(n-1) - 1. \tag{1.3}$$

We construct the **disk tree A_∞ algebra over $\mathbb{Z}_2$** as follows, using the **Novikov ring**

$$\Lambda = \left\{ \sum c_i e^{\nu_i} \,|\, c_i \in \mathbb{Z}_2, \nu_i \in [0, \infty) \right\}.$$

Here e is the quantum variable. In the sum, there are finitely many non-zero c_i with $\nu_i \leq N$ for any N. Let C be the complex generated by Morse critical points

$$C = \sum_{p \in \operatorname{crit}(L)} \Lambda \langle p \rangle.$$

The **total complex** $\widetilde{C} = \bigoplus_{k=0}^{\infty} \bigotimes^k C$ has an obvious bilinear form

$$\left\langle \sum \lambda_P P, \sum \lambda'_P P \right\rangle := \sum \lambda_P \lambda'_P \in \Lambda,$$

where we sum over pure tensors $P = p_k \otimes \cdots \otimes p_1$ with $p_i \in \operatorname{crit}(L)$. For each $k \geq 0$, define the **k-th multiplication** $m^k : \bigotimes^k C \to C$ as follows. Given a pure tensor of critical points $R = r_k \otimes \cdots \otimes r_1$ and a critical point s, we denote by μ^* the integer such that $\mathrm{ind}_{\bar\partial_J}([R,s;\mu^*]) = 0$, which we calculate from (1.3). We denote the perturbed section $f := \bar\partial_J + s$, with perturbation given by Theorem 1.7. Hence the solution set $f^{-1}(0) \cap \mathfrak{X}([R,s;\mu^*],\nu)$ is a compact 0-dimensional manifold. We define the coefficient $\langle m^k(R), s\rangle$ by taking the $\mathbb{Z}_2$ count of the above solution set

$$\langle m^k(R), s\rangle := \sum_{\nu\geq 0} n_{\mathbb{Z}_2}\big(f^{-1}(0)\cap \mathfrak{X}([R,s;\mu^*],\nu)\big)\, e^{\nu}. \tag{1.4}$$

Naturally, we have $m^k(R) = \sum_{s\in\mathrm{crit}(L)} \langle m^k(R), s\rangle s$. We then extend the multiplications to $\widetilde{m} : \widetilde{C} \to \widetilde{C}$ in the standard way ([13]) as follows. Given a pure tensor $P = p_k \otimes \cdots \otimes p_1$, we denote its length by $|P| = k$. We define

$$\widetilde{m}(P) = \sum_{P = P_2\otimes P'\otimes P_1} P_2 \otimes m^{|P'|}(P') \otimes P_1, \quad (|P'|, |P_i| \text{ potentially } 0) \tag{1.5}$$

and then extend Λ-linearly to combinations of pure tensors. The extension $\widetilde{m}$ satisfies the $\boldsymbol{A_\infty}$ **algebra equation**

$$\widetilde{m}\circ\widetilde{m} = 0 \tag{1.6}$$

since the boundary of a 1-dimensional solution set is precisely given by disk trees with once broken Morse flow lines, which are the concatenations of two index 0 solutions. Thus $(\widetilde{C}, \widetilde{m})$ is a **curved** $\boldsymbol{A_\infty}$ **algebra** (curved means m^0 does not necessarily vanish).

Thus we finish constructing an A_∞ algebra $(\widetilde{C}, \widetilde{m})$ by choosing any compatible almost-complex structure J and some sc^+ perturbation s. In order for $(\widetilde{C}, \widetilde{m})$ to be a symplectic invariant, we would like to show that the A_∞ algebra is independent of the choice of pairs (J, s). More precisely, given two such pairs (J_{-1}, s_{-1}) and (J_1, s_1) and let $(\widetilde{C}, \widetilde{m}_{-1})$ and $(\widetilde{C}, \widetilde{m}_1)$ be their respective A_∞ algebra, we need to construct an A_∞ isomorphism between $(\widetilde{C}, \widetilde{m}_{-1})$ and $(\widetilde{C}, \widetilde{m}_1)$.

To prove the above result, we first find a smooth 1-parameter family of almost-complex structures $(J_t)_{t\in[-1,1]}$ between J_{-1} and J_1. We can do this since the space of almost-complex structures compatible with ω is connected. The difficult part is finding a 1-parameter family of $\circ$-compatible perturbations $(s_t)_{t\in[-1,1]}$ so that the family of sections

$$F : [-1,1]\times\mathfrak{X} \to \mathfrak{Y}, \quad (t,x) \mapsto (\bar\partial_{J_t} + s_t)(x)$$

achieves a certain special transversality condition. The transversality problem is complicated by the presence of index -1 solutions at certain irregular $t \in (-1,1)$: index -1 solutions can potentially concatenate with itself for arbitrarily many times and the results are still index -1 solutions. In this thesis, we deal with a special case when F is assumed to be $\boldsymbol{\circ}$**-transverse**, which limits the number of ways an index -1 can concatenate with itself (Theorem 1.8).

Theorem 1.8. *Suppose there exists a 1-parameter family of $\circ$-compatible sc^+ perturbations so that the family $F(t,x) = (\bar{\partial}_{J_t} + s_t)(x)$ is $\circ$-transverse, then there exists an A_∞ isomorphism $\widetilde{\varphi} : (\widetilde{C}, \widetilde{m}_{-1}) \to (\widetilde{C}, \widetilde{m}_1)$.*

2. Quotient Space of Disk Trees

Let (M, ω) be a symplectic manifold of dimension $2n$ with $\omega|_{\pi_2(M)} = 0$, and L a Lagrangian submanifold of M. In addition, we choose a compatible almost complex structure J on M, and a Morse-Smale pair (f, g) on the Lagrangian L.

In this section, we define the quotient space of disk trees

$$\mathfrak{X}([p_k \otimes \cdots \otimes p_1, q; \mu], \nu).$$

We first go over the standard notions of an ordered tree and a Morse trajectory space.

2.1. Ordered Tree.

A **tree** T is a connected graph with no cycle. Let us denote by V the set of vertices, and by E the set of edges. A **rooted tree** is a tree with a designated **root vertex** $\mathrm{rt(T)} \in \mathrm{V}$. Let the edges be oriented towards the root. The orientation on the edges induces a partial ordering on V, known as the **tree order**, where the root $\mathrm{rt(T)}$ is the maximal element. This allows us to think of E as a subset of $(\mathrm{V} \times \mathrm{V}) \setminus \Delta$. More precisely, for two adjacent vertices v and w with $\mathrm{v} < \mathrm{w}$, we denote by (v, w) the edge going from v to w; we call v a **child** of w, and w a **parent** of v.

With respect to a given vertex v, there are two types of edges: the set of **incoming edges** and **outgoing edges** of v are given by

$$\mathrm{E^{in}(v)} := \{(\mathrm{w}, \mathrm{v}) \in \mathrm{E}\}, \quad \mathrm{E^{out}(v)} := \{(\mathrm{v}, \mathrm{w}) \in \mathrm{E}\}, \tag{2.1}$$

respectively. The set of all edges of v is given by $\mathrm{E(v)} := \mathrm{E^{in}(v)} \cup \mathrm{E^{out}(v)}$. A vertex v is called a **leaf** if $\mathrm{E^{in}(v)} = \emptyset$. We note that $\mathrm{E^{out}(v)}$ is empty if v is the root, and has a single element otherwise. We define the **valency** of a vertex v to be the cardinality of $\mathrm{E(v)}$, denoted as $|\mathrm{v}|$.

If T is a rooted tree and each set $\mathrm{E^{in}(v)}$ has an order, then T is called an **ordered tree**. Then we can order the set $\mathrm{E(v)}$ as

$$\mathrm{E(v)} = \left\{ \mathrm{e}^0(\mathrm{v}), \ldots, \mathrm{e}^{|\mathrm{v}|-1}(\mathrm{v}) \right\}. \tag{2.2}$$

For $\mathrm{v} \neq \mathrm{rt(T)}$, we choose $\mathrm{e}^0(\mathrm{v})$ to be the outgoing edge of v. This order induces an ordering of all children $\{\mathrm{w} \in \mathrm{V} \mid (\mathrm{w}, \mathrm{v}) \in \mathrm{E^{in}(v)}\}$. Combining this order with the tree order induced by the orientation of the edges, we get a **lexicographical ordering** on the set of all vertices. $\zeta : \mathrm{T} \to \mathrm{T}'$ is a **an ordered tree isomorphism** if it is a bijection between V and V' which preserves the lexicographical ordering, and induces a bijection between E and E'.

It is straightforward to verify the following uniqueness result for ordered tree isomorphism.

Proposition 2.1. *If $\zeta : \mathrm{T} \to \mathrm{T}'$ and $\bar{\zeta} : \mathrm{T} \to \mathrm{T}'$ are both ordered tree isomorphisms, then $\zeta = \bar{\zeta}$.*

2.2. **Morse Trajectory Space.**

Another essential element of the quotient space of disk trees is the compactified Morse moduli spaces. Here we give a short introduction. For more details, we refer the readers to [20]. Let (f, g) be a Morse-Smale pair on Lagrangian L. For $p \in \mathrm{crit}(L)$, we denote by $|p|$ the Morse index of p.

For $p^- \neq p^+ \in \mathrm{crit}(L)$, we define the space of Morse flow lines from p^- to p^+ modulo the $\mathbb{R}$-shift action as

$$\mathcal{M}(p^-, p^+) := \{\gamma : \mathbb{R} \to L \mid \dot{\gamma} = -\nabla f(\gamma), \lim_{s\to\pm\infty} \gamma(s) = p^{\pm}\}/\mathbb{R}.$$

For $p^- = p^+$, we set $\mathcal{M}(p^-, p^+) = \emptyset$. We recall from Morse theory that $\mathcal{M}(p^-, p^+) = \emptyset$ if $|p^-| \leq |p^+|$. Now we define the space of half-infinite Morse flow lines.

$$\mathcal{M}(p^-, L) := \{\gamma : (-\infty, 0] \to L \mid \dot{\gamma} = -\nabla f(\gamma), \lim_{s\to-\infty} \gamma(s) = p^-\},$$

$$\mathcal{M}(L, p^+) := \{\gamma : [0, \infty) \to L \mid \dot{\gamma} = -\nabla f(\gamma), \lim_{s\to\infty} \gamma(s) = p^+\}.$$

Lastly, the space of finite Morse flow lines is

$$\mathcal{M}(L, L) := \{\gamma : [0, a] \to L \mid a \geq 0, \dot{\gamma} = -\nabla f(\gamma)\}.$$

The above spaces of Morse flow lines admit a compactification by adding the "broken Morse flow lines". To be precise, let $U^{\pm}$ be either L or a single critical point, then we define the **Morse trajectory space** as

$$\overline{\mathcal{M}}(U^-, U^+) := \bigcup_{k\in\mathbb{N}_0,\, |p_1|>\cdots>|p_k|} \mathcal{M}(U^-, p_1) \times \mathcal{M}(p_1, p_2) \times \cdots \times \mathcal{M}(p_k, U^+).$$

An element $\underline{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_k)$ is called a **generalized trajectory**, which represents a **broken flow line** when $k \geq 1$. The Morse trajectory space $\overline{\mathcal{M}}(U^-, U^+)$ can be given the smooth structure of a compact smooth manifold with boundary and corners.

To specify the topology of $\overline{\mathcal{M}}(U^-, U^+)$, we define the following length.

Definition 2.2. We define the **renormalized length** $\ell(\underline{\gamma})$ of a generalized trajectory $\underline{\gamma}$ by,

$$\ell(\underline{\gamma}) := \begin{cases} \frac{a}{1+a} & \text{if } \underline{\gamma} = \gamma_0 : [0, a] \to L, \\ 1 & \text{if } \underline{\gamma} \text{ is broken, or unbroken but infinite or half-infinite.} \end{cases}$$

Note that the renormalized length can distinguish trajectories in $\mathcal{M}(L, L)$ with image being a single critical point p and broken trajectories in $\mathcal{M}(L, p) \times \mathcal{M}(p, L)$ with the same image. We now give $\overline{\mathcal{M}}(U^-, U^+)$ the topology induced by the following metric.

$$d_{\overline{\mathcal{M}}}(\underline{\gamma}, \underline{\eta}) := d_{\mathrm{Hausdorff}}(\overline{\mathrm{im}\underline{\gamma}}, \overline{\mathrm{im}\underline{\eta}}) + |\ell(\underline{\gamma}) - \ell(\underline{\eta})|, \tag{2.3}$$

where $\overline{\mathrm{im}\underline{\gamma}} = \overline{\mathrm{im}\gamma_0} \cup \cdots \cup \overline{\mathrm{im}\gamma_k}$. We recall the definition of Hausdorff distance in the appendix (Definition 13.5).

For a broken flow line $\underline{\hat{\gamma}} = (\hat{\gamma}_0, \hat{\gamma}_1, \ldots, \hat{\gamma}_k)$ and small $\varepsilon > 0$, there is a diffeomorphism (onto its image) called **Morse gluing map** (see [20])

$$\begin{aligned} (2.4) \qquad B_\varepsilon(\hat{\gamma}_0) \times [0,\varepsilon) \times B_\varepsilon(\hat{\gamma}_1) \times \cdots \times [0,\varepsilon) \times B_\varepsilon(\hat{\gamma}_k) &\to \overline{\mathcal{M}} \\ (\gamma_0, r_1, \gamma_1, \ldots, r_k, \gamma_k) &\mapsto \sharp_{\underline{r}}(\underline{\gamma}), \end{aligned}$$

where $\underline{r} = (r_1, \ldots, r_k)$ are the gluing parameters. The trivially glued Morse trajectory $\sharp_{\underline{0}}(\underline{\gamma})$ is equal to $\underline{\gamma}$, and if all r_i are positive, the glued Morse trajectory $\sharp_{\underline{r}}(\underline{\gamma})$ is unbroken. Moreover, for trajectories $\underline{\gamma} = (\underline{\gamma}^1, \underline{\gamma}', \underline{\gamma}^2)$ and parameters $\underline{r} = (\underline{r}^1, \underline{r}', \underline{r}^2)$, we have associativity $\sharp_{\underline{r}}(\underline{\gamma}) = \sharp_{(\underline{r}^1, \underline{r}^2)}(\underline{\gamma}^1, \sharp_{\underline{r}'}(\underline{\gamma}'), \underline{\gamma}^2)$.

This map later plays a crucial role in our A_∞ algebra.

Definition 2.3. We define the **evaluation map** on these Morse trajectory spaces as follows.

- Define $\mathrm{ev}^- \times \mathrm{ev}^+ : \overline{\mathcal{M}}(p^-, p^+) \to \{p^-\} \times \{p^+\}$ by $\mathrm{ev}^-(\underline{\gamma}) = p^-$, and $\mathrm{ev}^+(\underline{\gamma}) = p^+$.
- Define $\mathrm{ev}^- \times \mathrm{ev}^+ : \overline{\mathcal{M}}(L, p^+) \to L \times \{p^+\}$ by $\mathrm{ev}^-(\gamma_0, \ldots, \gamma_k) = \gamma_0(0)$, and $\mathrm{ev}^+(\underline{\gamma}) = p^+$.
- Define $\mathrm{ev}^- \times \mathrm{ev}^+ : \overline{\mathcal{M}}(p^-, L) \to \{p^-\} \times L$ by $\mathrm{ev}^-(\underline{\gamma}) = p^-$, and $\mathrm{ev}^+(\gamma_0, \ldots, \gamma_k) = \gamma_k(0)$.
- Define $\mathrm{ev}^- \times \mathrm{ev}^+ : \overline{\mathcal{M}}(L, L) \to L \times L$ by $\mathrm{ev}^-(\gamma_0, \ldots, \gamma_k) = \gamma_0(0)$, and $\mathrm{ev}^+(\gamma_0, \ldots, \gamma_k) = \gamma_k(0)$ for $k \geq 1$ or $\mathrm{ev}^+(\gamma_0 : [0,a] \to L) = \gamma_0(a)$.

The evaluation map $\mathrm{ev}^- \times \mathrm{ev}^+$ is smooth.

2.3. The Quotient Space of Disk Trees.

We now define the quotient space of disk trees as a set. Let $p_k, \ldots, p_1, q$ be critical points, μ an integer, and ν a non-negative real number. The **quotient space of disk trees** with incoming critical points $p_k, \ldots, p_1$, outgoing critical point q, and $\mu \in \mathbb{Z}, \nu \geq 0$ is denoted by

$$(2.5) \quad \mathfrak{X}([p_k \otimes \cdots \otimes p_1, q; \mu], \nu) := \{(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}) \,|\, (1) - (4) \text{ satisfied}\}/_{\sim_{\mathrm{bihol}}}.$$

We define conditions (1) – (4) and the equivalence relation $\sim_{\mathrm{bihol}}$ as follows.

(1) T is an ordered tree with the following requirements.

- The root $\mathrm{rt}(\mathrm{T})$ has a single edge, i.e., $|\mathrm{rt}(\mathrm{T})| = 1$.
- The tree T is equipped with a choice of a partition $\mathrm{V}^{\mathrm{m}} \sqcup \mathrm{V}^{\mathrm{c}} = \mathrm{V}$, into the set of **main vertices** V^{m}, and the set of **critical vertices** V^{c}, such that $\mathrm{rt}(\mathrm{T}) \in \mathrm{V}^{\mathrm{c}}$, and $\mathrm{V}^{\mathrm{c}} \setminus \{\mathrm{rt}(\mathrm{T})\}$ consists of exactly k leaves (same k as in (2.5)).

(2) $\underline{\gamma} = (\underline{\gamma}_{\mathrm{e}})_{\mathrm{e} \in \mathrm{E}}$ is a tuple of generalized Morse trajectories with the following requirements.

For each critical vertex $\mathrm{v} \in \mathrm{V}^{\mathrm{c}}$, we assign $p_{\mathrm{v}} \in \mathrm{crit} f$ in the following way. The set of critical leaves $\mathrm{V}^{\mathrm{c}} \setminus \{\mathrm{rt}(\mathrm{T})\}$ has exactly k elements, with an order induced from the lexicographical order of T. Hence $\mathrm{V^c} \setminus \{\mathrm{rt(T)}\} = \{\mathrm{w}_k, \ldots, \mathrm{w}_1\}$. We define

$$p_{\mathrm{w_i}} := p_i, \quad p_{\mathrm{rt(T)}} := q,$$

where p_i's and q are the Morse critical points given in (2.5). For an edge $\mathrm{e} = (\mathrm{v}^-, \mathrm{v}^+)$, the generalized Morse trajectories $\underline{\gamma}_{\mathrm{e}}$ are required to have ends on these critical points as follows.

$$\underline{\gamma}_{\mathrm{e}} \in \begin{cases} \overline{\mathcal{M}}(p_{\mathrm{v}^-}, p_{\mathrm{v}^+}), & \text{for } \mathrm{v}^-, \mathrm{v}^+ \in \mathrm{V^c}, \\ \overline{\mathcal{M}}(p_{\mathrm{v}^-}, L), & \text{for } \mathrm{v}^- \in \mathrm{V^c}, \mathrm{v}^+ \in \mathrm{V^m}, \\ \overline{\mathcal{M}}(L, p_{\mathrm{v}^+}), & \text{for } \mathrm{v}^- \in \mathrm{V^m}, \mathrm{v}^+ \in \mathrm{V^c}, \\ \overline{\mathcal{M}}(L, L), & \text{for } \mathrm{v}^-, \mathrm{v}^+ \in \mathrm{V^m}. \end{cases}$$

(3) $(\underline{x}, \underline{u}) = (\underline{x}_{\mathrm{v}}, u_{\mathrm{v}})_{\mathrm{v} \in \mathrm{V^m}}$ is a tuple of boundary marked points and disk maps with the following requirements.

- For each main vertex $\mathrm{v} \in \mathrm{V^m}$ and each edge $\mathrm{e} \in \mathrm{E(v)}$, we associate a **boundary marked point** $x_{\mathrm{v,e}} \in \partial D$. Furthermore, these marked points $x_{\mathrm{v,e}}$ are distinct, with $x_{\mathrm{v,e^0(v)}}, \ldots, x_{\mathrm{v,e^{|v|-1}(v)}}$ positioned *counter-clockwise* on ∂D (see the labeling in (2.2)). We denote this ordered set of boundary marked points by

$$\underline{x}_{\mathrm{v}} = \{x_{\mathrm{v,e}} \in \partial D \,|\, \mathrm{e} \in \mathrm{E(v)}\}.$$

- For each main vertex $\mathrm{v} \in \mathrm{V^m}$, the disk map u_{v} has regularity $H^{3,\delta_0}((D, \underline{x}_{\mathrm{v}}), M)$. (Roughly speaking, u_{v} has three weak derivatives and δ exponential decay near the marked points $\underline{x}_{\mathrm{v}}$. See Section 3.3 for the precise definition.) Moreover, u_{v} satisfies the **Lagrangian boundary condition**,

$$u_{\mathrm{v}}(\partial D) \subset L.$$

  Then it follows that u_{v} represents a relative homology class in $H_2(M, L)$. We define $\omega(u_{\mathrm{v}})$ to be the coupling

$$\omega(u_{\mathrm{v}}) := \langle [u_{\mathrm{v}}]_{H_2}, [\omega]_{H^2} \rangle, \tag{2.6}$$

  and we require $\omega(u_{\mathrm{v}}) \geq 0$.

It is sometimes more convenient to use an alternative way to index boundary marked points and disk maps by edges $\mathrm{e} = (\mathrm{v}^-, \mathrm{v}^+)$:

$$x_{\mathrm{e}}^{\pm} := x_{\mathrm{v}^{\pm},\mathrm{e}}, \quad u_{\mathrm{e}}^{\pm} := u_{\mathrm{v}^{\pm}}. \tag{2.7}$$

(4) The tuple $(\mathrm{T}, \underline{\gamma}, (\underline{x}, \underline{u}))$ satisfies the following additional requirements.

- A main vertex $\mathrm{v} \in \mathrm{V^m}$ is a **ghost vertex** if $\omega(u_{\mathrm{v}}) = 0$. We impose the **stability condition**, that is,

$$\text{for a ghost vertex v, we have } |\mathrm{v}| \geq 3. \tag{2.8}$$

- For each edge $\mathrm{e} = (\mathrm{v}^-, \mathrm{v}^+) \in \mathrm{E}$, we impose the **coincidence condition** defined by the evaluation map (Definition 2.3),

$$u_{\mathrm{e}}^{\pm}(x_{\mathrm{e}}^{\pm}) = \mathrm{ev}^{\pm}(\underline{\gamma}_{\mathrm{e}}) \text{ if } \mathrm{v}^{\pm} \in \mathrm{V}^{\mathrm{m}}. \tag{2.9}$$

- Denote the **Maslov index** of u_{v} with respect to L by $\mu(u_{\mathrm{v}}) \in \mathbb{Z}$. (See Appendix C.3 of [15] for a complete exposition of boundary Maslov index.) For $\mu \in \mathbb{Z}$ and $\nu \geq 0$ in (2.5), we impose the **topology condition**

$$\sum_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}} \mu(u_{\mathrm{v}}) = \mu, \quad \sum_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}} \omega(u_{\mathrm{v}}) = \nu. \tag{2.10}$$

Lastly, we define the equivalence relation $\sim_{\mathrm{bihol}}$.

Definition 2.4. We say $(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$ is **equivalent to** $(\mathrm{T}', \underline{\gamma}', \underline{x}', \underline{u}')$ **via a biholomorphism** if there exists a tree isomorphism $\zeta : \mathrm{T} \to \mathrm{T}'$ and a tuple of disk automorphisms $\underline{\psi} = (\psi_{\mathrm{v}})_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}}$ (see Section 13.1) such that

- $\underline{\gamma}_{\mathrm{e}} = \underline{\gamma}'_{\mathrm{e}'}$ for every $\mathrm{e} \in \mathrm{E}$ and $\mathrm{e}' = \zeta(\mathrm{e})$,
- $\psi_{\mathrm{v}}(x_{\mathrm{v},\mathrm{e}}) = x'_{\mathrm{v}',\mathrm{e}'}$ for every $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}, \mathrm{e} \in \mathrm{E}(\mathrm{v})$ and $\mathrm{v}' = \zeta(\mathrm{v}), \mathrm{e}' = \zeta(\mathrm{e})$,
- $u_{\mathrm{v}} \circ \psi_{\mathrm{v}}^{-1} = u'_{\mathrm{v}'}$ for every $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$ and $\mathrm{v}' = \zeta(\mathrm{v})$.

In short, we write $(\zeta, \underline{\psi})(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}) = (\mathrm{T}', \underline{\gamma}', \underline{x}', \underline{u}')$. Moreover, we write an equivalence class under the relation $\sim_{\mathrm{bihol}}$ as

$$[\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}].$$

Finally, we define the **quotient space of disk trees** by taking the union

$$\mathfrak{X} := \bigsqcup \mathfrak{X}([p_k \otimes \cdots \otimes p_1, q; \mu], \nu) \tag{2.11}$$

over all $k \in \mathbb{N}_0$, all critical points p_i and q, and all $\mu \in \mathbb{Z}$ and $\nu \geq 0$.

The following result shows that each element $(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$ has trivial isotropy group.

Proposition 2.5. *Suppose there are two biholomorphisms* $(\zeta, \underline{\psi})$ *and* $(\bar{\zeta}, \underline{\bar{\psi}})$ *which satisfy*

$$(\zeta, \underline{\psi})(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}) = (\bar{\zeta}, \underline{\bar{\psi}})(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}).$$

Then we have $(\zeta, \underline{\psi}) = (\bar{\zeta}, \underline{\bar{\psi}})$.

We shall prove this result later by using a transversal constraint to pass onto the Deligne-Mumford space. Most importantly, we shall give $\mathfrak{X}$ an M-polyfold structure.

3. Polyfold Smoothness and H^{3,δ_0} Maps

As we shall see later, an M-polyfold is locally modeled on certain subsets of sc-Banach spaces, a type of Banach space with an infinite nested sequence of Banach spaces with certain properties. In [7], Hofer, Wysocki, and Zehnder define a smooth structure on sc-Banach spaces. In this section, we define the topology of the space of boundary marked points and disk maps using sc-Banach spaces.

3.1. **Sc-Banach Spaces.**

We first define the notion of an sc-Banach space, which provides a foundation for the polyfold smooth structure. We refer the readers to [7] for an in-depth introduction.

Definition 3.1. An **sc-structure** on a Banach space E consists of a nested sequence of Banach spaces $(E_m)_{m\in\mathbb{N}_0}$ with $E_0 = E$, and each E_{m+1} is a linear subspace of E_m but with possibly different norms. Moreover, this sequence of Banach spaces must satisfy the following two conditions.

(1) The inclusion maps $E_{m+1} \to E_m$ are compact operators.
(2) The space $E_\infty := \bigcap_{m\in\mathbb{N}_0} E_m$ is dense in every E_m.

A Banach space E equipped with an sc-structure is called an **sc-Banach space**.

Here we present two examples of sc-Banach spaces which are crucial to the quotient space of disk maps: one for the regularity of a disk map *away* from the marked points, and another for the regularity *near* the marked points under certain strip coordinates.

Example 3.2. Let $V \subset \mathbb{R}^k$ be a bounded domain with Lipshitz boundary in the Euclidean space, and $H^m(V, \mathbb{C}^n)$ the Sobolev-m function space. The space $E = H^3(V, \mathbb{C}^n)$ is a sc-Banach space with sc-structure $E_m = H^{3+m}(V, \mathbb{C}^n)$. Indeed, it follows from the compact embedding theorem that the inclusion $E_{m+1} \to E_m$ is compact. Lastly, we have $E_\infty = C^\infty(V, \mathbb{C}^n)$ and the space of smooth functions is dense in every E_m.

The reason for choosing H^3 space as the 0-level sc-structure is that the Sobolev embedding theorem guarantees $H^3 \hookrightarrow C^1$, and later on we shall use the first derivative of disk maps in our construction.

The following example describes the sc-structure of maps on the infinite strip that take value in $\mathbb{C}^n$ with boundary condition on $\mathbb{R}^n$. This is a local version of the Lagrangian boundary condition.

Example 3.3. We define the **weighted Sobolev space**

$$H^{m,\delta}(\mathbb{R} \times [0, \pi], \mathbb{C}^n; \mathbb{R}^n)$$

as follows. Let f be a function whose weak derivative $\partial^\alpha f$ is locally represented by an L^2 function, where the multi-index α is of order $|\alpha|$ less than m. Then the norm $\|f\|_{H^{m,\delta}}$ is defined by

$$\|f\|^2_{H^{m,\delta}} := \sum_{|\alpha|\leq m} \int_{\mathbb{R}\times[0,\pi]} |\partial^\alpha f(s,t)|^2 e^{2\delta|s|} ds dt. \tag{3.1}$$

The space $H^{m,\delta}(\mathbb{R} \times [0, \pi], \mathbb{C}^n; \mathbb{R}^n)$ consists of all functions f whose norm $\|f\|_{H^{m,\delta}}$ is finite with each $f(s, 0), f(s, \pi) \in \mathbb{R}^n$. We refer to δ as the **weight**.

Let E be $H^{3,\delta_0}(\mathbb{R} \times [0, \pi], \mathbb{C}^n; \mathbb{R}^n)$ with $\delta_0 > 0$. Given a strictly increasing sequence of weights δ_m with $\delta_0 < \delta_1 < \cdots$, the nested sequence (E_m) with $E_m = H^{3+m,\delta_m}(\mathbb{R} \times [0, \pi], \mathbb{C}^n; \mathbb{R}^n)$ is an sc-Banach structure. Indeed, one can use the compact embedding for bounded domains and the increasing weights to show that the embedding $E_{m+1} \to E_m$ is compact. Lastly, the E_∞ space is the set of smooth functions with the proper decay; it is dense in every E_m because the space of compactly support smooth functions is dense.

As a variation of the weighted Sobolev space, we define the **weighted Sobolev space with limits**

$$H^{m,\delta}_{\lim}(\mathbb{R} \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$$

to consist of functions f with limits $c \in \mathbb{R}^n$ such that $f - c$ lies in $H^{m,\delta}(\mathbb{R} \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$. We define the norm

$$\|f\|^2_{H^{m,\delta}_{\lim}} := |c|^2 + \|f - c\|^2_{H^{m,\delta}}.$$

Let E be $H^{3,\delta_0}_{\lim}(\mathbb{R} \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$ with $\delta_0 > 0$. With strictly increasing weights $\delta_0 < \delta_1 < \cdots$, the sequence $E_m = H^{3+m,\delta_m}_{\lim}(\mathbb{R} \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$ is an sc-Banach structure.

We now introduce the concepts of sc-operator and partial quadrant.

Definition 3.4. Let E and F be sc-Banach spaces. A linear map $T : E \to F$ is an **sc-operator** if we have $T(E_m) \subset F_m$ for all $m \in \mathbb{N}_0$ and the restrictions $T : E_m \to F_m$ are continuous. An sc-operator is an **sc-isomorphism** if its inverse is also an sc-operator. Lastly, a subset $C \subset E$ is called a **partial quadrant** if after applying an sc-isomorphism it is of the form $[0,\infty)^k \times W$ for some sc-Banach space W.

Going beyond linear maps, we now discuss the notions of continuity and differentiability in sc-Banach spaces. Let U and V be open subsets of partial quadrants C and D in sc-Banach space E and F, respectively. We denote their m-level by $U_m := U \cap E_m$ and $V_m := V \cap F_m$.

Definition 3.5. A map $f : U \to V$ is $\boldsymbol{sc^0}$ if we have $f(U_m) \subset V_m$ for all $m \in \mathbb{N}_0$ and the restrictions $f : U_m \to V_m$ are continuous.

We denote by U^k the k-level set U_k with sc-structure $U^k_m := U_{k+m}$. Define the **tangent space** TU by

$$TU := U^1 \times E.$$

Definition 3.6. An sc^0 map $f : U \to V$ is said to be $\boldsymbol{sc^1}$ if for every $x \in U^1$ there exists a bounded linear operator $Df(x) : E_0 \to F_0$ such that the following holds.

(1) For $h \in E_1$ with $x + h \in C$, we have the limit

$$\lim_{|h|_1 \to 0} \frac{1}{|h|_1} |f(x+h) - f(x) - Df(x)h|_0 = 0.$$

(2) The *tangent map* $Tf : TU \to TV$ defined by

$$(x, h) \mapsto (f(x), Df(x)h)$$

is sc^0.

We say f is sc^2 if the tangent map Tf is sc^1. Inductively, this defines the notion of sc^k. We say f is **sc-smooth** (or $\boldsymbol{sc^\infty}$) if f is sc^k for all k.

One key distinction between sc-differentiability and classical differentiability is that the continuity of $(x, h) \mapsto Df(x)h$ as in sc^1 is *weaker* than the continuity of $x \mapsto Df(x)$ in operator norm as in C^1; Proposition 4.2 of [7] shows that the reparametrization action on a cylinder is sc-smooth but not classically smooth. We refer the readers to Proposition 1.9 and 1.10 of [7] for a thorough study of the relationship between the sc and classical differentiability. Furthermore, Theorem 1.11 of [7] shows that this generalized smooth structure has a chain rule. Lastly, an M-polyfold is locally modeled on certain subsets of sc-Banach spaces called sc-retracts (Section 6.1). These subsets are the images of certain sc-smooth retractions, and they are not generally open in their ambient sc-Banach spaces.

3.2. **Boundary Marked Points and Strip Coordinates.**

As we shall see in the Section 4, we define the topology of the quotient space of disk trees $\mathfrak{X}$ using gluing. The construction of a glued disk map involves viewing neighborhoods of marked points in the disk as half infinite strips, truncating them to certain lengths, gluing them pair-wise, and interpolating the maps on these glued strips. In this section, we define the space of boundary marked points and the strip coordinates near them.

We denote the **space of boundary marked points** by

$$\underline{MP}(\partial D) := \left\{ \underline{x} \subset \partial D \;\middle|\; \begin{array}{c} \underline{x} = (x_0, \ldots, x_k) \text{ for } k \geq 0, \\ \text{distinct and ordered counter-clockwise} \end{array} \right\}. \tag{3.2}$$

The underline in $\underline{MP}(\partial D)$ emphasizes that the boundary marked points are ordered. This space has a metric

$$d(\underline{x}, \underline{x}') = \begin{cases} \max_i d_{\partial D}(x_i, x_i') & \text{if } n(\underline{x}) = n(\underline{x}'), \\ 100 & \text{if } n(\underline{x}) \neq n(\underline{x}'), \end{cases}$$

where $n(\underline{x})$ denotes the cardinality of $\underline{x}$. We denote the ε-neighborhood of $\underline{\hat{x}}$ in $\underline{MP}(\partial D)$ by

$$\mathcal{U}_\varepsilon(\underline{\hat{x}}).$$

We now introduce the notion of strip coordinates near a boundary marked point. Let us denote the half-infinite intervals by

$$\mathbb{R}^+ := [0, \infty), \quad \mathbb{R}^- := (-\infty, 0].$$

Definition 3.7. Let x be a boundary marked point. A biholomorphism into a closed neighborhood $N(x) \subset D$ of x

$$h^+ : \mathbb{R}^+ \times [0, \pi] \to N(x) \setminus \{x\}$$
$$(\text{or } h^- : \mathbb{R}^- \times [0, \pi] \to N(x) \setminus \{x\})$$

is called **positive (or negative) strip coordinates near $\boldsymbol{x}$** if it is of the form

$$h^\pm := f \circ p^\pm,$$

where

- $p^\pm : \mathbb{R}^\pm \times [0, \pi] \to \{0 < |z| \leq 1, \mathrm{Im} z \geq 0\}$ are given by
$$p^+(z) := -e^{-z}, \quad p^-(z) := e^z,$$
- f is a Möbius transformation that maps the extended upper half plane $\{\mathrm{Im} z \geq 0\} \cup \{\infty\}$ to the disk D with $f(0) = x$.

The closed neighborhood $N(x)$ is called a **strip neighborhood of $\boldsymbol{x}$**.

We also consider a family of such strip coordinates.

Definition 3.8. Let $\hat{x}$ be a boundary marked point. Suppose there is a smooth family of Möbius transformations f_x for $x \in \mathcal{U}_\varepsilon(\hat{x})$ such that

- each f_x maps the extended upper half plane $\{\mathrm{Im} z \geq 0\} \cup \{\infty\}$ to the disk D, and
- $f_x(0) = x$ and $f_x(\infty)$ is independent of x (i.e., $f_x(\infty) = f_{\hat{x}}(\infty)$).

Then we call $h^\pm(x, \cdot) = f_x \circ p^\pm$ a **family of strip coordinates near $\boldsymbol{\hat{x}}$**. For each $x \in \mathcal{U}_\varepsilon(\hat{x})$, we have strip neighborhood $N(x)$.

Note that the limit of a family of strip coordinates is the same as the boundary point x

$$\lim_{z \to \pm\infty} h^\pm(x, z) = x.$$

For $R \geq 0$, we denote the **shrunk strip neighborhood** by

$$N(x; -R) := \begin{cases} \{x\} \cup h^+(x, [R, \infty) \times [0, \pi]), \\ \{x\} \cup h^-(x, (-\infty, -R] \times [0, \pi]). \end{cases} \tag{3.3}$$

Note that $N(x; 0) = N(x)$ and $N(x; -R)$ shrinks to the marked point $\{x\}$ as $R \to \infty$.

Remark 3.9. If $h^\pm(x, \cdot)$ is a family of strip coordinates near $\hat{x}$ and ψ is a disk automorphism, then $k^\pm(x, \cdot) := \psi \circ h^\pm(\psi^{-1}(x), \cdot)$ is a family of strip coordinates near $\psi(\hat{x})$.

Remark 3.10. Recall from (2.2), for a main vertex $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$, $\mathrm{e}^0(\mathrm{v})$ is the outgoing edge, and $\mathrm{e}^1(\mathrm{v}), \ldots, \mathrm{e}^k(\mathrm{v})$ are incoming edges. For boundary marked points $\underline{x} = (x_0, \ldots, x_k)$, we assign negative strip coordinates h_0^- near the outgoing marked point x_0, and for each $i \geq 1$ we assign positive strip coordinates h_i^+ near the incoming marked point x_i. Thus we have a tuple of strip coordinates $\underline{h} = (h_0^-, h_1^+, \ldots, h_k^+)$ near $\underline{x}$.

In our topology of the quotient space of disk trees, we allow the boundary marked points to vary while fix the complex structure on D to be the standard complex structure. However, in the polyfold theory for Gromov-Witten in [8], the marked points are fixed, whereas the domain complex structure can vary. These two approaches are equivalent. In the following lemma, we explicitly constructs a family of diffeomorphisms $\nu_{\underline{x}} : D \to D$ defined for each $\underline{x} \in \mathcal{U}_\varepsilon(\hat{\underline{x}})$. It maps the fixed marked points $\hat{\underline{x}}$ on the disk $(D, \nu_{\underline{x}}^* i)$ with varying complex structure to the varying marked points $\underline{x}$ on the standard disk (D, i). We shall use this family of diffeomorphisms in the construction for the topology of the space of boundary marked points and H^{3,δ_0} disk maps in Section 3.3.

Fix boundary marked points $\hat{\underline{x}} = (\hat{x}_0, \dots, \hat{x}_k)$, and let $B(\hat{x}_i) \subset D$ be mutually disjoint open subsets of D containing $\hat{x}_i$. Let

$$\underline{h} = (h_0^-(x_0, \cdot), h_1^+(x_1, \cdot), \dots, h_k^+(x_k, \cdot))$$

be a tuple of families of strip coordinates near $\hat{\underline{x}}$. Choose $\varepsilon > 0$ small enough so that for all $\underline{x} \in \mathcal{U}_\varepsilon(\hat{\underline{x}})$, each strip neighborhood $N(x_i)$ is contained in $B(\hat{x}_i)$.

Lemma 3.11. *There exists a smooth family of diffeomorphisms $\nu_{\underline{x}} : D \to D$ defined for each $\underline{x} \in \mathcal{U}_\varepsilon(\hat{\underline{x}})$ with the following properties.*

- $\nu_{\underline{x}}(z) = z$ *for* $z \notin \bigsqcup_{i=0}^k B(\hat{x}_i)$.
- *If $\underline{x}$ is such that $x_i = \hat{x}_i$ for some i, then $\nu_{\underline{x}}(z) = z$ for $z \in B(\hat{x}_i)$.*
- $\nu_{\underline{x}}(z) = f_{x_i} \circ f_{\hat{x}_i}^{-1}(z)$ *for $z \in N(\hat{x}_i)$, where f_{x_i} is the Möbius transformation in Definition 3.8.*

In particular, we have $\nu_{\hat{\underline{x}}} = \mathrm{Id}$. Moreover, $\nu_{\underline{x}}(\hat{x}_i) = x_i$, and $\nu_{\underline{x}}$ maps the strip neighborhood $N(\hat{x}_i)$ biholomorphically to $N(x_i)$. It also transports strip coordinates in the sense that $\nu_{\underline{x}} \circ h_i^\pm(\hat{x}_i, \cdot) = h_i^\pm(x_i, \cdot)$

We can construct such a family of diffeomorphisms via integrating vector fields. In the fixed marked points and varying complex structure model, the last property translates to saying the complex structure $(D, \nu_{\underline{x}}^* i)$ is the standard i near the fixed marked points $\hat{\underline{x}}$. As we shall see, the last property is crucial in the topology of the space of boundary marked points and H^{3,δ_0} disk maps.

3.3. $\mathcal{W}$ the Space of Boundary Marked Points and H^{3,δ_0} Maps.

In the description of the quotient space of disk trees, the space of boundary marked points and disk maps plays an important role. In this section, we define a topology on this space.

Recall that L is the given Lagrangian submanifold of the symplectic manifold M.

Fix $m \geq 3$ and weight $\delta \in (0,1)$. Let $\underline{x} = (x_0, \dots, x_k) \in \underline{MP}(\partial D)$ be boundary marked points.

Definition 3.12. Choose open neighborhoods U_i of $u(x_i)$ in M, along with C^∞ charts $\varphi_i : U_i \to B_1(0) \subset \mathbb{C}^n$ with $\varphi_i(u(x_i)) = 0$ and $\varphi_i(L \cap U_i) = \mathbb{R}^n \cap$

$B_1(0)$. Also choose strip coordinates $h_i^\pm$ such that the strip neighborhoods $N(x_i)$ satisfies $u(N(x_i)) \subset U_i$. We define $\mathrm{Map}^{m,\delta}((D,\underline{x}),M;L)$ the space of **$H^{m,\delta}$ maps with marked points $\underline{x}$ and boundary condition L** to be the set of $u : D \to M$ with the following properties.

- The restriction $u|_{D\setminus\bigsqcup_i N(x_i;-1)}$ belongs to $H^m(D \setminus \bigsqcup_i N(x_i;-1))$, where each $N(x_i;-1)$ is the shrunk strip neighborhood (see (3.3)).
- On the strip neighborhood $N(x_i)$, the local expression in strip coordinates $\varphi_i \circ u \circ h_i^\pm$ belongs to $H^{m,\delta}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$ (see Example 3.3).
- $u(z) \in L$ for $z \in \partial D$.

One can show that the above space is well-defined, i.e., independent of choices of charts $\underline{\varphi}$ and strip coordinates $\underline{h}$. It is independent of choices of charts by Proposition 13.18. Moreover, suppose we have two strip coordinates $h_i^\pm$ and $h_i'^\pm$. By Lemma 13.17, the map $(h_i^\pm)^{-1} \circ h_i'^\pm$ satisfies the conditions in Proposition 13.16. Then it follows from Proposition 13.16 that $u \circ h_i^\pm$ belongs to $H^{m,\delta}$ if and only if $u \circ h_i'^\pm = (u \circ h_i^\pm) \circ ((h_i^\pm)^{-1} \circ h_i'^\pm)$ belongs to $H^{m,\delta}$. Thus the above space is independent of choices of strip coordinates.

Remark 3.13. We choose $\mathrm{Map}^{3,\delta_0}$ to be our base regularity. To give the readers an idea of the smoothness of maps $u \in \mathrm{Map}^{3,\delta_0}$, the Sobolev embedding theorem guarantees that u is C^1 away from the marked points $\underline{x}$. However, one can show that the weight $\delta_0 < 1$ is not large enough to grant C^1 regularity at $\underline{x}$, but it does guarantee C^0. In our future construction, we shall take advantage of the first derivative in the interior of the disk.

Remark 3.14. Here we highlight a small difference in convention with the work of Hofer, Wysocki, and Zehnder. In our Definition 3.7, coordinates $p^\pm$ are defined on half-infinite strips $\mathbb{R}^\pm \times [0,\pi]$ by

$$p^+(s+it) = -e^{-(s+it)}, \quad p^-(s+it) = e^{s+it}.$$

In the setup under Theorem 2.1 of [8] however, coordinates $p^\pm$ are defined on half-infinite cylinders $\mathbb{R}^\pm \times S^1$ with $S^1 := [0,1]/_{0\sim 1}$, and they are given by

$$p^+(s+it) = -e^{-2\pi(s+it)}, \quad p^+(s+it) = e^{2\pi(s+it)}.$$

The factor of 2π scales the weight δ accordingly. In particular, the sc-smoothness result in Theorem 2.7 of [7] requires the weight $\delta < 2\pi$. Here we require $\delta < 1$.

As we have seen in Section 2.3, the space of boundary marked points and H^{3,δ_0} disk maps plays an essential role in the quotient space of disk trees. We now define this space and give it a topology later on.

Definition 3.15. We define $\mathcal{W}$ the space of **boundary marked points and H^{3,δ_0} maps** by

$$\mathcal{W} := \{(\underline{x}, u) \,|\, \underline{x} \in \underline{MP}(\partial D), u \in \mathrm{Map}^{3,\delta_0}((D,\underline{x}),M;L)\}.$$

In order to define a topology on $\mathcal{W}$, we first consider the space of sections at a given disk map.

Fix $m \geq 3$ and weight $\delta \in (0,1)$, and fix boundary marked points $\underline{x} = (x_0, \dots, x_k) \in \underline{MP}(\partial D)$ and disk map $u \in \mathrm{Map}^{m,\delta}((D, \underline{x}), M; L)$. Let $u^*TM \to D$ be the pullback bundle. We now define the Banach space of $H^{m,\delta}$ sections of the pullback bundle.

Definition 3.16. Choose C^∞ charts $\underline{\varphi}$ and strip coordinates $\underline{h}$ as in Definition 3.12. We define $\mathrm{Sec}^{m,\delta}((D, \underline{x}), u^*TM; TL)$ the Banach space of $\boldsymbol{H^{m,\delta}}$ **sections of** $\boldsymbol{u^*TM}$ **with marked points** $\underline{\boldsymbol{x}}$ **and boundary condition** $\boldsymbol{TL}$ to be the set of $\xi : D \to u^*TM$ with the following properties.

- The restriction $\xi|_{D \setminus \bigsqcup_i N(x_i;-1)}$ belongs to $H^m(D \setminus \bigsqcup_i N(x_i;-1))$, where each $N(x_i;-1)$ is the shrunk strip neighborhood.
- On the strip neighborhood $N(x_i)$, the local expression in strip coordinates

$$\xi_i^{h^\pm}(z) := D\varphi_i(u(h_i^\pm(z))(\xi(h_i^\pm(z)) \tag{3.4}$$

  on $\mathbb{R}^\pm \times [0,\pi]$ belongs to the weighted Sobolev space with limits $\xi_i^{h^\pm} \in H^{m,\delta}_{\mathrm{lim}}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$ (see Example 3.3).
- $\xi(z) \in T_{u(z)}L$ for $z \in \partial D$.

Choose a Riemannian metric g on M, we now define a norm

$$\|\xi\|^2 := \left\|\xi|_{D \setminus \bigsqcup_i N(x_i;-1)}\right\|^2_{H^m} + \sum_{i=0}^{k} \left\|\xi_i^{h^\pm}\right\|^2_{H^{m,\delta}_{\mathrm{lim}}}, \tag{3.5}$$

where the norm $\left\|\xi|_{D \setminus \bigsqcup_i N(x_i;-1)}\right\|_{H^m}$ is defined using the metric g and its covariant derivative in the standard way.

Similar to Definition 3.12, one can show that the above space is well-defined, i.e., independent of choices of charts $\underline{\varphi}$ and strip coordinates $\underline{h}$. Moreover, different choices of metric g, charts, and strip coordinates give equivalent norms (3.5).

Remark 3.17. Suppose disk map u lies in $\mathrm{Map}^{3+m,\delta_m}((D, \underline{x}), M; L)$ for all $m \geq 0$, with strictly increasing sequence of weights $\delta_0 < \delta_1 < \cdots < 1$. The nested sequence

$$E_m := \mathrm{Sec}^{3+m,\delta_m}((D, \underline{x}), u^*TM; TL)$$

defines an sc-structure on $\mathrm{Sec}^{3,\delta_0}((D, \underline{x}), u^*TM; TL)$. We shall use this sc-Banach space to construct an M-polyfold atlas in Section 6. Note that such sc-structure is not defined for general disk map $u \in \mathrm{Map}^{3,\delta_0}$ because local expression in (3.4) involves u as well.

We now proceed to define a topology on $\mathcal{W}$ by defining a neighborhood basis around each $(\hat{\underline{x}}, \hat{u}) \in \mathcal{W}$. Firstly, by Lemma 4.3.3 of [15], there is a metric on M such that L is totally geodesic.

Lemma 3.18. *There exists a Riemannian metric g on M such that the Lagrangian L becomes totally geodesic, i.e., every geodesic of the submanifold L is a geodesic of the ambient manifold M.*

Choose such a metric g. For a section $\xi \in \mathrm{Sec}^{3,\delta_0}((D, \underline{\hat{x}}), \hat{u}^*TM; TL)$, we use the exponential map of (M, g) to define a disk map $\exp_{\hat{u}} \circ \xi : D \to M$

$$\exp_{\hat{u}} \circ \xi(z) = \exp_{\hat{u}(z)}(\xi(z)). \tag{3.6}$$

The map $\exp_{\hat{u}} \circ \xi$ belongs to $\mathrm{Map}^{3,\delta_0}((D, \underline{\hat{x}}), M; L)$. Indeed, using standard analysis of Sobolev spaces and the smoothness of the exponential map, one can verify the first two properties of Definition 3.12. Moreover, for a boundary point $z \in \partial D$ we have $\exp_{\hat{u}} \circ \xi(z) \in L$, since $\xi(z) \in T_{\hat{u}(z)}L$ and L is totally geodesic.

Using the family of marked points varying diffeomorphisms $\nu_{\underline{x}}$ in Lemma 3.11, we define an ε-neighborhood of $(\underline{\hat{x}}, \hat{u})$

$$\mathcal{U}_\varepsilon(\underline{\hat{x}}, \hat{u}) := \left\{ \left(\underline{x}, \exp_{\hat{u}} \circ \xi \circ \nu_{\underline{x}}^{-1}\right) \;\middle|\; \begin{array}{c} \xi \in \mathrm{Sec}^{3,\delta_0}((D, \underline{\hat{x}}), \hat{u}^*TM; TL), \\ \underline{x} \in \mathcal{U}_\varepsilon(\underline{\hat{x}}),\ \|\xi\| < \varepsilon \end{array} \right\}. \tag{3.7}$$

One can verify $\exp_{\hat{u}} \circ \xi \circ \nu_{\underline{x}}^{-1} \in \mathrm{Map}^{3,\delta_0}((D, \underline{x}), M; L)$ by using the property that the diffeomorphism $\nu_{\underline{x}}$ maps each strip neighborhood $N(\hat{x}_i)$ to $N(x_i)$ biholomorphically (see Lemma 3.11).

This collection of neighborhoods defines a topology on $\mathcal{W}$.

Proposition 3.19. *The collection*

$$\{\mathcal{U}_\varepsilon(\underline{\hat{x}}, \hat{u}) \mid (\underline{\hat{x}}, \hat{u}) \in \mathcal{W}, \varepsilon > 0\}$$

forms a basis in $\mathcal{W}$.

Similar to showing that Definition 3.12 is well-defined, the above result follows from Lemma 13.17 and Proposition 13.16. We equip $\mathcal{W}$ with this topology. As we shall see in Proposition 6.5, the space $\mathcal{W}$ can be given the structure of an M-polyfold without boundary and the disk automorphism action

$$\mathrm{Aut}(D) \times \mathcal{W} \to \mathcal{W}, \quad (\psi, (\underline{x}, u)) \mapsto (\psi(\underline{x}), u \circ \psi^{-1})$$

is sc-smooth.

4. Gluing and the Topology of the Quotient Space of Disk Trees

In this section, we construct a neighborhood basis for the quotient space of disk trees $\mathfrak{X}$ (see (2.11)) around any element κ. Fix a representative $\hat{\tau} = (\hat{\mathrm{T}}, \underline{\hat{\gamma}}, \underline{\hat{x}}, \underline{\hat{u}})$ of κ. In the following text, we shall define this neighborhood basis by constructing the **gluing map**

$$\begin{aligned} [\#] : \mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\hat{\tau}) &\to \mathfrak{X} \\ (\underline{r}, (\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})) &\mapsto [\#_{\underline{r}}(\hat{\mathrm{T}}), \#_{\underline{r}}(\underline{\gamma}), \#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{u})]. \end{aligned} \tag{4.1}$$

Firstly, we denote the set of **nodal edges** of $\hat{\tau}$ by

$$\hat{\mathrm{E}}^{\mathrm{nd}} := \{\hat{\mathrm{e}} \in \hat{\mathrm{E}} \,|\, \ell(\underline{\hat{\gamma}}_{\hat{\mathrm{e}}}) = 0\}. \tag{4.2}$$

Note that for each edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, we have a nodal map $(\hat{u}^-_{\hat{\mathrm{e}}}, \hat{u}^+_{\hat{\mathrm{e}}})$, i.e., $\hat{u}^-_{\hat{\mathrm{e}}}(\hat{x}^-_{\hat{\mathrm{e}}}) = \hat{u}^+_{\hat{\mathrm{e}}}(\hat{x}^+_{\hat{\mathrm{e}}})$.

Definition 4.1. For $\varepsilon > 0$, define $\mathcal{U}_\varepsilon(\hat{\tau})$ the ***ε*-neighborhood of $\hat{\tau}$** to be the set of $(\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})$ that satisfy the following conditions.

- For each edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}$, $\underline{\gamma}_{\hat{\mathrm{e}}}$ lies in $B^{\overline{\mathcal{M}}}_\varepsilon(\underline{\hat{\gamma}}_{\hat{\mathrm{e}}})$, the ε-neighborhood of $\underline{\hat{\gamma}}_{\hat{\mathrm{e}}}$ in the Morse trajectory space $\overline{\mathcal{M}}$ (see (2.3)). Moreover, we fix $\ell(\underline{\gamma}_{\hat{\mathrm{e}}}) = 0$ for all nodal edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$.
- For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, $(\underline{x}_{\hat{\mathrm{v}}}, u_{\hat{\mathrm{v}}})$ lies in $\mathcal{U}_\varepsilon(\underline{\hat{x}}_{\hat{\mathrm{v}}}, \hat{u}_{\hat{\mathrm{v}}}) \subset \mathcal{W}$ (see (3.7)).
- The coincidence condition (2.9) is satisfied.

In order to construct the gluing map [#], we first fix the following smooth cut-off function $\beta : \mathbb{R} \to [0,1]$ such that

- $\beta(s) = 1$ for $s \leq -1$ and $\beta(s) = 0$ for $s \geq 1$,
- $\beta'(s) < 0$ for $s \in (-1, 1)$,
- $\beta(s) + \beta(-s) = 1$ for all s.

As we shall see, we will use β to carry out interpolation on strips. In addition, we choose a smooth increasing function $\iota : (-\infty, 0] \to (-\infty, 0]$ such that

- $\iota^{(k)}(0) = 0$ for all $k \geq 0$,
- $\lim_{s\to-\infty} \iota(s) = -\infty$.

As we shall see, we will use ι to adjust the speed of the Morse flow displacement. Later on, this adjustment will be essential in proving the sc-smoothness of the transition maps of our M-polyfold charts and proving the sc-smoothness of the $\bar{\partial}_J$ section.

Remark 4.2. We proceed to make the following preparatory choices.

(1) Choose a continuous decreasing function called a **gluing profile** $R : (0, 1] \to [0, \infty)$ such that $\lim_{r\to 0} R(r) = \infty$. ((1) is a universal choice, independent of the element $\kappa \in \mathfrak{X}$ around which the neighborhood is based.)
(2) For each nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, choose an open neighborhood $U_{\hat{\mathrm{e}}}$ of $\hat{u}^-_{\hat{\mathrm{e}}}(\hat{x}^-_{\hat{\mathrm{e}}}) = \hat{u}^+_{\hat{\mathrm{e}}}(\hat{x}^+_{\hat{\mathrm{e}}})$ in M, along with a C^∞ chart $\varphi_{\hat{\mathrm{e}}} : U_{\hat{\mathrm{e}}} \to B_1(0) \subset \mathbb{C}^n$ such that $\varphi_{\hat{\mathrm{e}}}(\hat{u}^-_{\hat{\mathrm{e}}}(\hat{x}^-_{\hat{\mathrm{e}}})) = 0$ and $\varphi_{\hat{\mathrm{e}}}(L \cap U_{\hat{\mathrm{e}}}) = \mathbb{R}^n \cap B_1(0)$.
(3) For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$ and edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}(\mathrm{v})$, choose an open neighborhood $B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) \subset D$ of $\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ and a family of strip coordinates $h^\pm_{\hat{\mathrm{e}}}$ near $\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ (as in Remark 3.10) such that
    - for each vertex $\hat{\mathrm{v}}$, the open neighborhoods $B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$ are mutually disjoint and contain the strip neighborhoods $N(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) \subset B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$,
    - for each nodal edge $\hat{\mathrm{e}}$, we have $\hat{u}^\pm_{\hat{\mathrm{e}}}\big(\overline{B(\hat{x}^\pm_{\hat{\mathrm{e}}})}\big) \subset \varphi^{-1}_{\hat{\mathrm{e}}}(B_{1/2}(0))$.

We now specify how small ε should be for the neighborhood $\mathcal{U}_\varepsilon(\hat{\tau})$. Firstly, let

$$\sigma : \mathbb{R} \times L \to L \tag{4.3}$$

be the flow of the Morse-Smale pair (f, g). In other words, the curve $s \mapsto \sigma^s(p)$ is an unbroken Morse flow.

Remark 4.3. In order for the gluing construction [#] to make sense, we need to choose a positive number $\hat{\varepsilon} = \hat{\varepsilon}(\hat{\tau})$ such that

(1) for each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}(\mathrm{v})$, and $(\underline{x}_{\hat{\mathrm{v}}}, u_{\hat{\mathrm{v}}}) \in \mathcal{U}_\varepsilon(\underline{\hat{x}}_{\hat{\mathrm{v}}}, \hat{u}_{\hat{\mathrm{v}}})$, we have $N(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) \subset B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$ and for nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$ we have $u^{\pm}_{\hat{\mathrm{e}}}\big(\overline{B(\hat{x}^{\pm}_{\hat{\mathrm{e}}})}\big) \subset \varphi_{\hat{\mathrm{e}}}^{-1}(B_{1/2}(0))$ (see Remark 4.2 (3)),
(2) for $r \in (-\hat{\varepsilon}, 0]$ and $p \in \varphi_{\hat{\mathrm{e}}}^{-1}(B_{1/2}(0))$, we have $|\varphi_{\hat{\mathrm{e}}}(\sigma^{\iota(r)}(p)) - \varphi_{\hat{\mathrm{e}}}(p)| < 1/2$, and
(3) for each non-nodal edge $\hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{nd}}$ and $\underline{\gamma}_{\hat{\mathrm{e}}} \in B_\varepsilon^{\overline{\mathcal{M}}}(\underline{\hat{\gamma}}_{\hat{\mathrm{e}}})$, we have $\ell(\underline{\gamma}_{\hat{\mathrm{e}}}) \neq 0$.

From here on, we always assume $\varepsilon < \hat{\varepsilon}(\hat{\mu})$.

Given a tuple of real numbers $\underline{\hat{r}} = (\hat{r}_{\hat{\mathrm{e}}})_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}}$, we denote its ε-neighborhood

$$\mathcal{U}_\varepsilon(\underline{\hat{r}}) := \textstyle\prod_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}} (\hat{r}_{\hat{\mathrm{e}}} - \varepsilon, \hat{r}_{\hat{\mathrm{e}}} + \varepsilon).$$

Now let $\underline{0}$ be the zero tuple (zero for each nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$). We introduce a tuple of **gluing parameters** $\underline{r} \in \mathcal{U}_\varepsilon(\underline{0})$. Now for $(\underline{r}, (\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})) \in \mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\hat{\tau})$, we construct the gluing

$$\#(\underline{r}, (\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})) = (\#_{\underline{r}}(\hat{\mathrm{T}}), \#_{\underline{r}}(\underline{\gamma}), \#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{u})) \tag{4.4}$$

as follows.

(1) The glued tree $\#_{\underline{r}}(\hat{\mathrm{T}})$.

We define the set of **gluing edges** to be

$$\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} := \{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \,|\, r_{\hat{\mathrm{e}}} > 0\}. \tag{4.5}$$

We now define an equivalence relation $\sim_{\underline{r}}$ on the set of vertices $\hat{\mathrm{V}}$: we say $\hat{\mathrm{v}} \sim_{\underline{r}} \hat{\mathrm{w}}$ if $\hat{\mathrm{v}}$ and $\hat{\mathrm{w}}$ can be connected by a sequence of gluing edges in $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}$. Define the set of glued vertices by

$$\#_{\underline{r}}(\hat{\mathrm{V}}) := \hat{\mathrm{V}}/_{\sim_{\underline{r}}},$$

and define the root by $\mathrm{rt}(\#_{\underline{r}}(\hat{\mathrm{T}})) := [\mathrm{rt}(\hat{\mathrm{T}})]_{\underline{r}}$. Moreover, we define the set of glued edges by

$$\#_{\underline{r}}(\hat{\mathrm{E}}) := \{([\hat{\mathrm{v}}]_{\underline{r}}, [\hat{\mathrm{w}}]_{\underline{r}}) \in \#_{\underline{r}}(\hat{\mathrm{V}}) \times \#_{\underline{r}}(\hat{\mathrm{V}}) \,|\, (\hat{\mathrm{v}}, \hat{\mathrm{w}}) \in \hat{\mathrm{E}},\ [\hat{\mathrm{v}}]_{\underline{r}} \neq [\hat{\mathrm{w}}]_{\underline{r}}\}.$$

Clearly, there is an edge identification map

$$\begin{gathered} \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} \xrightarrow{\sim} \#_{\underline{r}}(\hat{\mathrm{E}}) \\ \hat{\mathrm{e}} = (\hat{\mathrm{v}}, \hat{\mathrm{w}}) \mapsto ([\hat{\mathrm{v}}]_{\underline{r}}, [\hat{\mathrm{w}}]_{\underline{r}}) = \mathrm{e}. \end{gathered} \tag{4.6}$$

Let $\#_{\underline{r}}(\hat{\mathrm{T}})$ have the order induced from $\hat{\mathrm{T}}$. Lastly, we define the set of critical vertices by $\#_{\underline{r}}(\hat{\mathrm{V}})^{\mathrm{c}} := \{[\hat{\mathrm{v}}]_{\underline{r}} \in \#_{\underline{r}}(\hat{\mathrm{V}}) \,|\, \hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{c}}\}$. Note that for an edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$, if either $\hat{\mathrm{v}}^-$ or $\hat{\mathrm{v}}^+$ is a critical vertex of $\hat{\mathrm{T}}$, then $\ell(\underline{\hat{\gamma}}_{\hat{\mathrm{e}}}) = 1$ by the definition of renormalized length. This shows that a critical vertex is not glued with other vertices. Hence $\#_{\underline{r}}(\hat{\mathrm{V}})^{\mathrm{c}} \simeq \hat{\mathrm{V}}^{\mathrm{c}}$.

One can check that all requirements described in Section 2.3 are satisfied. This finishes the construction of the glued tree $\#_{\underline{r}}(\hat{\mathrm{T}})$.

(2) The displaced Morse trajectories $\#_{\underline{r}}(\underline{\gamma})$.

Using the edge identification map (4.6), for each non-gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}$ we denote by $\mathrm{e} \in \#_{\underline{r}}(\hat{\mathrm{E}})$ its corresponding glued edge. Hence it suffices to define $\#_{\underline{r}}(\underline{\gamma})_{\mathrm{e}}$ for all non-gluing edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}$. Note that the set $\hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}$ can be decomposed into $\hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} = (\hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}) \sqcup (\hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}})$, i.e., the set of non-nodal edges and the set of non-gluing nodal edges.

For $\hat{\mathrm{e}} \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}$, we keep the Morse trajectory as it is

$$\#_{\underline{r}}(\underline{\gamma})_{\mathrm{e}} := \underline{\gamma}_{\hat{\mathrm{e}}}. \tag{4.7}$$

For $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}$, we have $r_{\hat{\mathrm{e}}} \leq 0$ and By Definition 4.1 the generalized Morse trajectory $\underline{\gamma}_{\hat{\mathrm{e}}}$ is a finite *constant* Morse flow line. Recall the flow $\psi : \mathbb{R} \times L \to L$ from (4.3). We define $\#_{\underline{r}}(\underline{\gamma})_{\mathrm{e}}$ to be the finite Morse flow line

$$\#_{\underline{r}}(\underline{\gamma})_{\mathrm{e}} : [\iota(r_{\hat{\mathrm{e}}}), 0] \to L, \quad s \mapsto \sigma^s(\gamma_{\hat{\mathrm{e}}}(0)). \tag{4.8}$$

(3) The glued boundary marked points and the glued disk maps $(\#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{u}))$.

First of all, we study the glued tree $\#_{\underline{r}}(\hat{\mathrm{T}})$ more closely.

Definition 4.4. For each vertex $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$, we denote by $\hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}} \subset \hat{\mathrm{V}}$ the set of representatives of v under the relation $\sim_{\underline{r}}$. Let $\hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}} \subset \hat{\mathrm{E}}$ be the set of edges between vertices in $\hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$. We call $\hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$ and $\hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}$ the **gluing vertices of v** and the **gluing edges of v**, respectively. Moreover, they form a subtree $\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}}$ and we call it the **gluing tree of v**.

We observe that the set of all vertices can be written as $\hat{\mathrm{V}} = \bigsqcup_{\mathrm{v}} \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$, and the set of all gluing edges can be written as $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} = \bigsqcup_{\mathrm{v}} \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}$. Lastly, the order of $\hat{\mathrm{T}}$ induces an order on the gluing tree $\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}}$. Hence the maximal element of $\hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$ is naturally the root $\mathrm{rt}(\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}})$.

For each main vertex $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$, we now construct a Riemann surface $\#_{\underline{r},\underline{x}}(\underline{D})_{\mathrm{v}}$ by gluing disks with standard complex structures $D_{\hat{\mathrm{v}}}$ for all $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$. For each gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}$, the quantity $R(r_{\hat{\mathrm{e}}})$ (see Remark 4.2 (1)) is called the **gluing length**, which we abbreviate as $R_{\hat{\mathrm{e}}}$. Let us denote by $L_{R_{\hat{\mathrm{e}}}}$ the left translation in the s coordinate by $R_{\hat{\mathrm{e}}}$,

$$L_{R_{\hat{\mathrm{e}}}} : [0, R_{\hat{\mathrm{e}}}] \times [0, \pi] \to [-R_{\hat{\mathrm{e}}}, 0] \times [0, \pi], \quad (s,t) \mapsto (s - R_{\hat{\mathrm{e}}}, t). \tag{4.9}$$

We can use the map $L_{R_{\hat{e}}}$ to glue the truncated strips $[0, R_{\hat{e}}] \times [0, \pi]$ and $[-R_{\hat{e}}, 0] \times [0, \pi]$ together. More precisely, the **glued strip** is the quotient space

$$([0, R_{\hat{e}}] \times [0, \pi]) \sqcup ([-R_{\hat{e}}, 0] \times [0, \pi])/_{L_{R_{\hat{e}}}}. \tag{4.10}$$

Clearly, the glued strip has a complex structure induced from that of $[0, R_{\hat{e}}] \times [0, \pi]$ and $[-R_{\hat{e}}, 0] \times [0, \pi]$, and it is biholomorphic to $[0, R_{\hat{e}}] \times [0, \pi]$.

Let $\hat{e} = (\hat{v}^-, \hat{v}^+) \in \hat{E}^{g}_{\underline{r}}$ be an arbitrary gluing edge. By abuse of notation, we identify $(R_{\hat{e}}, \infty) \times [0, \pi]$ and $[0, R_{\hat{e}}] \times [0, \pi]$ with their image under $h^+_{\hat{e}}(x^+_{\hat{e}}, \cdot)$ in $D_{\hat{v}^+}$, and identify $(-\infty, -R_{\hat{e}}) \times [0, \pi]$ and $[-R_{\hat{e}}, 0] \times [0, \pi]$ with their image under $h^-_{\hat{e}}(x^-_{\hat{e}}, \cdot)$ in $D_{\hat{v}^-}$. We now define the **glued disk** $\#_{\underline{r},\underline{x}}(\underline{D})_{v}$ to be the quotient space

(4.11)
$$\frac{\bigsqcup_{\hat{v} \in \hat{V}^{g}_{v}} D_{\hat{v}} \setminus \bigsqcup_{\hat{e} \in \hat{E}^{g}_{v}} ((R_{\hat{e}}, \infty) \times [0, \pi] \cup \{x^+_{\hat{e}}\}) \sqcup (\{x^-_{\hat{e}}\} \cup (-\infty, -R_{\hat{e}}) \times [0, \pi])}{(L_{R_{\hat{e}}} : [0, R_{\hat{e}}] \times [0, \pi] \to [-R_{\hat{e}}, 0] \times [0, \pi])_{\hat{e} \in \hat{E}^{g}_{v}}}.$$

The standard complex structures on each disk naturally give rise to a complex structure on the glued disk. We now define the set of **glued boundary marked points** by

$$\#_{\underline{r}}(\underline{x})_{v,e} := x_{\hat{v},\hat{e}} \text{ for } v \in \hat{V}^{g}_{v} \text{ and } e \notin \hat{E}^{g}_{v}, \tag{4.12}$$

where the glued edge e corresponds to the non-gluing edge $\hat{e}$ by map (4.6). The Riemann surface $\#_{\underline{r},\underline{x}}(\underline{D})_{v}$ has an orientation, and we order $\#_{\underline{r}}(\underline{x})_{v}$ counter-clockwise.

We now define the glued disk map $\#_{\underline{r},\underline{x}}(\underline{u})_{v}$. Our first step is to define it on the glued strips. For each gluing edge $\hat{e} \in \hat{E}^{g}_{v}$, we denote the local expressions in strip coordinates by

$$u^{h^+}_{\hat{e}} := \varphi_{\hat{e}} \circ u^+_{\hat{e}} \circ h^+_{\hat{e}}(x^+_{\hat{e}}, \cdot), \quad u^{h^-}_{\hat{e}} := \varphi_{\hat{e}} \circ u^-_{\hat{e}} \circ h^-_{\hat{e}}(x^-_{\hat{e}}, \cdot). \tag{4.13}$$

Since $u^{\pm}_{\hat{e}}(N(x^{\pm}_{\hat{e}})) \subset \varphi^{-1}_{\hat{e}}(B_{1/2}(0))$ (by the choice in Remark 4.3 (1)), the above local expressions are well-defined. Note that the functions $u^{h^+}_{\hat{e}}|_{[0,R_{\hat{e}}]\times[0,\pi]}$ and $u^{h^-}_{\hat{e}}|_{[-R_{\hat{e}},0]\times[0,\pi]}$ give rise to functions on the glued strip. Now viewing both of them as functions on $[0, R_{\hat{e}}] \times [0, \pi]$, they are given by

$$u^{h^+}_{\hat{e}}(s, t), \quad u^{h^-}_{\hat{e}}(\cdot - R_{\hat{e}})(s, t) := \bar{u}^-_{\hat{e}}(s - R_{\hat{e}}, t).$$

The gluing is simply an interpolation of these two functions using the cut-off function β before Remark 4.2: define the gluing of $u^{h^+}_{\hat{e}}$ and $u^{h^-}_{\hat{e}}$ with gluing parameter $r_{\hat{e}}$ to be a map from $[0, R_{\hat{e}}] \times [0, \pi]$ to $\mathbb{C}^n$,

$$\oplus_{r_{\hat{e}}}(u^{h^+}_{\hat{e}}, u^{h^-}_{\hat{e}}) := \beta(\cdot - R_{\hat{e}}/2)\, u^{h^+}_{\hat{e}} + (1 - \beta(\cdot - R_{\hat{e}}/2))\, u^{h^-}_{\hat{e}}(\cdot - R_{\hat{e}}). \tag{4.14}$$

Now we have a glued function $\oplus_{r_{\hat{e}}}(u^{h^+}_{\hat{e}}, u^{h^-}_{\hat{e}})$ on the glued strip, with its image contained in $B_1(0) \subset \mathbb{C}^n$. By the choice of charts (see Remark 4.2 (3)), $\varphi^{-1}_{\hat{e}} \circ \oplus_{r_{\hat{e}}}(u^{h^+}_{\hat{e}}, u^{h^-}_{\hat{e}})$ is a map from the glued strip to M, with the Lagrangian boundary condition satisfied on $[0, R_{\hat{e}}] \times \{0, \pi\}$.

Let $\hat{\mathrm{f}}$ be the outgoing edge of the root of the gluing tree $\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}}$. Thus we have $r_{\hat{\mathrm{f}}} \leq 0$. Our second step is to define the glued disk map on $N(x^-_{\hat{\mathrm{f}}})$. Define $\alpha^- := \beta(\cdot + 1)$. Note that $\alpha^-(s) = 1$ for $s \leq -2$ and $\alpha^-(s) = 0$ for $s \geq 0$. Recall the flow $\psi : \mathbb{R} \times L \to L$ from (4.3). By the choice of ε (Remark 4.3 (2)), the function

$$u^{h^-}_{\hat{\mathrm{f}}} + \alpha^- \cdot [\varphi_{\hat{\mathrm{f}}}(\sigma^{\iota(r_{\hat{\mathrm{f}}})}(u^-_{\hat{\mathrm{f}}}(x^-_{\hat{\mathrm{f}}}))) - \varphi_{\hat{\mathrm{f}}}(u^-_{\hat{\mathrm{f}}}(x^-_{\hat{\mathrm{f}}}))] \tag{4.15}$$

is well-defined on $\{x^-_{\hat{\mathrm{f}}}\} \cup \mathbb{R}^- \times [0,\pi] \simeq N(x^-_{\hat{\mathrm{f}}})$, with its image contained in $B_1(0) \subset \mathbb{C}^n$. Moreover, the choice of charts (Remark 4.2 (3)) ensures that Lagrangian boundary condition is satisfied. Note that its value at $x^-_{\hat{\mathrm{f}}}$ is $\sigma^{\iota(r_{\hat{\mathrm{f}}})}(u^-_{\hat{\mathrm{f}}}(x^-_{\hat{\mathrm{f}}})) = \mathrm{ev}^-(\#_{\underline{r}}(\underline{\gamma})_{\mathrm{f}})$ by (4.8), where ev^- is the evaluation map on the Morse trajectory space. This guarantees the coincidence condition (2.9).

In summary, the **glued disk map** $\#_{\underline{r},\underline{x}}(\underline{u})_{\mathrm{v}}$ on $\#_{\underline{r},\underline{x}}(\underline{D})_{\mathrm{v}}$ is defined by

(4.16)

$$\begin{cases} u_{\hat{\mathrm{v}}}, & \text{for } \hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}, \\ & \text{on } D_{\hat{\mathrm{v}}} \setminus \bigsqcup_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}} \sqcup \{\hat{\mathrm{f}}\}} N(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}}); \\ \oplus_{r_{\hat{\mathrm{e}}}}(u^{h^+}_{\hat{\mathrm{e}}}, u^{h^-}_{\hat{\mathrm{e}}}), & \text{for } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}, \\ & \text{on } ([0,R_{\hat{\mathrm{e}}}] \times [0,\pi]) \sqcup ([-R_{\hat{\mathrm{e}}},0] \times [0,\pi])/_{L_{R_{\hat{\mathrm{e}}}}}; \\ u^{h^-}_{\hat{\mathrm{f}}} + \alpha^- \cdot [\varphi_{\hat{\mathrm{f}}}(\sigma^{\iota(r_{\hat{\mathrm{f}}})}(u^-_{\hat{\mathrm{f}}}(x^-_{\hat{\mathrm{f}}}))) & \\ \qquad - \varphi_{\hat{\mathrm{f}}}(u^-_{\hat{\mathrm{f}}}(x^-_{\hat{\mathrm{f}}}))], & \text{on } \{x^-_{\hat{\mathrm{f}}}\} \cup \mathbb{R}^- \times [0,\pi] \simeq N(x^-_{\hat{\mathrm{f}}}). \end{cases}$$

Remark 4.5. Here the terminology "gluing" corresponds to "pre-gluing" in the classical approach to moduli space of J-holomorphic curves. To glue two J-holomorphic maps (rather than H^{3,δ_0} maps as in this paper), one need to first perform the interpolation known as "pre-gluing", and then apply the implicit function theorem to get a nearby J-holomorphic map.

One can verify that all of the requirements in Section 2.3 are satisfied for the element $[\#](\underline{r}, (\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})) := [\#_{\underline{r}}(\hat{\mathrm{T}}), \#_{\underline{r}}(\underline{\gamma}), \#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{u})]$. This defines the gluing map (4.1).

We define an ε-neighborhood of κ by

$$\mathfrak{U}_\varepsilon(\kappa; \hat{\tau}; R) := [\#]\left(\mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\hat{\tau})\right), \tag{4.17}$$

where $\hat{\tau}$ is the representative of σ. Later in the paper, we shall prove that this collection of neighborhoods defines a Hausdorff and second-countable topology.

Theorem 4.6. *For each gluing profile R, the collection*

$$\{\mathfrak{U}_\varepsilon(\kappa; \hat{\tau}; R) \,|\, \kappa \in \mathfrak{X},\ 0 < \varepsilon < \hat{\varepsilon}(\hat{\tau})\}$$

forms a basis in $\mathfrak{X}$ and defines a Hausdorff topology, and the topology is independent of the gluing profile R.

We shall prove Theorem 4.6 in Section 8.

Example 4.7. Let us consider the following simple case. Let κ be an element of the quotient space of disk trees and a representative $\hat{\tau}$, with the set of vertices $\hat{\mathrm{V}} = \{\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+, \hat{\mathrm{rt}}\}$ and the set of edges $\hat{\mathrm{E}} = \{(\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+), (\hat{\mathrm{v}}^+, \hat{\mathrm{rt}})\}$. We denote $\hat{\mathrm{e}} := (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$. Furthermore, let $\hat{\mathrm{v}}^-$ and $\hat{\mathrm{v}}^+$ be main vertices, with $\ell(\underline{\hat{\gamma}}_{\hat{\mathrm{e}}}) = 0$. Now consider a curve $r \mapsto [\#](r, (\hat{\mathrm{T}}, \underline{\hat{\gamma}}, (\underline{\hat{x}}, \underline{\hat{u}})))$ with $r \in (-\varepsilon, \varepsilon)$. For $r < 0$, we have a pair of maps $\left(\#_{r, \hat{x}^-_{\hat{\mathrm{e}}}}(\hat{u}^-_{\hat{\mathrm{e}}}), \hat{u}^+_{\hat{\mathrm{e}}}\right)$ with the Morse trajectory $\#_r(\underline{\hat{\gamma}})_{\mathrm{e}}$ between them. At $r = 0$, we get a nodal map $(\hat{u}^-_{\hat{\mathrm{e}}}, \hat{u}^+_{\hat{\mathrm{e}}})$. As r increases further, two maps "merged" into a single glued map $\#_{r, \underline{\hat{x}}}(\underline{\hat{u}})_{\mathrm{v}}$, with $\hat{\mathrm{v}}^-$ and $\hat{\mathrm{v}}^+$ glued to one vertex v. Curves of this forms are continuous in the topology of the quotient space of disk trees. Thus the "disk bubbling" lies in the interior of the quotient space of disk trees, and only the "breaking of the Morse flow line" lies on the boundary.

5. Transversal Constraint, Deligne-Mumford Space, and Stabilization

In order to prove the collection of neighborhoods $\{\mathfrak{U}_\varepsilon(\kappa; \hat{\tau})\}$ in (4.17) defines a topology and then construct an atlas on the quotient space of disk trees, we study a finite dimensional manifold called the Deligne-Mumford space. As we shall see, understanding the topology and the atlas of the Deligne-Mumford space provides essential tools to aid the study of the quotient space of disk trees.

5.1. **Transversal Constraint.**

Transversal constraints help us set up the M-polyfold atlas for the quotient space of disk trees. With transversal constraints, we obtain a local map which takes the infinite dimensional quotient space of disk trees to the finite dimensional Deligne-Mumford space.

We first define the notion of transversal constraint for a single disk map.

Definition 5.1. Given a disk map $u \in \mathrm{Map}^{3,\delta_0}((D, \underline{x}), M; L)$ (Definition 3.12), suppose there is an interior point $o \in D^\circ$ such that the derivative $Du(o)$ is injective. A codimension 2 submanifold Σ of M is an **transversal constraint** at o if we have

$$\mathrm{im}(Du(o)) \oplus T_{u(o)}\Sigma = T_{u(o)}M.$$

For a disk tree, one can construct a transversal constraint for each non-ghost vertex.

Lemma 5.2. *Let* $\tau = (\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$ *be a representative of a disk tree.*

(1) *For each non-ghost vertex* v *(see (2.8)), there exists an interior point* $o_{\mathrm{v}} \in D^\circ$ *such that the derivative* $Du_{\mathrm{v}}(o_{\mathrm{v}})$ *is injective.*
(2) *Given a Riemannian metric* g *on* M*, for each point* $o_{\mathrm{v}} \in D^\circ$ *where the derivative* $Du_{\mathrm{v}}(o_{\mathrm{v}})$ *is injective, there exists a transversal constraint* Σ_{v} *at* o_{v} *that is totally geodesic, i.e., every geodesic of* Σ_{v} *is a geodesic of* M*.*

Proof. For a non-ghost vertex v, the derivative Du_{v} must be injective somewhere in D° because otherwise $\int_D u_{\mathrm{v}}^* \omega = \omega(u_{\mathrm{v}}) = 0$.

Let o_{v} be a point where the derivative $Du_{\mathrm{v}}(o_{\mathrm{v}})$ is injective. We choose a codimension 2 linear subspace $X_{u_{\mathrm{v}}(o_{\mathrm{v}})}$ of the tangent space $T_{u_{\mathrm{v}}(o_{\mathrm{v}})}M$ which is transversal to the image $\mathrm{im}(Du_{\mathrm{v}}(o_{\mathrm{v}}))$. Now define

$$\Sigma_{\mathrm{v}} := \{\exp_{u_{\mathrm{v}}(o_{\mathrm{v}})}(\xi) \mid \xi \in X_{u_{\mathrm{v}}(o_{\mathrm{v}})}, |\xi| < \sigma\}.$$

For σ small enough, Σ_{v} is a totally geodesic submanifold. □

We now introduce a more global version of the transversal constraint.

Definition 5.3. Let $\tau = (\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$ be a representative of a disk tree. A (potentially disconnected) codimension 2 submanifold Σ of M is an **everywhere transversal constraint** for τ if

- for each main vertex $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$, the submanifold Σ is transversal to u_{v}, with each pre-image $u_{\mathrm{v}}^{-1}(\Sigma)$ being a finite subset of D°, and
- for each non-ghost vertex v (see (2.8)), the pre-image $u_{\mathrm{v}}^{-1}(\Sigma)$ is non-empty.

We now prove that there exists an everywhere transversal constraint Σ simultaneously for two disk tree representatives, and we can make Σ contained in an arbitrarily small neighborhood around finitely many intersection points. Later on we shall use this result to prove trivial isotropy result (Proposition 2.5) and the Hausdorff property (Theorem 4.6).

Proposition 5.4. *Let $\tau = (\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$ and $\tau' = (\mathrm{T}', \underline{\gamma}', \underline{x}', \underline{u}')$ be representatives of two disk trees. There is a finite set of intersection points $Q \subset M$ such that*

- *for each main vertex $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$ and $\mathrm{v}' \in \mathrm{V}'^{\mathrm{m}}$, we have $u_{\mathrm{v}}^{-1}(Q) \subset D^\circ$ and $u'^{-1}_{\mathrm{v}'}(Q) \subset D^\circ$, and*
- *for every open set U containing the finite set Q, there exists an everywhere transversal constraint $\Sigma \subset U$ for both τ and τ', and for each main vertex $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$ and $\mathrm{v}' \in \mathrm{V}'^{\mathrm{m}}$, we have $u_{\mathrm{v}}^{-1}(Q) = u_{\mathrm{v}}^{-1}(\Sigma)$ and $u'^{-1}_{\mathrm{v}'}(Q) = u'^{-1}_{\mathrm{v}'}(\Sigma)$.*

Proof. We claim that there is a finite set $Q \subset M$ such that

- for each main vertex $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$, the derivative Du_{v} is injective at $u_{\mathrm{v}}^{-1}(Q)$ with $u_{\mathrm{v}}^{-1}(Q)$ a finite subset of D°, and for each main vertex $\mathrm{v}' \in \mathrm{V}'^{\mathrm{m}}$ the same is true for $u'_{\mathrm{v}'}$, and
- for each non-ghost vertex v, the pre-image $u_{\mathrm{v}}^{-1}(Q)$ is non-empty, and for each non-ghost vertex v' the same is true for $u'_{\mathrm{v}'}$.

We choose the set of intersection points Q as follows. For any main vertex $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$, the curve $u_{\mathrm{v}}|_{\partial D}$ is continuous and piece-wise C^1 by H^{3,δ_0} regularity (see Remark 3.13), so $u_{\mathrm{v}}(\partial D)$ has Hausdorff dimension at most one. The same is true for any main vertex $\mathrm{v}' \in \mathrm{V}'^{\mathrm{m}}$. Denote

$$B := \left(\bigcup_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}} u_{\mathrm{v}}(\partial D)\right) \cup \left(\bigcup_{\mathrm{v}' \in \mathrm{V}'^{\mathrm{m}}} u'_{\mathrm{v}'}(\partial D)\right).$$

Moreover, let A_{v} be the set of points in D° where the derivative Du_{v} fails to be injective. Since $u_{\mathrm{v}}|_{D^\circ}$ is C^1, it follows from Sard's theorem ([17]) that the two dimensional Hausdorff measure (defined in [17]) of $u_{\mathrm{v}}(A_{\mathrm{v}})$ is 0. We define $A'_{\mathrm{v}'} \subset D^\circ$ similarly, and the same is true for $u'_{\mathrm{v}'}(A'_{\mathrm{v}'})$. Denote

$$C := \left(\bigcup_{\mathrm{v}\in \mathrm{V}^{\mathrm{m}}} u_{\mathrm{v}}(A_{\mathrm{v}})\right) \cup \left(\bigcup_{\mathrm{v}'\in \mathrm{V}'^{\mathrm{m}}} u'_{\mathrm{v}'}(A'_{\mathrm{v}'})\right).$$

The set $B \cup C \subset M$ has two dimensional Hausdorff measure 0, and we will avoid this set when we pick the set of intersection points Q.

Fix a non-ghost vertex v. The derivative Du_{v} must be injective somewhere in D° because otherwise $\int_D u_{\mathrm{v}}^*\,\omega = \omega(u_{\mathrm{v}}) = 0$. Hence the two dimensional Hausdorff measure of $u_{\mathrm{v}}(D)$ is positive, which implies $u_{\mathrm{v}}(D)\setminus(B\cup C)$ is non-empty. By construction, any $q \in u_{\mathrm{v}}(D) \setminus (B \cup C)$ satisfies that

- for all main vertices $\mathrm{w} \in \mathrm{V}^{\mathrm{m}}$, the derivative Du_{w} is injective at $u_{\mathrm{w}}^{-1}(q) \subset D^\circ$, and for all main vertices $\mathrm{w}' \in \mathrm{V}'^{\mathrm{m}}$ the same is true for $u'_{\mathrm{w}'}$, and
- $u_{\mathrm{v}}^{-1}(q)$ is non-empty.

We prove that each $u_{\mathrm{w}}^{-1}(q)$ is finite: otherwise $u_{\mathrm{w}}^{-1}(q)$ must cluster at some $z \in D_{\mathrm{w}}$. By continuity, we have $z \in u_{\mathrm{w}}^{-1}(q)$. Since $Du_{\mathrm{w}}(z)$ is injective, the map u_{w} is locally injective near z, which contradicts with the assumption that z is a cluster point.

Similarly for a fixed non-ghost vertex v$'$, any $q' \in u'_{\mathrm{v}'}(D)\setminus(B\cup C)$ satisfies similar conditions as above with $u'^{-1}_{\mathrm{v}'}(q')$ being non-empty. Now by taking the union of q and q' for all non-ghost vertices, we get the desired finite set $Q \subset M$.

Let U be an open set containing the set of intersection points Q. We now set up an everywhere transversal constraint in U. For each $q \in Q$, we choose a codimension 2 linear subspace X_q of T_qM, which is transversal to all $\mathrm{im}(Du_{\mathrm{v}}(z))$ for main vertices v and $z \in u_{\mathrm{v}}^{-1}(q)$, and transversal to all $\mathrm{im}(Du'_{\mathrm{v}'}(z))$ for main vertices v$'$ and $z \in u'^{-1}_{\mathrm{v}'}(q)$. Choose a Riemannian metric g on M and define

$$\Sigma_q := \{\exp_q(\xi) \,|\, \xi \in X_q, |\xi| < \sigma\}.$$

For σ small enough, Σ_q is a submanifold contained in the neighborhood U and we have $u_{\mathrm{v}}^{-1}(\Sigma_q) = u_{\mathrm{v}}^{-1}(q)$ and $u'^{-1}_{\mathrm{v}'}(\Sigma_q) = u'^{-1}_{\mathrm{v}'}(q)$. One can choose such Σ_q for every $q \in Q$. The submanifold $\Sigma := \bigcup_{q\in Q} \Sigma_q$ is our desired everywhere transversal constraint. □

Let $\tau = (\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$ be a representative of a disk tree, and let Σ be an everywhere transversal constraint for τ. We can then define a tuple $(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O})$, where each $\ell_{\mathrm{e}} := \ell(\underline{\gamma}_{\mathrm{e}})$ is the renormalized length in (2.2) and each $O_{\mathrm{v}} := u_{\mathrm{v}}^{-1}(\Sigma)$ is the pre-image. In the next section, we shall study the space of all such tuples $(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O})$.

5.2. **Deligne-Mumford Space.**

The Deligne-Mumford space provides essential tools for examining the topology and the atlas of the quotient space of disk trees. We refer the readers to [19], [18], [16], [14], and [10] for studies of other cases of the Deligne-Mumford space.

The **Deligne-Mumford space** with k incoming critical vertices is denoted by

$$\mathfrak{DM}(k) := \{(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O}) \mid (1) - (4) \text{ satisfied}\}/_{\sim_{\mathrm{bihol}}}.$$

We specify condition (1) – (4) as follows and the equivalence relation $\sim_{\mathrm{bihol}}$ as follows.

(1) T is an ordered tree satisfying the same requirement as in (1) of Section 2.3, with critical vertices V^{c} consisting of k leaves and the root $\mathrm{rt}(\mathrm{T})$.

(2) $\underline{\ell} = (\ell_{\mathrm{e}})_{\mathrm{e} \in \mathrm{E}}$ is a tuple of edge lengths with $\ell_{\mathrm{e}} \in [0,1]$. Moreover, for an edge $\mathrm{e} = (\mathrm{v}^-, \mathrm{v}^+)$ with either v^- or v^+ being a critical vertex, we require $\ell_{\mathrm{e}} = 1$.

(3) $(\underline{x}, \underline{O}) = (\underline{x}_{\mathrm{v}}, O_{\mathrm{v}})_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}}$ is a tuple of boundary marked points and interior marked points, where $\underline{x}_{\mathrm{v}}$ is ordered and O_{v} is unordered.

(4) The tuple $(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O})$ satisfies the **stability condition**, that is,

$$\text{for a main vertex } \mathrm{v} \in \mathrm{V}^{\mathrm{m}} \text{ with } O_{\mathrm{v}} = \emptyset, \text{ we have } |\mathrm{v}| \geq 3. \tag{5.1}$$

Lastly, we define the equivalence relation $\sim_{\mathrm{bihol}}$ similarly to Definition 2.4.

Definition 5.5. We say $(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O})$ is **equivalent to** $(\mathrm{T}', \underline{\ell}', \underline{x}', \underline{O}')$ **via a biholomorphism** if there exists a tree isomorphism $\zeta : \mathrm{T} \to \mathrm{T}'$ and a tuple of disk automorphisms $\underline{\psi} = (\psi_{\mathrm{v}})_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}}$ such that

- $\ell_{\mathrm{e}} = \ell'_{\mathrm{e}'}$ for every $\mathrm{e} \in \mathrm{E}$ and $\mathrm{e}' = \zeta(\mathrm{e})$,
- $\psi_{\mathrm{v}}(x_{\mathrm{v},\mathrm{e}}) = x'_{\mathrm{v}',\mathrm{e}'}$ for every $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}, \mathrm{e} \in \mathrm{E}(\mathrm{v})$ and $\mathrm{v}' = \zeta(\mathrm{v}), \mathrm{e}' = \zeta(\mathrm{e})$,
- $\psi_{\mathrm{v}}(O_{\mathrm{v}}) = O'_{\mathrm{v}'}$ for every $\mathrm{v} \in \mathrm{V}^{\mathrm{m}}$ and $\mathrm{v}' = \zeta(\mathrm{v})$.

In short, we write $(\zeta, \underline{\psi})(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O}) = (\mathrm{T}', \underline{\ell}', \underline{x}', \underline{O}')$. Moreover, we write an equivalence class under the relation $\sim_{\mathrm{bihol}}$ as

$$[\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O}].$$

Finally, we define the **Deligne-Mumford space** by taking the union

$$\mathfrak{DM} := \bigsqcup_{k \in \mathbb{N}_0} \mathfrak{DM}(k).$$

The following result shows that each element $(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O})$ has trivial isotropy group, and as a result, the Deligne-Mumford space has the structure of a manifold with boundary (Theorem 5.17) instead of an orbifold.

Proposition 5.6. *Suppose there are two biholomorphisms $(\zeta, \underline{\psi})$ and $(\bar{\zeta}, \underline{\bar{\psi}})$ which satisfy*

$$(\zeta, \underline{\psi})(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O}) = (\bar{\zeta}, \underline{\bar{\psi}})(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O}).$$

Then we have $(\zeta, \underline{\psi}) = (\bar{\zeta}, \underline{\bar{\psi}})$.

Proof. It suffices to show that if we have $(\zeta, \underline{\psi})(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O}) = (\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O})$, then $(\zeta, \underline{\psi})$ is the identity biholomorphism. Since $\zeta : \mathrm{T} \to \mathrm{T}$ is an ordered tree automorphism, Lemma 2.1 shows that ζ is the identity map. Furthermore the stability condition (5.1) and Lemma 13.1 imply that each ψ_{v} is the identity map. □

We now use the above result and the existence of everywhere transversal constraint to prove that the quotient space of disk trees has trivial isotropy group (Proposition 2.5).

Proof of Proposition 2.5. It suffices to show that if we have equivalence $(\zeta, \underline{\psi})(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}) = (\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$, then $(\zeta, \underline{\psi})$ is the identity biholomorphism. By Proposition 5.4, there exists an everywhere transversal constraint Σ for $(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u})$. We define a tuple $(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O})$, where each $\ell_{\mathrm{e}} := \ell(\underline{\gamma}_{\mathrm{e}})$ is the renormalized length in (2.2) and each $O_{\mathrm{v}} := u_{\mathrm{v}}^{-1}(\Sigma)$ is the pre-image. Note that $(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O})$ satisfies the stability condition (5.1) by the definition of everywhere transversal constraint (Definition 5.3) and the disk tree stability condition (2.8). Moreover, we have $(\zeta, \underline{\psi})(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O}) = (\mathrm{T}, \underline{\ell}, \underline{x}, \underline{O})$. Proposition 5.6 implies that $(\zeta, \underline{\psi})$ is the identity biholomorphism. □

Remark 5.7. The main difference between the Deligne-Mumford space of marked points trees and a classical Deligne-Mumford space is the extra data of edge lengths $\underline{\ell}$. Classically, $x_{\mathrm{e}}^{\pm}$ are called *boundary marked points* if $\mathrm{e} \notin \hat{\mathrm{E}}^{\mathrm{nd}}$, and *nodal points* if $\mathrm{e} \in \hat{\mathrm{E}}^{\mathrm{nd}}$; in our paper, we refer to both these cases as boundary marked points. In addition, a minor difference is that classically, an edge between a critical vertex and a main vertex would be represented by a half infinite edge (see [14]); in our paper, we require those edges have length 1.

5.3. **Gluing and the Topology of the Deligne-Mumford Space.**

Given an element σ of $\mathfrak{DM}$, we fix a representative $\hat{\mu} = (\hat{\mathrm{T}}, \underline{\hat{\ell}}, \underline{\hat{x}}, \underline{\hat{O}})$ of σ. We now proceed to construct a neighborhood basis $\mathfrak{U}_{\varepsilon}(\sigma; \hat{\mu})$ for the topology of $\mathfrak{DM}$ by gluing around $\hat{\mu}$, in an analogous way as in Section 4.

We first define the **space of (unordered) interior marked points** by

$$MP(D^{\circ}) := \{O \subset D^{\circ} \,|\, n(O) < \infty\}, \tag{5.2}$$

where $n(Z)$ denotes the cardinality of Z.

First of all, the space of unordered interior marked points $MP(D^{\circ})$ has a metric defined by the Hausdorff distance (see Definition 13.5),

$$d(O, O') = \begin{cases} d_{\mathrm{Hausdorff}}(O, O') & \text{if } n(O) = n(O'), \\ 100 & \text{if } n(O) \neq n(O'). \end{cases}$$

This gives the product metric on the space of boundary marked points (3.2) and interior marked points $\underline{MP}(\partial D) \times MP(D^{\circ})$,

$$d((\underline{x}, O), (\underline{x}', O')) = \max\{d(\underline{x}, \underline{x}'), d(O, O')\},$$

and we denote the ε-neighborhood of $(\underline{\hat{x}}, \hat{O})$ in $\underline{MP}(\partial D) \times MP(D^\circ)$ by

$$\mathcal{U}_\varepsilon(\underline{\hat{x}}, \hat{O}).$$

We now construct the following gluing map analogous to (4.1)

$$\begin{aligned} (5.3) \qquad [\#] : \mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\hat{\mu}) &\to \mathfrak{DM} \\ (\underline{r}, (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})) &\mapsto [\#_{\underline{r}}(\hat{\mathrm{T}}), \#_{\underline{r}}(\underline{\ell}), \#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{O})]. \end{aligned}$$

Firstly, we denote the set of **nodal edges** of $\hat{\mu}$ analogous to (4.2)

$$(5.4) \qquad \hat{\mathrm{E}}^{\mathrm{nd}} := \{\hat{\mathrm{e}} \in \hat{\mathrm{E}} \,|\, \hat{\ell}_{\hat{\mathrm{e}}} = 0\}.$$

Definition 5.8. For $\varepsilon > 0$, we define $\mathcal{U}_\varepsilon(\hat{\mu})$ the ***ε*-neighborhood of *$\hat{\mu}$*** to be the set of $(\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})$ that satisfy the following conditions.

- For each $\hat{\mathrm{e}} \in \hat{\mathrm{E}}$, $\ell_{\hat{\mathrm{e}}}$ lies in $(\hat{\ell}_{\hat{\mathrm{e}}} - \varepsilon, \hat{\ell}_{\hat{\mathrm{e}}} + \varepsilon) \cap [0, 1]$. Moreover, we fix $\ell_{\hat{\mathrm{e}}} = 0$ for all nodal edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, and fix $\ell_{\hat{\mathrm{e}}} = 1$ for edges $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$ with either $\hat{\mathrm{v}}^-$ or $\hat{\mathrm{v}}^+$ being a critical vertex.
- For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, $(\underline{x}_{\hat{\mathrm{v}}}, O_{\hat{\mathrm{v}}})$ lies in $\mathcal{U}_\varepsilon(\underline{\hat{x}}_{\hat{\mathrm{v}}}, \hat{O}_{\hat{\mathrm{v}}})$.

Remark 5.9. Similar to Remark 4.2, before constructing the map $[\#]$ in (5.3) we make the following preparatory choices.

(1) Choose a gluing profile $R : (0, 1] \to [0, \infty)$, i.e., a continuous decreasing function such that $\lim_{r \to 0} R(r) = \infty$ ((1) is a universal choice, independent of $\sigma \in \mathfrak{DM}$ around which the neighborhood is based).

(2) For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$ and edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}(\hat{\mathrm{v}})$, choose an open neighborhood $B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) \subset D$ of $\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ and a family of strip coordinates $h^\pm_{\hat{\mathrm{e}}}$ near $\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ (as in Remark 3.10) such that for fixed vertex $\hat{\mathrm{v}}$, the open neighborhoods $B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$ are mutually disjoint and also disjoint from the interior marked points $\hat{O}_{\hat{\mathrm{v}}}$, and they contain the strip neighborhoods $N(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) \subset B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$.

Remark 5.10. Similar to Remark 4.3, we choose a positive number $\hat{\varepsilon} = \hat{\varepsilon}(\hat{\mu})$ such that

(1) for each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}(\hat{\mathrm{v}})$, and $(\underline{x}_{\hat{\mathrm{v}}}, O_{\hat{\mathrm{v}}}) \in \mathcal{U}_{\hat{\varepsilon}}(\underline{\hat{x}}_{\hat{\mathrm{v}}}, \hat{O}_{\hat{\mathrm{v}}})$, we have $O_{\hat{\mathrm{v}}} \cap B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) = \emptyset$ and $N(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) \subset B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$ (see Remark 5.9 (2)), and

(2) for each non-nodal edge $\hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{nd}}$ and each $\ell_{\hat{\mathrm{e}}} \in (\hat{\ell}_{\hat{\mathrm{e}}} - \hat{\varepsilon}, \hat{\ell}_{\hat{\mathrm{e}}} + \hat{\varepsilon})$, we have $\ell_{\hat{\mathrm{e}}} \neq 0$.

From here on, we always assume $\varepsilon < \hat{\varepsilon}(\hat{\mu})$.

Similar to Section 4, let $\underline{0}$ be the zero tuple (zero for each nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$). We introduce a tuple of gluing parameters $\underline{r} \in \mathcal{U}_\varepsilon(\underline{0})$. Now for $(\underline{r}, (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})) \in \mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\hat{\mu})$, we construct the gluing

$$(5.5) \qquad \#(\underline{r}, (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})) = (\#_{\underline{r}}(\hat{\mathrm{T}}), \#_{\underline{r}}(\underline{\ell}), \#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{O}))$$

as follows.

(1) The glued tree $\#_{\underline{r}}(\hat{\mathrm{T}})$ is defined by collapsing the *gluing edges* $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} := \{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \,|\, r_{\hat{\mathrm{e}}} > 0\}$ as in Section 4.

(2) The displaced edge lengths $\#_{\underline{r}}(\underline{\ell})$.

Using the edge identification map (4.6), for each non-gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}$ we denote by $\mathrm{e} \in \#_{\underline{r}}(\hat{\mathrm{E}})$ its corresponding glued edge. Moreover, we have decomposition $\hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} = (\hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}) \sqcup (\hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}})$ into non-nodal edges and non-gluing nodal edges.

For $\hat{\mathrm{e}} \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}$, we keep the edge length as it is

$$\#_{\underline{r}}(\underline{\ell})_{\mathrm{e}} := \ell_{\hat{\mathrm{e}}}, \tag{5.6}$$

For $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}$, we define

$$\#_{\underline{r}}(\underline{\ell})_{\mathrm{e}} := \frac{-\iota(r_{\hat{\mathrm{e}}})}{1-\iota(r_{\hat{\mathrm{e}}})}, \tag{5.7}$$

where $\iota : (-\infty, 0] \to (-\infty, 0]$ is the smooth function chosen before Remark 4.2. Note that this definition is consistent with the renormalized length (Definition 2.2) of the Morse trajectory (4.8).

(3) The glued boundary and interior marked points $(\#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{O}))$.

For each main vertex $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$, recall from (4.12) that the **glued boundary marked points** are given by

$$\#_{\underline{r}}(\underline{x})_{\mathrm{v,e}} := x_{\hat{\mathrm{v}},\hat{\mathrm{e}}} \text{ for } \hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}} \text{ and } \hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}, \tag{5.8}$$

where the edge e corresponds to the non-gluing edge ê by map (4.6). They are defined on the glued disk $\#_{\underline{r},\underline{x}}(\underline{D})_{\mathrm{v}}$, which we constructed by taking the union of disks $D_{\hat{\mathrm{v}}}$ for all $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$ and then discarding parts of the strip neighborhoods of $x^{\pm}_{\hat{\mathrm{e}}}$ for gluing edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}$ (see (4.11)).

Furthermore, we define the **glued interior marked points** by taking the union of the interior marked points $O_{\hat{\mathrm{v}}}$, which are disjoint from strip neighborhoods by Remark 5.10 (1).

$$\#_{\underline{r},\underline{x}}(\underline{O})_{\mathrm{v}} := \textstyle\bigsqcup_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}} O_{\hat{\mathrm{v}}}. \tag{5.9}$$

One can verify that all of the requirements in Section 5.2 are satisfied for the element $[\#](\underline{r}, (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})) := [\#_{\underline{r}}(\hat{\mathrm{T}}), \#_{\underline{r}}(\underline{\ell}), \#_{\underline{r}}(\underline{x}), \#_{\underline{r},\underline{x}}(\underline{O})]$. This defines the gluing map (5.3).

We now define an ε-neighborhood of σ by

$$\mathfrak{U}_{\varepsilon}(\sigma; \hat{\mu}; R) := [\#]\left(\mathcal{U}_{\varepsilon}(\underline{0}) \times \mathcal{U}_{\varepsilon}(\hat{\mu})\right), \tag{5.10}$$

where $\hat{\mu}$ is the representative of σ. In Section 7, we prove that this collection of neighborhoods defines a Hausdorff topology.

Theorem 5.11. *For each gluing profile R, the collection*

$$\{\mathfrak{U}_{\varepsilon}(\sigma; \hat{\mu}, R) \,|\, \sigma \in \mathfrak{DM},\, 0 < \varepsilon < \hat{\varepsilon}(\hat{\mu})\}$$

forms a basis in $\mathfrak{DM}$ and defines a Hausdorff topology, and it is independent of the gluing profile R.

5.4. **The Atlas of the Deligne-Mumford space.**

In this section, we give the Deligne-Mumford space $\mathfrak{DM}$ a manifold with boundary structure. Given an element σ of $\mathfrak{DM}$, we fix a representative $\hat{\mu} = (\hat{\mathrm{T}}, \underline{\hat{\ell}}, \underline{\hat{x}}, \underline{\hat{O}})$ of σ. We now construct a smooth chart around σ by using a variation of the gluing map $[\#] : \mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\hat{\mu}) \to \mathfrak{DM}$ in (5.3). More precisely, we use the same map $[\#]$ but restrict the domain to a subset of $\mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\hat{\mu})$.

In order to have the Euclidean space as the chart domain, we must consider the space of *ordered* interior marked points.

We define the **space of (ordered) interior marked points** by

$$\underline{MP}(D^\circ) := \{\underline{o} \subset D^\circ \,|\, \underline{o} = (o_1, \ldots, o_k) \text{ for } k \geq 0,\ o_j \text{ distinct}\}.$$

Note that the underline in $\underline{MP}(D^\circ)$ indicates that the interior marked points are ordered, as opposed to the unordered version $MP(D^\circ)$ in (5.2). This space has a metric

$$d(\underline{o}, \underline{o}') = \begin{cases} \max_j d(o_j, o'_j), & \text{if } n(\underline{o}) = n(\underline{o}'), \\ 100, & \text{if } n(\underline{o}) \neq n(\underline{o}'), \end{cases}$$

where $n(\underline{o})$ denotes the cardinality of $\underline{o}$. Similar to Section 5.3, this gives the product metric on the space of boundary marked points (3.2) and interior marked points $\underline{MP}(\partial D) \times \underline{MP}(D^\circ)$,

$$d((\underline{x}, \underline{o}), (\underline{x}', \underline{o}')) = \max\{d(\underline{x}, \underline{x}'), d(\underline{o}, \underline{o}')\},$$

and we denote the ε-neighborhood of $(\underline{\hat{x}}, \underline{\hat{o}})$ in $\underline{MP}(\partial D) \times \underline{MP}(D^\circ)$ by

$$\mathcal{U}_\varepsilon(\underline{\hat{x}}, \underline{\hat{o}}).$$

For the representative $\hat{\mu} = (\hat{\mathrm{T}}, \underline{\hat{\ell}}, (\underline{\hat{x}}, \underline{\hat{O}}))$, we pick an *arbitrary* ordering of the interior marked points. More precisely, for each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, we fix a set of ordered interior marked points $\underline{\hat{o}}_{\hat{\mathrm{v}}} \in \underline{MP}(D^\circ)$ with

$$\{\hat{o}_{\hat{\mathrm{v}},j}\} = \hat{O}_{\hat{\mathrm{v}}}. \tag{5.11}$$

By abuse of notation, we denote the representative with ordering

$$\hat{\mu} := (\hat{\mathrm{T}}, \underline{\hat{\ell}}, \underline{\hat{x}}, \underline{\hat{o}}).$$

We now define a subset of the marked points $(\underline{\hat{x}}, \underline{\hat{o}})$ which parametrizes the disk automorphism group $\mathrm{Aut}(D)$.

Definition 5.12. For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, we choose an index set of boundary marked points and interior marked points

$$A_{\hat{\mathrm{v}}} \subset \hat{\mathrm{E}}(\hat{\mathrm{v}}) \sqcup \{1, \ldots, n(\underline{\hat{o}}_{\hat{\mathrm{v}}})\}$$

such that each index set $A_{\hat{\mathrm{v}}}$ is either of the form

- $A_{\hat{\mathrm{v}}} = \{\hat{\mathrm{e}}, j\}$ with $1 \leq j \leq n(\underline{\hat{o}}_{\hat{\mathrm{v}}})$, or
- $A_{\hat{\mathrm{v}}} = \{\hat{\mathrm{e}}^{i_0}, \hat{\mathrm{e}}^{i_1}, \hat{\mathrm{e}}^{i_2}\}$.

We can indeed make such a choice due to the stability condition (5.1). We call such an index set $\underline{A} = (A_{\hat{\mathrm{v}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}}$ an **automorphism index set**. For marked points $(\underline{x}, \underline{o}) \in \mathcal{U}_\varepsilon(\hat{\underline{x}}, \hat{\underline{o}})$, we define its **automorphism components** by

$$\pi^{\mathrm{a}}(\underline{x}, \underline{o}) := \left((x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})_{\hat{\mathrm{e}} \in A_{\hat{\mathrm{v}}}}, (o_{\hat{\mathrm{v}},j})_{j \in A_{\hat{\mathrm{v}}}}\right)_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}}.$$

Remark 5.13. We call them automorphism components because by Proposition 13.8 they parametrize the disk automorphism group. More precisely, we can denote by $\psi_{\pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}}, \underline{o}_{\hat{\mathrm{v}}})}$ the unique disk automorphism with

$$\psi_{\pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}}, \underline{o}_{\hat{\mathrm{v}}})}(\pi^{\mathrm{a}}(\hat{\underline{x}}_{\hat{\mathrm{v}}}, \hat{\underline{o}}_{\hat{\mathrm{v}}})) = \pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}}, \underline{o}_{\hat{\mathrm{v}}}). \tag{5.12}$$

Proposition 13.8 implies that $\psi_{\pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}}, \underline{o}_{\hat{\mathrm{v}}})}$ parametrizes the disk automorphism group for all possible $\pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}}, \underline{o}_{\hat{\mathrm{v}}})$.

We now define the neighborhood slice of $\hat{\mu}$ by fixing the automorphism components of elements of neighborhood $\mathcal{U}_\varepsilon(\hat{\mu})$ (see Definition 5.8).

Definition 5.14. Let $\hat{\mu} = (\hat{\mathrm{T}}, \hat{\underline{\ell}}, \hat{\underline{x}}, \hat{\underline{o}})$ be an arbitrary representative, and fix an automorphism index set $\underline{A}$. Define $\mathcal{U}^{\mathrm{slc}}_\varepsilon(\hat{\mu})$ the **$\boldsymbol{\varepsilon}$-neighborhood slice of $\hat{\boldsymbol{\mu}}$** to be the subset of $\mathcal{U}_\varepsilon(\hat{\mu})$ consisting of $(\mathrm{T}, \underline{\ell}, \underline{x}, \underline{o})$ such that

$$\pi^{\mathrm{a}}(\underline{x}, \underline{o}) = \pi^{\mathrm{a}}(\hat{\underline{x}}, \hat{\underline{o}}).$$

Similarly, we denote $\mathcal{U}^{\mathrm{slc}}_\varepsilon(\hat{\underline{x}}, \hat{\underline{o}})$ to be the subset of $\mathcal{U}_\varepsilon(\hat{\underline{x}}, \hat{\underline{o}})$ consisting of $(\underline{x}, \underline{o})$ that satisfies the above condition.

Remark 5.15. Note that any marked points $(\underline{x}_{\hat{\mathrm{v}}}, \underline{o}_{\hat{\mathrm{v}}}) \in \mathcal{U}_\varepsilon(\hat{\underline{x}}_{\hat{\mathrm{v}}}, \hat{\underline{o}}_{\hat{\mathrm{v}}})$ are equivalent to the marked points $\psi^{-1}_{\pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}}, \underline{o}_{\hat{\mathrm{v}}})}(\underline{x}_{\hat{\mathrm{v}}}, \underline{o}_{\hat{\mathrm{v}}})$ that lies in a neighborhood slice $\mathcal{U}^{\mathrm{slc}}_{\varepsilon'}(\hat{\underline{x}}_{\hat{\mathrm{v}}}, \hat{\underline{o}}_{\hat{\mathrm{v}}})$. On the other hand, if marked points $(\underline{x}_{\hat{\mathrm{v}}}, \underline{o}_{\hat{\mathrm{v}}}), (\underline{x}'_{\hat{\mathrm{v}}}, \underline{o}'_{\hat{\mathrm{v}}}) \in \mathcal{U}^{\mathrm{slc}}_\varepsilon(\hat{\underline{x}}_{\mathrm{v}}, \hat{\underline{o}}_{\mathrm{v}})$ are equivalent via a disk automorphism ψ, then we have

$$\psi(\pi^{\mathrm{a}}(\hat{\underline{x}}_{\hat{\mathrm{v}}}, \hat{\underline{o}}_{\hat{\mathrm{v}}})) = \pi^{\mathrm{a}}(\hat{\underline{x}}_{\hat{\mathrm{v}}}, \hat{\underline{o}}_{\hat{\mathrm{v}}}),$$

and thus $\psi = \mathrm{Id}$. Hence restricting to the neighborhood slice in effect quotients out the $\mathrm{Aut}(D)$ action.

We now define the chart of $\mathfrak{DM}$ by restricting the gluing map $[\#]$ (5.3) to a neighborhood slice.

$$\begin{aligned} \theta : \mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}^{\mathrm{slc}}_\varepsilon(\hat{\mu}) &\to \mathfrak{DM} \\ (\underline{r}, \mu) &\mapsto [\#](\underline{r}, \mu). \end{aligned} \tag{5.13}$$

Remark 5.16. By construction $\mathcal{U}^{\mathrm{slc}}_\varepsilon(\hat{\mu})$ can be given the structure of an open subset of $[0, \infty)^N$. Indeed, the neighborhood slice $\mathcal{U}^{\mathrm{slc}}_\varepsilon(\hat{\mu})$ is the product of

- intervals $(\hat{\ell}_{\hat{\mathrm{e}}} - \varepsilon, \hat{\ell}_{\hat{\mathrm{e}}} + \varepsilon) \cap [0, 1]$ for non-nodal edges $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \notin \hat{\mathrm{E}}^{\mathrm{nd}}$ with both $\hat{\mathrm{v}}^-$ and $\hat{\mathrm{v}}^+$ being main vertices, and
- $\mathcal{U}^{\mathrm{slc}}_\varepsilon(\hat{\underline{x}}_{\hat{\mathrm{v}}}, \hat{\underline{o}}_{\hat{\mathrm{v}}})$ for main vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$.

In particular, for each non-automorphism component of the boundary marked points, the neighborhood $B^{\partial D}_\varepsilon(\hat{x}_{\hat{v},\hat{e}})$ can be parametrized as $\{e^{i\eta}\,|\,\eta\in(\hat{\eta}-\varepsilon,\hat{\eta}+\varepsilon)\}$.

Note that by the choice in Remark 5.10 (2), the aforementioned interval $(\hat{\ell}_{\hat{e}}-\varepsilon,\hat{\ell}_{\hat{e}}+\varepsilon)\cap[0,1]$ is either of the form $(\hat{\ell}_{\hat{e}}-\varepsilon,\hat{\ell}_{\hat{e}}+\varepsilon)$ or $(\hat{\ell}_{\hat{e}}-\varepsilon,1]$ (but never $[0,\hat{\ell}_{\hat{e}}+\varepsilon)$), and the latter gives the neighborhood of the boundary strata of $\mathfrak{DM}$.

The following result shows that the collection of θ's forms an atlas with the exponential gluing profile. We shall prove this result in Section 7.

Theorem 5.17. *Let the gluing profile R be given by the exponential gluing profile $\varphi(r)=e^{\frac{1}{r}}-e$. Then the collection of charts $\theta:\mathcal{U}_\varepsilon(\underline{0})\times\mathcal{U}^{\mathrm{slc}}_\varepsilon(\hat{\mu})\to\mathfrak{DM}$ forms a smooth manifold with boundary and corners atlas.*

The next result is simply a consequence of the Hausdorff topology (Theorem 5.11) and Theorem 5.17.

Theorem 5.18. *The Deligne-Mumford space $\mathfrak{DM}$ is a manifold with boundary and corners.*

5.5. **Stabilization.**

Ultimately, we study the finite dimensional Deligne-Mumford space because it facilitates the study of the infinite dimensional quotient space of disk trees. We now introduce the bridge that connects the two spaces. We define the stabilization on a neighborhood of a representative of a disk tree, which adds extra interior marked points on the domain, thus obtaining a stable element in the Deligne-Mumford space.

Let $\hat{\tau}=(\hat{\mathrm{T}},\underline{\hat{\gamma}},\underline{\hat{x}},\underline{\hat{u}})$ be a representative of a disk tree. Choose a set of interior marked points $\underline{\hat{o}}=(\hat{o}_{\hat{v},j})_{\hat{v}\in\hat{V}^{\mathrm{m}}}$ at which each derivative $D\hat{u}_{\hat{v}}$ is injective, such that the cardinality has $n(\underline{\hat{o}}_{\hat{v}})\geq 1$ for all non-ghost vertices $\hat{v}$. Also choose a transversal constraint $\Sigma_{\hat{v},j}$ at $\hat{o}_{\hat{v},j}$ as guaranteed by Lemma 5.2. Then Proposition 13.13 shows that there is $\varepsilon>0$ and a neighborhood $B(\hat{o}_{\hat{v},j})$ such that for each $(\underline{x}_{\hat{v}},u_{\hat{v}})\in\mathcal{U}_\varepsilon(\underline{\hat{x}}_{\hat{v}},\hat{u}_{\hat{v}})$, there exists *precisely one point* $o(u_{\hat{v}})_j$ in $B(\hat{o}_{\hat{v},j})$ with

$$u_{\hat{v}}(o(u_{\hat{v}})_j)\in\Sigma_{\hat{v},j}. \tag{5.14}$$

Definition 5.19. For each tuple $(\hat{\mathrm{T}},\underline{\gamma},\underline{x},\underline{u})$ in the neighborhood $\mathcal{U}_\varepsilon(\hat{\tau})$, we define its **stabilization induced by $\underline{\boldsymbol{\Sigma}}$ near $\underline{\hat{\boldsymbol{o}}}$** as

$$\mathrm{st}(\hat{\mathrm{T}},\underline{\gamma},\underline{x},\underline{u})=(\hat{\mathrm{T}},\underline{\ell},\underline{x},\underline{o}),$$

where each $\ell_{\hat{e}}:=\ell(\underline{\gamma}_{\hat{e}})$ is the renormalized length in (2.2), and we define $o_{\hat{v},j}:=o(u_{\hat{v}})_j$ as in (5.14).

Note that by construction each stabilization $\mathrm{st}(\tau)$ satisfies the stability condition (5.1) in the Deligne-Mumford space. Hence $\mathrm{st}(\tau)$ is a representative of an element of $\mathfrak{DM}$.

We now show that the stabilization defines a continuous map from a neighborhood of $\hat{\tau}$ to a neighborhood of $\mathrm{st}(\hat{\tau})$.

Lemma 5.20. *There are $\hat{\varepsilon} > 0, \hat{\nu} > 0$ such that the stabilization* st *defines a continuous map*

$$\mathrm{st} : \mathcal{U}_{\hat{\varepsilon}}(\hat{\tau}) \to \mathcal{U}_{\hat{\nu}}(\mathrm{st}(\hat{\tau})).$$

The above result follows from Proposition 13.13 and the continuity of the renormalized length ℓ on the space of Morse trajectory space.

We shall use stabilization when constructing the M-polyfold chart of the quotient space of disk trees. Similarly, an *everywhere* transversal constraint (Definition 5.3) defines an *everywhere* stabilization. We shall use the everywhere stabilization when proving the Hausdorff property of the quotient space of disk trees.

Let $\hat{\tau} = (\hat{\mathrm{T}}, \underline{\hat{\gamma}}, \underline{\hat{x}}, \underline{\hat{u}})$ be a representative of a disk tree, and choose an everywhere transversal constraint Σ for $\hat{\tau}$.

Definition 5.21. For each tuple $(\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u})$ in the neighborhood $\mathcal{U}_{\varepsilon}(\hat{\tau})$, we define its **everywhere stabilization induced by $\boldsymbol{\Sigma}$** as

$$\mathrm{st}(\hat{\mathrm{T}}, \underline{\gamma}, \underline{x}, \underline{u}) = (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O}),$$

where each $\ell_{\hat{\mathrm{e}}} := \ell(\underline{\gamma}_{\hat{\mathrm{e}}})$ is the renormalized length in (2.2) and each $O_{\hat{\mathrm{v}}} := u_{\hat{\mathrm{v}}}^{-1}(\Sigma)$ is the pre-image.

Similar to Lemma 5.20, the everywhere stabilization defines a continuous map.

Lemma 5.22. *There are $\hat{\varepsilon} > 0, \hat{\nu} > 0$ such that the everywhere stabilization* st *defines a continuous map*

$$\mathrm{st} : \mathcal{U}_{\hat{\varepsilon}}(\hat{\tau}) \to \mathcal{U}_{\hat{\nu}}(\mathrm{st}(\hat{\tau})).$$

As in Lemma 5.20, the continuity result follows from Proposition 13.13 and the continuity of the renormalized length ℓ.

We now show that there exists an everywhere transversal constraint whose corresponding everywhere stabilization map behave well with nearby equivalence. Later on we shall use this result to prove the Hausdorff property of the quotient space of disk trees.

Proposition 5.23. *Let $\hat{\tau}$ and $\hat{\tau}'$ be representatives of two disk trees. There exists an everywhere transversal constraint Σ for $\hat{\tau}$ and $\hat{\tau}'$, with induced everywhere stabilization maps*

$$\mathrm{st} : \mathcal{U}_{\hat{\varepsilon}}(\hat{\tau}) \to \mathcal{U}_{\hat{\nu}}(\mathrm{st}(\hat{\tau})), \quad \mathrm{st}' : \mathcal{U}_{\hat{\varepsilon}'}(\hat{\tau}') \to \mathcal{U}_{\hat{\nu}'}(\mathrm{st}'(\hat{\tau}'))$$

such that if there are $(\underline{r}, \tau) \in \mathcal{U}_{\hat{\varepsilon}}(\underline{0}) \times \mathcal{U}_{\hat{\varepsilon}}(\hat{\tau})$ and $(\underline{r}', \tau') \in \mathcal{U}_{\hat{\varepsilon}'}(\underline{0}) \times \mathcal{U}_{\hat{\varepsilon}'}(\hat{\tau}')$ and a biholomorphism $(\zeta, \underline{\psi})$ with

$$(\zeta, \underline{\psi})(\#(\underline{r}, \tau)) = \#(\underline{r}', \tau'),$$

then the same biholomorphism yields equivalence

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mathrm{st}(\tau))) = \#(\underline{r}', \mathrm{st}'(\tau')).$$

Proof. Recall from Remark 4.2 (2), for each nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$ we chose an open neighborhood $U_{\hat{\mathrm{e}}}$ around the image $\hat{u}^+_{\hat{\mathrm{e}}}(\hat{x}^+_{\hat{\mathrm{e}}}) = \hat{u}^-_{\hat{\mathrm{e}}}(\hat{x}^-_{\hat{\mathrm{e}}})$ in M, and similarly for each nodal edge $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{nd}}$ we chose an open neighborhood $U'_{\hat{\mathrm{e}}'}$. On the other hand, by Proposition 5.4 we choose a finite set of intersection points $Q \subset M$. For main vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$ and $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{m}}$ we have $\hat{u}_{\hat{\mathrm{v}}}^{-1}(Q) \subset D^\circ$ and $\hat{u}'^{-1}_{\hat{\mathrm{v}}'}(Q) \subset D^\circ$. Because these pre-images are contained in the *interior* of disks, the set Q does not intersect $\hat{u}^\pm_{\hat{\mathrm{e}}}(\hat{x}^\pm_{\hat{\mathrm{e}}})$ or $\hat{u}'^\pm_{\hat{\mathrm{e}}'}(\hat{x}'^\pm_{\hat{\mathrm{e}}'})$. Thus we can choose an open set U containing Q which does not intersect open neighborhoods $U_{\hat{\mathrm{e}}}$ or $U'_{\hat{\mathrm{e}}'}$. Then Proposition 5.4 shows that there exists an everywhere transversal constraint $\Sigma \subset U$ for $\hat{\tau}$ and $\hat{\tau}'$.

Thus we find an everywhere transversal constraint Σ for $\hat{\tau}$ and $\hat{\tau}'$ which avoids all neighborhoods $U_{\hat{\mathrm{e}}}$ and $U'_{\hat{\mathrm{e}}'}$ for nodal edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$ and $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{nd}}$. Now let $\hat{\nu}, \hat{\nu}'$ be as given in Lemma 5.22. Choose $\hat{\varepsilon}$ to be the minimum of the $\hat{\varepsilon}$ in Lemma 5.22 and the $\hat{\varepsilon}(\hat{\tau})$ in Remark 4.3, and choose $\hat{\varepsilon}'$ similarly.

Recall from the gluing construction that the plus gluing (4.14) and the map displacement (4.15) occur within such neighborhoods $U_{\hat{\mathrm{e}}}$ and $U'_{\hat{\mathrm{e}}'}$. Therefore $\#_{\underline{r},\underline{x}}(\underline{u})^{-1}_{\mathrm{v}}(\Sigma)$ lies outside of the glued strips of the domain $\#_{\underline{r},\underline{x}}(D)_{\mathrm{v}}$; furthermore, the pre-image of the glued map $\#_{\underline{r},\underline{x}}(\underline{u})^{-1}_{\mathrm{v}}(\Sigma)$ is equal to the glued pre-images $\#_{\underline{r},\underline{x}}(\underline{u}^{-1}(\Sigma))_{\mathrm{v}}$. Also we have $\#_{\underline{r}',\underline{x}'}(\underline{u}')^{-1}_{\mathrm{v}'}(\Sigma) = \#_{\underline{r}',\underline{x}'}(\underline{u}'^{-1}(\Sigma))_{\mathrm{v}'}$.

Now assume we have $(\zeta, \underline{\psi})(\#(\underline{r}, \tau)) = \#(\underline{r}', \tau')$. Then for main glued vertices $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$ and $\mathrm{v}' \in \#_{\underline{r}'}(\hat{\mathrm{V}}')$ with $\zeta(\mathrm{v}) = \mathrm{v}'$, we have

$$\#_{\underline{r},\underline{x}}(\underline{u})_{\mathrm{v}} \circ \psi_{\mathrm{v}}^{-1} = \#_{\underline{r}',\underline{x}'}(\underline{u}')_{\mathrm{v}'}.$$

Hence we have equality of pre-images $\psi_{\mathrm{v}}(\#_{\underline{r},\underline{x}}(\underline{u})^{-1}_{\mathrm{v}}(\Sigma)) = \#_{\underline{r}',\underline{x}'}(\underline{u}')^{-1}_{\mathrm{v}'}(\Sigma)$, and the above analysis establishes the equivalence of interior marked points

$$\psi_{\mathrm{v}}(\#_{\underline{r},\underline{x}}(\underline{u}^{-1}(\Sigma))_{\mathrm{v}}) = \#_{\underline{r}',\underline{x}'}(\underline{u}'^{-1}(\Sigma))_{\mathrm{v}'}. \tag{5.15}$$

Lastly, for corresponding edges e and e′ with $\zeta(\mathrm{e}) = \mathrm{e}'$, we have equality of displaced Morse trajectories $\#_{\underline{r}}(\underline{\gamma})_{\mathrm{e}} = \#_{\underline{r}'}(\underline{\gamma}')_{\mathrm{e}'}$. Then it follows from the construction in (5.6) and (5.7) that we have equality of the displaced renormalized lengths

$$\#_{\underline{r}}(\ell(\underline{\gamma}))_{\mathrm{e}} = \#_{\underline{r}'}(\ell(\underline{\gamma}'))_{\mathrm{e}'}. \tag{5.16}$$

By Definition 5.21, (5.15), and (5.16), we have equivalence

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mathrm{st}(\tau))) = \#(\underline{r}', \mathrm{st}'(\tau')).$$

□

6. The Atlas of the Quotient Space of Disk Trees

In this section, we give the quotient space of disk trees $\mathfrak{X}$ an M-polyfold with boundary and corners structure, which we shall define in Section 6.1. Given an element κ of $\mathfrak{X}$, we fix a representative $\hat{\tau} = (\hat{\mathrm{T}}, \underline{\hat{\gamma}}, \underline{\hat{x}}, \underline{\hat{\mu}})$. We now construct a chart around κ by using a variation of the gluing map $[\#] : \mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\hat{\tau}) \to \mathfrak{X}$ in (4.1). Note that the gluing map $[\#]$ itself is not a chart. It is not injective due to (1) the $\mathrm{Aut}(D)$ action on each disk, and (2) the interpolation used in the gluing (4.14). We deal with problem (1) by restricting to a neighborhood slice (see Section 5.4 and especially Remark 5.15), and problem (2) by restricting to a subset of maps called "splicing core" (see Section 6.2). Hofer, Wysocki, and Zehnder built M-polyfold theory with problem (2) in mind, adopting splicing cores as local models of the M-polyfold.

6.1. M-Polyfold Local Models and Splicing.

We now give a brief introduction to the local models of M-polyfold, and we refer the readers to Section 3.1 for the preliminary concepts of sc-structure and Section 2.2 of [7] for a complete exposition on the local models.

Definition 6.1. Let U be an open subset of partial quadrant C of an sc-Banach space E. An sc^∞ map $r : U \to U$ is an **sc-smooth retraction** if r satisfies $r \circ r = r$, and we call its image $O = r(U)$ an **sc-smooth retract**.

We define the **tangent space** of O by $TO := Tr(TU)$, where TU is the tangent space of U and Tr is the tangent map. (See Definition 3.6.) Lemma 1.14 of [7] shows that the notion of tangent space is well-defined, i.e., if we have another sc-smooth retraction $s : V \to V$ with the same image $s(V) = O$, then they produce the same tangent space $Tr(TU) = Ts(TV)$.

We shall use sc-smooth retracts as local models for M-polyfold. Hence we specify the notion of sc-smooth map between retracts.

Definition 6.2. Let $O \subset E$ and $O' \subset E'$ be sc-smooth retracts in sc-Banach spaces E and E', respectively. A map $f : O \to O'$ between two sc-smooth retracts is $\boldsymbol{sc^1}$ if the composite $f \circ r$ is sc^1, where U is an open subset of the sc-Banach space E and $r : U \to U$ is an sc-smooth retraction onto $r(U) = O$.

Lemma 1.16 of [7] shows that this notion is well-defined, i.e., it is independent of the choice of the retraction r. Similarly to Section 3.1, we can define the notion of sc-smoothness for maps between retracts. Moreover, Theorem 1.17 of [7] proves the chain rule for such maps.

We now introduce the local degeneracy index, which describes the boundary and corners structure. Let C be a partial quadrant in an sc-Banach space E. After applying an sc-isomorphism we can assume $C = [0, \infty)^k \times W$ for some sc-Banach space W (see Definition 3.4). We then define the **local degeneracy index** $d_C : C \to \mathbb{N}_0$ as the cardinality of the set

$$d_C(a_1, \ldots, a_k, w) := n(\{i \in \{1, \ldots, k\} \mid a_i = 0\}). \tag{6.1}$$

We say x lies in the interior of C if $d_C(x) = 0$, and lies on the boundary of C if $d_C(x) \geq 1$.

A local M-polyfold model is an sc-retract that behaves well with the local degeneracy index.

Definition 6.3. A **local M-polyfold model** is a triple (O, C, E), where O is an sc-smooth retract of a partial quadrant C in an sc-Banach space E with the following conditions.

(1) There is an open subset U of the partial quadrant C and an sc-smooth retraction $r : U \to U$ with $O = r(U)$.
(2) For each smooth point $x \in O_\infty$, the kernel of the map $\mathrm{Id} - Dr(x)$ has an sc-complement contained in C.
(3) For each $x \in O$, there exists a sequence of smooth points $x_n \in O_\infty$ converging to x with $d_C(x_n) = d_C(x)$.

We now define an M-polyfold with boundary and corners.

Definition 6.4. $\mathfrak{X}$ is an **M-polyfold with boundary and corners** if $\mathfrak{X}$ is a metrizable topological space with maximal atlas $\{(\phi, \mathfrak{U}, (O, C, E)\}$, where each $\mathfrak{U} \subset \mathfrak{X}$ is an open subset and $\phi : O \to \mathfrak{U}$ is a homeomorphism; moreover, if two charts $\phi : O \to \mathfrak{U}$ and $\phi' : O' \to \mathfrak{U}'$ have a non-empty target intersection $\mathfrak{U} \cap \mathfrak{U}' \neq \emptyset$, then the transition map $\phi'^{-1} \circ \phi : \phi^{-1}(\mathfrak{U} \cap \mathfrak{U}') \to \phi'^{-1}(\mathfrak{U} \cap \mathfrak{U}')$ is an sc-diffeomorphism.

We now give an example of an M-polyfold without boundary, $\mathcal{W}$ the space of boundary marked points and H^{3,δ_0} maps given in Definition 3.15. This is a simple M-polyfold whose local models are not only sc-smooth retracts, but open subsets of sc-Banach spaces. Furthermore, we explore the sc-smoothness of the action by disk automorphisms $\mathrm{Aut}(D)$ (see Section 13.1).

Proposition 6.5. *$\mathcal{W}$ the space of boundary marked points and H^{3,δ_0} maps is an M-polyfold without boundary. For each disk map $\hat{u}$ belonging to the space $\mathrm{Map}^{3+m,\delta_m}((D, \underline{\hat{x}}), M; L)$ for all $m \geq 0$, the chart on the neighborhood $\mathcal{U}_\varepsilon(\underline{\hat{x}}, \hat{u})$ (see (3.7)) is given by*

$$\alpha : \mathcal{U}_\varepsilon(\underline{\hat{x}}) \times B_\varepsilon(0) \to \mathcal{U}_\varepsilon(\underline{\hat{x}}, \hat{u}), \quad (\underline{x}, \xi) \mapsto \left(\underline{x}, \exp_{\hat{u}} \circ \xi \circ \nu_{\underline{x}}^{-1}\right).$$

Here $B_\varepsilon(0) \subset \mathrm{Sec}^{3,\delta_0}((D, \underline{\hat{x}}), \hat{u}^*TM; TL)$ is the ε-neighborhood of the zero section. Moreover, the action

$$\mathrm{Aut}(D) \times \mathcal{W} \to \mathcal{W}, \quad (\psi, (\underline{x}, u)) \mapsto (\psi(\underline{x}), u \circ \psi^{-1})$$

is sc-smooth.

Proof. Suppose there is another chart

$$\alpha' : \mathcal{U}_{\varepsilon'}(\underline{\hat{x}}') \times B_{\varepsilon'}(0') \to \mathcal{U}_{\varepsilon'}(\underline{\hat{x}}', \hat{u}'), \quad (\underline{x}', \xi') \mapsto \left(\underline{x}', \exp_{\hat{u}'} \circ \xi' \circ \nu'^{-1}_{\underline{x}'}\right)$$

with families of strip coordinates $\underline{h}'$ near $\underline{\hat{x}}'$, and we assume there is non-empty intersection $\mathcal{U}_\varepsilon(\underline{\hat{x}}, \hat{u}) \cap \mathcal{U}_{\varepsilon'}(\underline{\hat{x}}', \hat{u}') \neq \emptyset$. The transition map $\alpha'^{-1} \circ \alpha$ is of the form

$$\begin{aligned} \alpha^{-1}(\mathcal{U}_\varepsilon(\underline{\hat{x}}, \hat{u}) \cap \mathcal{U}_{\varepsilon'}(\underline{\hat{x}}', \hat{u}')) &\to \alpha'^{-1}(\mathcal{U}_\varepsilon(\underline{\hat{x}}, \hat{u}) \cap \mathcal{U}_{\varepsilon'}(\underline{\hat{x}}', \hat{u}')) \\ (\underline{x}, \xi) &\mapsto \left(\underline{x}, \exp_{\hat{u}'}^{-1} \circ \exp_{\hat{u}} \circ \xi \circ \nu_{\underline{x}}^{-1} \circ \nu'_{\underline{x}}\right). \end{aligned}$$

Recall from Definition 3.16 that the space of H^{3,δ_0} sections are defined over two regions: the H^3 space on the region outside of strip neighborhoods, and the H^{3,δ_0} space on strip neighborhoods. The above map is sc-smooth for the H^3 space on the region outside of strip neighborhoods due to Proposition 13.14 and Proposition 13.15. In order to verify that it is sc-smooth on the

H^{3,δ_0} space on strip neighborhoods, we express the above map in each strip coordinates $h_i'^{\pm}(\hat{x}_i', \cdot)$

$$
\begin{aligned}
&\exp_{\hat{u}'}^{-1} \circ \exp_{\hat{u}} \circ \xi \circ \nu_{\underline{x}}^{-1} \circ \nu'_{\underline{x}} \circ h_i'^{\pm}(\hat{x}_i', \cdot) \\
&= \exp_{\hat{u}'}^{-1} \circ \exp_{\hat{u}} \circ \xi \circ h_i^{\pm}(\hat{x}_i, \cdot) \circ h_i^{\pm}(x_i, \cdot)^{-1} \circ h_i'^{\pm}(x_i, \cdot).
\end{aligned} \tag{6.2}
$$

The above equality follows Lemma 3.11, which implies

$$\nu_{\underline{x}} \circ h_i^{\pm}(\hat{x}_i, \cdot) = h_i^{\pm}(x_i, \cdot), \quad \nu'_{\underline{x}} \circ h_i'^{\pm}(\hat{x}_i, \cdot) = h_i'^{\pm}(x_i, \cdot).$$

Thus the map from $(\underline{x}, \xi \circ h_i^{\pm}(\hat{x}_i, \cdot))$ to the expression in (6.2) is sc-smooth due to Lemma 13.17, Proposition 13.16, and Proposition 13.18.

We prove the sc-smoothness of the $\mathrm{Aut}(D)$ action similarly. Let $\mathcal{U}_\varepsilon(\hat{\psi})$ be a neighborhood of a disk automorphism $\hat{\psi}$, and let $\underline{h}'$ be families of strip coordinates near marked points $\hat{\psi}(\hat{\underline{x}})$. The disk automorphism action

$$\mathcal{U}_\varepsilon(\hat{\psi}) \times \mathcal{U}_\varepsilon(\hat{\underline{x}}, \hat{u}) \to \mathcal{U}_{\varepsilon'}(\hat{\psi}(\hat{\underline{x}}), \hat{u} \circ \hat{\psi}^{-1}), \quad (\psi, (\underline{x}, u)) \mapsto (\psi(\underline{x}), u \circ \psi^{-1})$$

in local charts is given by

$$(\psi, (\underline{x}, \xi)) \mapsto \left(\psi(\underline{x}), \exp_{\hat{u} \circ \hat{\psi}^{-1}}^{-1} \circ \exp_{\hat{u}} \circ \xi \circ \nu_{\underline{x}}^{-1} \circ \psi^{-1} \circ \nu'_{\psi(\underline{x})}\right).$$

As before, the above map is sc-smooth for the H^3 space on the region outside of strip neighborhoods due to Proposition 13.14 and Proposition 13.15. We express this map in each strip coordinates $h_i'^{\pm}(\hat{\psi}(\hat{x}_i), \cdot)$

$$
\begin{aligned}
&\exp_{\hat{u} \circ \hat{\psi}^{-1}}^{-1} \circ \exp_{\hat{u}} \circ \xi \circ \nu_{\underline{x}}^{-1} \circ \psi^{-1} \circ \nu'_{\psi(\underline{x})} \circ h_i'^{\pm}(\hat{\psi}(\hat{x}_i), \cdot) \\
&= \exp_{\hat{u} \circ \hat{\psi}^{-1}}^{-1} \circ \exp_{\hat{u}} \circ \xi \circ h_i^{\pm}(\hat{x}_i, \cdot) \circ h_i^{\pm}(x_i, \cdot)^{-1} \circ \psi^{-1} \circ h_i'^{\pm}(\psi(x_i), \cdot).
\end{aligned}
$$

The map from $(\psi, (\underline{x}, \xi \circ h_i^{\pm}(\hat{x}_i, \cdot)))$ to the above expression is sc-smooth due to Lemma 13.17, Proposition 13.16, and Proposition 13.18. $\square$

As we shall see in the next section, the following type of retraction describes the local model near a strip nodal map.

Definition 6.6. Let V be an open subset of a partial quadrant D in an sc-Banach space W, and let E be another sc-Banach space. An sc-smooth map $\pi : V \times F \to F$ is called a **splicing** if $\pi_v : F \to F$ is a family of linear sc-projections, i.e., each π_v is an sc-operator (Definition 3.4) with $\pi_v \circ \pi_v = \pi_v$. We call the set

$$K = \{(v, f) \in V \times F \mid \pi_v(f) = f\}$$

the **splicing core**.

This defines a retraction $(v, f) \mapsto (v, \pi_v(f))$ with the splicing core K as its image. In the next section, we shall see the case that $V = (-\varepsilon, \varepsilon)$, each v is a gluing parameter, and each f is a strip nodal map.

6.2. **Splicing Arising from Plus Gluing.**

In order to construct an M-polyfold chart for the quotient space of disk trees, one natural candidate is the gluing map [#] given in (4.1). However, this map is not injective and one of the reasons is that the interpolation in the plus gluing $\oplus_r$ (4.14) is not injective. We resolve this problem by restricting the domain of the plus gluing to a subspace, which is obtained by effectively modding out the kernel of $\oplus_r$. This subspace is r dependent, and we shall give it the structure of a splicing core.

We shall stay close to Section 1.3 of [7] in this section. The main difference in the cylinder gluing of [7] is the presence of an extra angle-twisting parameter ϑ

$$[0,R]\times S^1 \to [-R,0]\times S^1, \quad (s,t)\mapsto (s-R, t-\vartheta),$$

as opposed to only having the translating parameter R in the strip gluing (4.9). Other than that, these two settings share many analytical similarities.

As a reminder, we denote $\mathbb{R}^+ := [0,\infty)$ and $\mathbb{R}^- := (-\infty,0]$. We first consider the domain of the plus gluing, the space of strip nodal maps which consists of $\mathbb{C}^n$-valued function pairs on strips $\mathbb{R}^\pm \times [0,\pi]$ with boundary condition on $\mathbb{R}^n$ and matching limits (see Example 3.3).

Definition 6.7. We define F^{nd} the **space of strip nodal maps** to be the function space of pairs (ξ^+,ξ^-) such that each $\xi^\pm$ can be written as

$$\xi^\pm = \eta^\pm + c,$$

with $\eta^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n)$ and $c\in\mathbb{R}^n$. Choose a strictly increasing sequence of weights $\delta_0<\delta_1<\cdots<1$, we give F^{nd} an sc-structure

$$\|(\xi^+,\xi^-)\|_m^2 := |c|^2 + \|\eta^+\|^2_{H^{3+m,\delta_m}} + \|\eta^-\|^2_{H^{3+m,\delta_m}},$$

with H^{3+m,δ_m} norm defined by

$$\|\eta^\pm\|^2_{H^{3+m,\delta_m}} := \sum_{|\alpha|\le 3+m}\int_{\mathbb{R}^\pm\times[0,\pi]} |\partial^\alpha \eta^\pm(s,t)|^2 e^{2\delta_m|s|}dsdt.$$

Let R be an arbitrary gluing profile. We shall define a splicing $(r,\xi^+,\xi^-)\mapsto \pi_r(\xi^+,\xi^-)$ for $r\in(-\varepsilon,\varepsilon)$ and $(\xi^+,\xi^-)\in F^{\mathrm{nd}}$. Recall the smooth cut-off function $\beta:\mathbb{R}\to[0,1]$ defined before Remark 4.2 with properties

- $\beta(s)=1$ for $s\le -1$ and $\beta(s)=0$ for $s\ge 1$,
- $\beta'(s)<0$ for $s\in(-1,1)$,
- $\beta(s)+\beta(-s)=1$ for all s.

We denote the shifted cut-off function by

$$\beta_{s_0}(s) := \beta(s-s_0).$$

For $r\in(0,\varepsilon)$, we abbreviate $R(r)$ as R. Recall from Section 4 that the **plus gluing** is given by

$$\oplus_r(\xi^+,\xi^-) = \beta_{R/2}\,\xi^+ + (1-\beta_{R/2})\,\xi^-(\cdot - R). \tag{6.3}$$

Note that the glued map $\oplus_r(\xi^+, \xi^-)$ lies in the sc-Banach space

$$H^3([0,R] \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n).$$

For a fixed r, the plus gluing map $\oplus_r$ is not injective. Hence we define its "complementary map" the **minus gluing** $\ominus_r$ and later on we shall show that the combined map $(\oplus_r, \ominus_r)$ is an isomorphism.

$$\begin{aligned}\ominus_r(\xi^+, \xi^-) = &-(1-\beta_{R/2})\big(\xi^+ - \tfrac{1}{2}([\xi^+]_{R/2} + [\xi^-]_{-R/2})\big) \\ &+ \beta_{R/2}\big(\xi^-(\cdot - R) - \tfrac{1}{2}([\xi^+]_{R/2} + [\xi^-]_{-R/2})\big),\end{aligned} \tag{6.4}$$

where $[\xi^+]_{R/2}$ and $[\xi^-]_{-R/2}$ are the real parts of the integrals

$$[\xi^+]_{R/2} := \mathrm{Re}\int_0^\pi \xi^+(R/2, t)dt, \quad [\xi^-]_{-R/2} := \mathrm{Re}\int_0^\pi \xi^-(-R/2, t)dt.$$

Note that $\ominus_r(\xi^+, \xi^-)$ has limit at positive infinity $-c + \frac{1}{2}([\xi^+]_{R/2} + [\xi^-]_{-R/2})$, and limit at negative infinity $c - \frac{1}{2}([\xi^+]_{R/2} + [\xi^-]_{-R/2})$, where c is the common limit of ξ^+ and ξ^-. Hence it lies in the space

$$H^{3,\delta_0}_{\text{op-lim}}(\mathbb{R} \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n),$$

the space of maps with opposite limits at positive and negative infinity. To define this space precisely, fix a smooth cut-off function γ with $\gamma(s) = 1$ for $s \geq 1$ and $\gamma(s) = -1$ for $s \leq -1$. We say ζ belongs to the space of opposite limits $H^{3,\delta_0}_{\text{op-lim}}(\mathbb{R} \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$ if there is $c \in \mathbb{R}^n$ with $\zeta - \gamma \cdot c$ being in $H^{3,\delta_0}(\mathbb{R} \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$. We give this space an sc-structure by defining

$$\|\zeta\|_m^2 := |c|^2 + \|\zeta - \gamma \cdot c\|^2_{H^{3+m,\delta_m}}.$$

Thus the **total gluing** $\boxdot_r = (\oplus_r, \ominus_r)$ maps from F^{nd} to the direct sum

$$G_r := H^3([0,R] \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n) \oplus H^{3,\delta_0}_{\text{op-lim}}(\mathbb{R} \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n).$$

The following result shows that the total gluing is a linear sc-isomorphism. To achieve the injectivity of the plus gluing map $\oplus_r$, we restrict its domain to a complementary subspace of $\ker(\oplus_r)$ in F^{nd}.

Lemma 6.8. *Let R be any gluing profile in Remark 4.2 (1). For $r \in (0,\varepsilon)$, the total gluing $\boxdot_r : F^{\mathrm{nd}} \to G_r$ is a linear sc-isomorphism. In particular for $r > 0$, we have the decomposition $F^{\mathrm{nd}} = \ker(\ominus_r) \oplus \ker(\oplus_r)$, and hence the restriction*

$$\oplus_r : \ker(\ominus_r) \to H^3([0,R] \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$$

is a linear sc-isomorphism.

This result is analogous to Theorem 1.27 of [7]. To prove the total gluing $\boxdot_r$ is a linear sc-isomorphism one simply need to solve a linear equation. We shall prove this result in Section 8.1

Hence for $r \in (0,\varepsilon)$, we define $\pi_r : F^{\mathrm{nd}} \to F^{\mathrm{nd}}$ to be the linear projection onto $\ker(\ominus_r)$ along $\ker(\oplus_r)$. More precisely, we define

$$\pi_r(\xi^+, \xi^-) = (\zeta^+, \zeta^-), \tag{6.5}$$

with (ζ^+, ζ^-) satisfying

$$\oplus_r(\zeta^+, \zeta^-) = \oplus_r(\xi^+, \xi^-),$$
$$\ominus_r(\zeta^+, \zeta^-) = 0.$$

The following result shows that the family of projections π_r extends smoothly to $r \in (-\varepsilon, 0]$ as the identity map.

Theorem 6.9. *Let the gluing profile R be given by the exponential gluing profile $\varphi(r) = e^{\frac{1}{r}} - e$. Let $\pi_r : F^{\mathrm{nd}} \to F^{\mathrm{nd}}$ be the linear projection of F^{nd} onto $\ker(\ominus_r)$ defined in (6.5) for $r \in (0, \varepsilon)$, and $\pi_r = \mathrm{Id}$ for $r \in (-\varepsilon, 0]$, then the map $(r, \xi^+, \xi^-) \mapsto \pi_r(\xi^+, \xi^-)$ is an sc-smooth splicing.*

This result is analogous to Theorem 1.28 of [7]. We shall prove this result in Section 8.1.

Note that the above splicing core $\{(r, \xi^+, \xi^-) \,|\, \pi_r(\xi^+, \xi^-) = (\xi^+, \xi^-)\}$ is of the form

$$K = \Big((-\varepsilon, 0] \times F^{\mathrm{nd}}\Big) \cup \Big(\bigcup_{r \in (0,\varepsilon)} \{r\} \times \ker(\ominus_r)\Big). \tag{6.6}$$

6.3. **M-Polyfold Charts for the Quotient Space of Disk Trees.**

Fix a strictly increasing sequence of weights $\delta_0 < \delta_1 < \cdots < 1$. Given a disk tree $\kappa \in \mathfrak{X}$ whose disk maps are of H^{3+m,δ_m} for all m, we fix a representative $\hat{\tau} = (\hat{\mathrm{T}}, \underline{\hat{\gamma}}, \underline{\hat{x}}, \underline{\hat{u}})$. For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, the space of sections $\mathrm{Sec}^{3,\delta_0}((D, \underline{\hat{x}}_{\hat{\mathrm{v}}}), \hat{u}_{\hat{\mathrm{v}}}^* TM; TL)$ has an sc-structure (see Remark 3.17). We shall use this sc-Banach space and the splicing arising from plus gluing (Section 6.2) to construct an M-polyfold chart around κ.

Remark 6.10. We choose a set of structures which augment the choices made in Remark 4.2.

(1) Choose a continuous decreasing function called a **gluing profile** $R : (0, 1] \to [0, \infty)$ such that $\lim_{r \to 0} R(r) = \infty$. ((1) is a universal choice.)
(2) Choose a metric g on M such that L is totally geodesic (Lemma 3.18), and moreover for each edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$, if $\hat{\mathrm{v}}^\pm$ is a main vertex, then the metric g is Euclidean around the image $\hat{u}^\pm_{\hat{\mathrm{e}}}(\hat{x}^\pm_{\hat{\mathrm{e}}})$.
(3) For each edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \in \hat{\mathrm{E}}$, if $\mathrm{v}^\pm$ is a main vertex, we choose an open neighborhood $U^\pm_{\hat{\mathrm{e}}}$ of $\hat{u}^\pm_{\hat{\mathrm{e}}}(\hat{x}^\pm_{\hat{\mathrm{e}}})$ in M, along with a C^∞ chart $\varphi^\pm_{\hat{\mathrm{e}}} : U^\pm_{\hat{\mathrm{e}}} \to B_1(0) \subset \mathbb{C}^n$ such that
   - the metric $g|_{U^\pm_{\hat{\mathrm{e}}}}$ is the pullback Euclidean metric $(\varphi^\pm_{\hat{\mathrm{e}}})^* g_{\mathrm{Eucl}}$,
   - $\varphi^\pm_{\hat{\mathrm{e}}}(\hat{u}^\pm_{\hat{\mathrm{e}}}(\hat{x}^\pm_{\hat{\mathrm{e}}})) = 0$ and $\varphi^\pm_{\hat{\mathrm{e}}}(L \cap U^\pm_{\hat{\mathrm{e}}}) = \mathbb{R}^n \cap B_1(0)$, and
   - for each nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, we have the same open neighborhoods $U^+_{\hat{\mathrm{e}}} = U^-_{\hat{\mathrm{e}}}$ and the same charts $\varphi^+_{\hat{\mathrm{e}}} = \varphi^-_{\hat{\mathrm{e}}}$, and we denote them by $U_{\hat{\mathrm{e}}}$ and $\varphi_{\hat{\mathrm{e}}}$.

(4) For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$ and edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}(\mathrm{v})$, choose an open neighborhood $B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) \subset D$ of $\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ and a family of strip coordinates $h^{\pm}_{\hat{\mathrm{e}}}$ near $\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ (as in Remark 3.10) such that
   - for each vertex $\hat{\mathrm{v}}$, the open neighborhoods $B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$ are mutually disjoint and contain the strip neighborhoods $N(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) \subset B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$,
   - for each edge $\hat{\mathrm{e}}$, we have $\hat{u}^{\pm}_{\hat{\mathrm{e}}}\big(\overline{B(\hat{x}^{\pm}_{\hat{\mathrm{e}}})}\big) \subset (\varphi^{\pm}_{\hat{\mathrm{e}}})^{-1}(B_{1/2}(0))$.

   Moreover for each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, choose a family of marked points varying diffeomorphism $\nu_{\underline{x}_{\hat{\mathrm{v}}}}$ in Lemma 3.11.

(5) For each non-ghost vertex $\hat{\mathrm{v}}$ (see (2.8)), we choose an interior point $\hat{o}_{\hat{\mathrm{v}}} \in D^{\circ}$ with injective derivative $D\hat{u}_{\hat{\mathrm{v}}}(\hat{o}_{\hat{\mathrm{v}}})$, and a transversal constraint $\Sigma_{\hat{\mathrm{v}}}$ at $\hat{o}_{\hat{\mathrm{v}}}$ that is totally geodesic with respect to the metric g (see Lemma 5.2).

(6) For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, choose an automorphism index set $A_{\hat{\mathrm{v}}}$ (see Definition 5.12) such that
   - for each non-ghost vertex $\hat{\mathrm{v}}$ we have $A_{\hat{\mathrm{v}}} = \{\hat{\mathrm{e}}, 1\}$ with edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}(\hat{\mathrm{v}})$, (note that $\hat{o}_{\hat{\mathrm{v}},1}$ is simply a formal way of indexing the only interior marked point $\hat{o}_{\hat{\mathrm{v}}}$), and
   - for each ghost vertex $\hat{\mathrm{v}}$ we have $A_{\hat{\mathrm{v}}} = \{\hat{\mathrm{e}}^{i_0}, \hat{\mathrm{e}}^{i_1}, \hat{\mathrm{e}}^{i_2}\} \subset \hat{\mathrm{E}}(\hat{\mathrm{v}})$.

The locally Euclidean condition on the metric g guarantees that for an edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^{-}, \hat{\mathrm{v}}^{+})$, if $\hat{\mathrm{v}}^{\pm}$ is a main vertex, then for any point $p \in U^{\pm}_{\mathrm{e}}$ and any tangent vector $v \in T_pM$, we have

$$\varphi^{\pm}_{\hat{\mathrm{e}}}(\exp_p(v)) = \varphi^{\pm}_{\hat{\mathrm{e}}}(p) + D\varphi^{\pm}_{\hat{\mathrm{e}}}(p)(v). \tag{6.7}$$

As we shall see, such linearity is important in the *injectivity* property of our charts (Remark 6.14).

To construct a chart, we shall perturb the disk maps $\underline{\hat{u}}$ in the direction of sections $\underline{\xi}$. Here we specify such space of admissible sections. We denote the product Banach space of H^{3,δ_0} sections (see Definition 3.16) by

$$E := \prod_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}} \mathrm{Sec}^{3,\delta_0}((D, \underline{\hat{x}}_{\hat{\mathrm{v}}}), \hat{u}^{*}_{\hat{\mathrm{v}}}TM; TL). \tag{6.8}$$

Since $\underline{\hat{u}}$ is H^{3+m,δ_m} for all m, by Remark 3.17 the space E is an sc-Banach space. As in (3.4), we denote the local expression of sections $\underline{\xi} \in E$ in the strip coordinates on $N(\hat{x}^{\pm}_{\hat{\mathrm{e}}})$ by

$$\xi^{h^{\pm}}_{\hat{\mathrm{e}}}(z) := D\varphi^{\pm}_{\hat{\mathrm{e}}}(\hat{u}^{\pm}_{\hat{\mathrm{e}}}(h^{\pm}_{\hat{\mathrm{e}}}(\hat{x}^{\pm}_{\hat{\mathrm{e}}}, z)))(\xi^{\pm}_{\hat{\mathrm{e}}}(h^{\pm}_{\hat{\mathrm{e}}}(\hat{x}^{\pm}_{\hat{\mathrm{e}}}, z))). \tag{6.9}$$

Definition 6.11. We define F **the slice subspace of sections** to be the subspace of E consisting of sections $\underline{\xi} \in E$ that satisfy the following conditions.

(1) For each non-ghost vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, we require $\xi_{\hat{\mathrm{v}}}(\hat{o}_{\hat{\mathrm{v}}}) \in T_{\hat{u}_{\hat{\mathrm{v}}}(\hat{o}_{\hat{\mathrm{v}}})}\Sigma_{\hat{\mathrm{v}}}$, where $\Sigma_{\hat{\mathrm{v}}}$ is the transversal constraint at $\hat{o}_{\hat{\mathrm{v}}}$ in Remark 6.10 (5).

(2) For each nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, we have $(\xi^{h^{+}}_{\hat{\mathrm{e}}}, \xi^{h^{-}}_{\hat{\mathrm{e}}}) \in F^{\mathrm{nd}}$, where the space of strip nodal maps F^{nd} is given in Definition 6.7.

(3) For each non-nodal edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}$, if $\hat{\mathrm{v}}^\pm$ is a main vertex, then we have $\xi_{\hat{\mathrm{e}}}^{h^\pm} \in H^{3,\delta_0}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$.

We now specify the domain of the splicing, which parametrizes the family of projections on the slice subspace of sections F. Firstly, we denote a tuple of generalized Morse trajectories

$$\underline{\hat{\rho}} = (\underline{\hat{\gamma}}_{\hat{\mathrm{e}}})_{\hat{\mathrm{e}} \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}},$$

forgetting every constant Morse trajectory of each nodal edge. We denote $\mathcal{U}_\varepsilon(\underline{\hat{\rho}})$ to be the ε-neighborhood of $\underline{\hat{\rho}}$, which consists of tuples of Morse trajectories $\underline{\rho} = (\underline{\rho}_{\hat{\mathrm{e}}})_{\hat{\mathrm{e}} \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}}$ with each $\underline{\rho}_{\hat{\mathrm{e}}}$ lies in the ε-neighborhood $B_\varepsilon^{\overline{\mathcal{M}}}(\underline{\hat{\rho}}_{\hat{\mathrm{e}}})$ in the Morse trajectory space (see (2.3)).

Next we denote by $\mathcal{U}_\varepsilon^{\mathrm{slc}}(\underline{\hat{x}})$ the ε-neighborhood slice of $\underline{\hat{x}}$. It consists of boundary marked points $\underline{x} \in \mathcal{U}_\varepsilon(\underline{\hat{x}})$ with fixed automorphism components $\pi^{\mathrm{a}}(\underline{x}) = \pi^{\mathrm{a}}(\underline{\hat{x}})$ (see Remark 6.10 (6) and Definition 5.12).

Now for $(\underline{r}, \underline{\rho}, \underline{x}) \in \mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\underline{\hat{\rho}}) \times \mathcal{U}_\varepsilon^{\mathrm{slc}}(\underline{\hat{x}})$, we define a splicing $\pi_{(\underline{r},\underline{\rho},\underline{x})} : F \to F$ by

$$\pi_{(\underline{r},\underline{\rho},\underline{x})}(\underline{\xi}) = \underline{\zeta}, \tag{6.10}$$

where for each nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, we define $(\zeta_{\hat{\mathrm{e}}}^{h^+}, \zeta_{\hat{\mathrm{e}}}^{h^-}) := \pi_{r_{\hat{\mathrm{e}}}}(\xi_{\hat{\mathrm{e}}}^{h^+}, \xi_{\hat{\mathrm{e}}}^{h^-})$ with $\pi_{r_{\hat{\mathrm{e}}}}$ being the splicing in Theorem 6.9, and define $\underline{\zeta}$ to be the same as $\underline{\xi}$ outside of strip neighborhoods $N(\hat{x}_{\hat{\mathrm{e}}}^\pm)$ of nodal edges.

We define $\mathcal{K}_\varepsilon^{\mathrm{slc}}(\underline{0}, \hat{\tau})$ the **ε-splicing core slice** to be

$$\mathcal{K}_\varepsilon^{\mathrm{slc}}(\underline{0}, \hat{\tau}) := \left\{ (\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}) \in \mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\underline{\hat{\rho}}) \times \mathcal{U}_\varepsilon^{\mathrm{slc}}(\underline{\hat{x}}) \times F \;\middle|\; \begin{array}{c} \pi_{(\underline{r},\underline{\rho},\underline{x})}(\underline{\xi}) = \underline{\xi}, \\ \|\underline{\xi}\| < \varepsilon \end{array} \right\}. \tag{6.11}$$

This splicing core slice shall be the domain of the M-polyfold charts. Note that even thought the set $\mathcal{U}_\varepsilon(\underline{0}) \times \mathcal{U}_\varepsilon(\underline{\hat{\rho}}) \times \mathcal{U}_\varepsilon^{\mathrm{slc}}(\underline{\hat{x}})$ is not a open subset of a partial quadrant in a sc-Banach space, it can be parametrized by an open subset of a partial quadrant in a Euclidean space. More precisely, the neighborhood $\mathcal{U}_\varepsilon(\underline{\hat{\rho}})$ can be parametrized by the Morse gluing (2.4) and $\mathcal{U}_\varepsilon^{\mathrm{slc}}(\underline{\hat{x}})$ can be parametrized as in Remark 5.16.

We now define the chart of the quotient space of disk trees $\mathfrak{X}$ by the following composition with the gluing map $[\#]$ in (4.1).

$$\begin{aligned} \Theta : \mathcal{K}_\varepsilon^{\mathrm{slc}}(\underline{0}, \hat{\tau}) &\to \mathfrak{X} \\ (\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}) &\mapsto [\#](\underline{r}, \tau(\underline{\rho}, \underline{x}, \underline{\xi})), \end{aligned} \tag{6.12}$$

and we define the tuple $\tau(\underline{\rho}, \underline{x}, \underline{\xi}) := (\hat{\mathrm{T}}, \underline{\gamma}(\underline{\rho}, \underline{\xi}), \underline{x}, \underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))$ as follows.

(1) The tuple of Morse trajectories $\underline{\gamma}(\underline{\rho}, \underline{\xi})$.

For each non-nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}$, we define $\underline{\gamma}(\underline{\rho}, \underline{\xi})_{\hat{\mathrm{e}}} := \underline{\rho}_{\hat{\mathrm{e}}}$.

For each nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, we define $\underline{\gamma}(\underline{\rho}, \underline{\xi})_{\hat{\mathrm{e}}}$ to be the *constant* Morse trajectory with value $\varphi_{\hat{\mathrm{e}}}^{-1}(c(\xi_{\hat{\mathrm{e}}}^{h^+}, \xi_{\hat{\mathrm{e}}}^{h^-}))$, where $c(\xi_{\hat{\mathrm{e}}}^{h^+}, \xi_{\hat{\mathrm{e}}}^{h^-})$ is the common limit of $\xi_{\hat{\mathrm{e}}}^{h^+}$, $\xi_{\hat{\mathrm{e}}}^{h^-}$ as in Definition 6.7.

(2) The tuple of boundary marked points and disk maps $(\underline{x}, \underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))$.

Let the boundary marked points $\underline{x}$ be as given.

In order to define disk maps $\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi})$, we first define cut-off functions $\alpha^{\pm}$

$$\alpha^-(s) := \beta(s+1), \quad \alpha^+(s) := \alpha^-(-s), \tag{6.13}$$

where β is chosen before Remark 4.2. Note that we have $\alpha^-(s) = 0$ for $s \geq 0$ and $\alpha^-(s) = 1$ for $s \leq -2$. Similarly we have $\alpha^+(s) = 0$ for $s \leq 0$ and $\alpha^+(s) = 1$ for $s \geq 2$. We shall use these cut-off functions to construct the maps $\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi})$ by changing the limits of $\underline{\hat{u}}$ on negative/positive infinite strips in order for them to coincide with the ends of Morse trajectories $\underline{\gamma}(\underline{\rho}, \underline{\xi})$.

For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, we define $u(\underline{\rho}, \underline{x}, \underline{\xi})_{\hat{\mathrm{v}}}$ outside of strip neighborhoods by exponentiating along $\xi_{\hat{\mathrm{v}}}$ and pre-composing with the marked points varying diffeomorphism $\nu_{\underline{x}_{\hat{\mathrm{v}}}}$ chosen in Remark 6.10 (4)

$$u(\underline{\rho}, \underline{x}, \underline{\xi})_{\hat{\mathrm{v}}}\big|_{D_{\hat{\mathrm{v}}} \setminus \bigsqcup_{\hat{\mathrm{e}}} N(x_{\hat{\mathrm{v}}, \hat{\mathrm{e}}})} := \exp_{\hat{u}_{\hat{\mathrm{v}}}} \circ\, \xi_{\hat{\mathrm{v}}} \circ \nu_{\underline{x}_{\hat{\mathrm{v}}}}^{-1}\big|_{D_{\mathrm{v}} \setminus \bigsqcup_{\hat{\mathrm{e}}} N(x_{\hat{\mathrm{v}}, \hat{\mathrm{e}}})}. \tag{6.14}$$

We now define the disk maps $\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi})$ on each strip neighborhood $N(x_{\mathrm{e}}^{\pm})$. Firstly denote the local expression of $\hat{u}_{\hat{\mathrm{e}}}^{\pm}$ in strip coordinates $h_{\mathrm{e}}^{\pm}(\hat{x}_{\mathrm{e}}^{\pm}, \cdot)$ by

$$\hat{u}_{\hat{\mathrm{e}}}^{h^{\pm}} := \varphi_{\hat{\mathrm{e}}}^{\pm} \circ \hat{u}_{\hat{\mathrm{e}}}^{\pm} \circ h_{\hat{\mathrm{e}}}^{\pm}(\hat{x}_{\hat{\mathrm{e}}}^{\pm}, \cdot). \tag{6.15}$$

We specify the local expression of $u(\underline{\rho}, \underline{x}, \underline{\xi})_{\hat{\mathrm{e}}}^{\pm}$ in strip coordinates $h_{\hat{\mathrm{e}}}^{\pm}(x_{\hat{\mathrm{e}}}^{\pm}, \cdot)$

$$u_{\hat{\mathrm{e}}}^{h^{\pm}} := \varphi_{\hat{\mathrm{e}}}^{\pm} \circ u(\underline{\rho}, \underline{x}, \underline{\xi})_{\hat{\mathrm{e}}}^{\pm} \circ h_{\hat{\mathrm{e}}}^{\pm}(x_{\hat{\mathrm{e}}}^{\pm}, \cdot) \tag{6.16}$$

as follows.

For each non-nodal edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}$, if $\hat{\mathrm{v}}^{\pm}$ is a main vertex, then define the local expression $u_{\hat{\mathrm{e}}}^{h^{\pm}}$ by adding the local expression $\xi_{\hat{\mathrm{e}}}^{h^{\pm}}$ in equation (6.9) and then change the limit

$$u_{\hat{\mathrm{e}}}^{h^{\pm}} := \hat{u}_{\hat{\mathrm{e}}}^{h^{\pm}} + \xi_{\hat{\mathrm{e}}}^{h^{\pm}} + \alpha^{\pm} \cdot \varphi_{\hat{\mathrm{e}}}^{\pm}(\mathrm{ev}^{\pm}(\underline{\rho}_{\hat{\mathrm{e}}})), \tag{6.17}$$

where $\mathrm{ev}^{\pm}$ is the evaluation map on the Morse trajectory space (Definition 2.3).

For each nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, we define the local expression $u_{\hat{\mathrm{e}}}^{h^{\pm}}$ by simply adding the local expression $\xi_{\hat{\mathrm{e}}}^{h^{\pm}}$

$$u_{\hat{\mathrm{e}}}^{h^{\pm}} := \hat{u}_{\hat{\mathrm{e}}}^{h^{\pm}} + \xi_{\hat{\mathrm{e}}}^{h^{\pm}}. \tag{6.18}$$

Note that in the above two local expressions, adding the section is the same as exponentiating along the section due to the linearity in equation (6.7). Hence the above construction on strip neighborhoods is compatible with the construction outside of strip neighborhoods (6.14). This shows that the marked points and disk map $(\underline{x}_{\hat{\mathrm{v}}}, u(\underline{\rho}, \underline{x}, \underline{\xi})_{\hat{\mathrm{v}}})$ lies in the H^{3,δ_0} space $\mathcal{W}$ (Definition 3.15).

We now check the coincidence condition (2.9) for the tuple $\tau(\underline{\rho}, \underline{x}, \underline{\xi}) = (\hat{\mathrm{T}}, \underline{\gamma}(\underline{\rho}, \underline{\xi}), \underline{x}, \underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))$. It suffices to verify the condition for each non-nodal edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}$, when $\hat{\mathrm{v}}^{\pm}$ is a main vertex. We have

$$\begin{aligned}\lim_{z \to \pm\infty} u_{\hat{\mathrm{e}}}^{h^{\pm}}(z) &= \lim_{z \to \pm\infty} \hat{u}_{\hat{\mathrm{e}}}^{h^{\pm}}(z) + \xi_{\hat{\mathrm{e}}}^{h^{\pm}}(z) + \alpha^{\pm}(z) \cdot \varphi_{\hat{\mathrm{e}}}^{\pm}(\mathrm{ev}^{\pm}(\underline{\rho}_{\hat{\mathrm{e}}})) \\ &= \varphi_{\hat{\mathrm{e}}}^{\pm}(\mathrm{ev}^{\pm}(\underline{\rho}_{\hat{\mathrm{e}}})).\end{aligned}$$

Furthermore, the tuple $\tau(\underline{\rho}, \underline{x}, \underline{\xi})$ lies in the neighborhood $\mathcal{U}_{\varepsilon}(\hat{\tau})$ (see Definition 4.1) since $(\underline{x}, \underline{u}(\underline{\rho}, \underline{\xi}, \underline{x}))$ lies in the neighborhood of $(\underline{\hat{x}}, \underline{\hat{u}})$ in (3.7). Hence we can apply the gluing map $[\#]$ on the tuple using the choices in Remark 6.10. This finishes the construction of the chart Θ in (6.12).

The following result shows that the collection of Θ's forms an atlas with the exponential gluing profile. We shall prove this result in Section 7.

Theorem 6.12. *Let the gluing profile R be given by the exponential gluing profile $\varphi(r) = e^{\frac{1}{r}} - e$. Then the collection of charts $\Theta : \mathcal{K}_{\varepsilon}^{\mathrm{slc}}(\underline{0}, \hat{\tau}) \to \mathfrak{X}$ forms an M-polyfold with boundary and corners atlas.*

The next result is a consequence of the Hausdorff topology (Theorem 4.6) and Theorem 6.12, and other secondary topological properties.

Theorem 6.13. *The quotient space of disk trees $\mathfrak{X}$ is an M-polyfold with boundary and corners.*

In order to have a metrizable space as specified in the definition of M-polyfolds (Definition 6.4), we also need the topology to be second countable and paracompact (Theorem 2.55 of [9]), and those topological properties can be proved similarly as Theorem 3.20 of [8].

Remark 6.14. The local expression of the glued map $\#_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))$ on the glued strip of an gluing edge $\mathrm{e} \in \hat{\mathrm{E}}_{\underline{r}}^{\mathrm{g}}$ is given by

$$\oplus_{r_{\mathrm{e}}}(u_{\hat{\mathrm{e}}}^{h^+}, u_{\hat{\mathrm{e}}}^{h^-}) = \oplus_{r_{\mathrm{e}}}(\hat{u}_{\hat{\mathrm{e}}}^{h^+} + \xi_{\hat{\mathrm{e}}}^{h^+}, \hat{u}_{\hat{\mathrm{e}}}^{h^-} + \xi_{\hat{\mathrm{e}}}^{h^-}) = \oplus_{r_{\mathrm{e}}}(\hat{u}_{\hat{\mathrm{e}}}^{h^+}, \hat{u}_{\hat{\mathrm{e}}}^{h^-}) + \oplus_{r_{\mathrm{e}}}(\xi_{\hat{\mathrm{e}}}^{h^+}, \xi_{\hat{\mathrm{e}}}^{h^-}). \tag{6.19}$$

The first equality follows from (6.18), and the second equality from the linearity of the plus gluing.

If two tuples $(\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}), (\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}') \in \mathcal{K}_{\varepsilon}^{\mathrm{slc}}(\underline{0}, \hat{\tau})$ give rise to the same glued maps

$$\#_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi})) = \#_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}')),$$

then for each gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}_{\underline{r}}^{\mathrm{g}}$, (6.19) shows that we have $\oplus_{r_{\mathrm{e}}}(\xi_{\hat{\mathrm{e}}}^{h^+}, \xi_{\hat{\mathrm{e}}}^{h^-}) = \oplus_{r_{\mathrm{e}}}(\xi_{\hat{\mathrm{e}}}'^{h^+}, \xi_{\hat{\mathrm{e}}}'^{h^-})$. It follows from the construction of the splicing core (Lemma 6.8) that we have $(\xi_{\hat{\mathrm{e}}}^{h^+}, \xi_{\hat{\mathrm{e}}}^{h^-}) = (\xi_{\hat{\mathrm{e}}}'^{h^+}, \xi_{\hat{\mathrm{e}}}'^{h^-})$. Also, one can follow through the construction and show $(\xi_{\hat{\mathrm{e}}}^{h^+}, \xi_{\hat{\mathrm{e}}}^{h^-}) = (\xi_{\hat{\mathrm{e}}}'^{h^+}, \xi_{\hat{\mathrm{e}}}'^{h^-})$ for other edges $\hat{\mathrm{e}}$, and ultimately $\underline{\xi} = \underline{\xi}'$. This gives the injectivity on the level of sections.

Remark 6.15. The above construction is valid even when the disk maps $\underline{\hat{u}}$ do not have regularity H^{3+m,δ_m} for all m. In that case, the space of sections E in (6.8) is a Banach space instead of an sc-Banach space (see Remark 3.17). One can still define the map $\Theta : \mathcal{K}^{\text{slc}}_{\varepsilon}(\underline{0}, \hat{\tau}) \to \mathfrak{X}$, except the set $\mathcal{K}^{\text{slc}}_{\varepsilon}(\underline{0}, \hat{\tau})$ is not an sc-smooth splicing core and Θ is not an M-polyfold chart.

Remark 6.16. The splicing (6.10) is a local M-polyfold model (Definition 6.3) since the boundary and corners structure arises from the Morse trajectory space (Section 2.2) but the splicing is independent of Morse trajectories $\underline{\rho}$.

7. Proof for the Topology and the Atlas of the Deligne-Mumford Space

7.1. Closedness of the Biholomorphic Equivalence in $\mathfrak{DM}$.

In this section we prove the Hausdorff property of the topology on $\mathfrak{DM}$ in Theorem 5.11. More precisely, we show that for any distinct pair $\sigma, \sigma' \in \mathfrak{DM}$, there exist neighborhoods $\mathfrak{U}_{\varepsilon}(\sigma; \hat{\mu})$ and $\mathfrak{U}_{\varepsilon'}(\sigma'; \hat{\mu}')$ defined in (5.10) with empty intersection. We shall prove it by using the following result regarding the closedness of the biholomorphic equivalence.

Proposition 7.1. *Given $(\underline{\bar{r}}, \bar{\mu}) \in \mathcal{U}_{\varepsilon}(\underline{0}) \times \mathcal{U}_{\varepsilon}(\hat{\mu})$ and $(\underline{\bar{r}}', \bar{\mu}') \in \mathcal{U}_{\varepsilon'}(\underline{0}) \times \mathcal{U}_{\varepsilon'}(\hat{\mu}')$, assume there are sequences $(\underline{r}_n, \mu_n) \to (\underline{\bar{r}}, \bar{\mu})$, $(\underline{r}'_n, \mu'_n) \to (\underline{\bar{r}}', \bar{\mu}')$, and a sequence of biholomorphisms $(\zeta_n, \underline{\psi}_n)$ with*

$$(\zeta_n, \underline{\psi}_n)(\#(\underline{r}_n, \mu_n)) = \#(\underline{r}'_n, \mu'_n).$$

Then there exists a biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ with

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\mu})) = \#(\underline{\bar{r}}', \bar{\mu}').$$

Proposition 7.1 is a weaker version of Theorem 7.11. We now show the Hausdorff property.

Proof of Hausdorff property. Suppose this topology is not Hausdorff. Then there are distinct $\sigma, \sigma' \in \mathfrak{DM}$ such that for all $\varepsilon, \varepsilon' > 0$, their neighborhoods $\mathfrak{U}_{\varepsilon}(\sigma; \hat{\mu})$ and $\mathfrak{U}_{\varepsilon'}(\sigma'; \hat{\mu}')$ have non-empty intersection. In other words, there exist sequences $(\underline{r}_n, \mu_n) \to (\underline{0}, \hat{\mu})$ and $(\underline{r}'_n, \mu'_n) \to (\underline{0}, \hat{\mu}')$, and a sequence of biholomorphisms $(\zeta_n, \underline{\psi}_n)$ such that $(\zeta_n, \underline{\psi}_n)(\#(\underline{r}_n, \mu_n)) = \#(\underline{r}'_n, \mu'_n)$. Applying Proposition 7.1, we get a biholomorphism $(\hat{\zeta}, \underline{\hat{\psi}})$ with $(\hat{\zeta}, \underline{\hat{\psi}})(\hat{\mu}) = \hat{\mu}'$. This contradicts with the assumption that $\sigma \neq \sigma'$. □

Now it suffices to prove Proposition 7.1. We shall construct the desired biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ by using "standard models" on the domain and the target glued disks.

7.1.1. *Standard Models for the Glued Disk.*

Fix a representative $\hat{\mu} = (\hat{\mathrm{T}}, \underline{\hat{\ell}}, \underline{\hat{x}}, \underline{\hat{O}})$ of an element σ of $\mathfrak{DM}$, and let $(\underline{r}, \mu) \in \mathcal{U}_{\varepsilon}(\underline{0}) \times \mathcal{U}_{\varepsilon}(\hat{\mu})$ be arbitrarily given. We recall it has the form $\mu = (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{O})$, whose underlying tree is the same as that of $\hat{\mu}$. For a main vertex $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$, recall that the glued disk $\#_{\underline{r}, \underline{x}}(\underline{D})_{\mathrm{v}}$ is constructed by

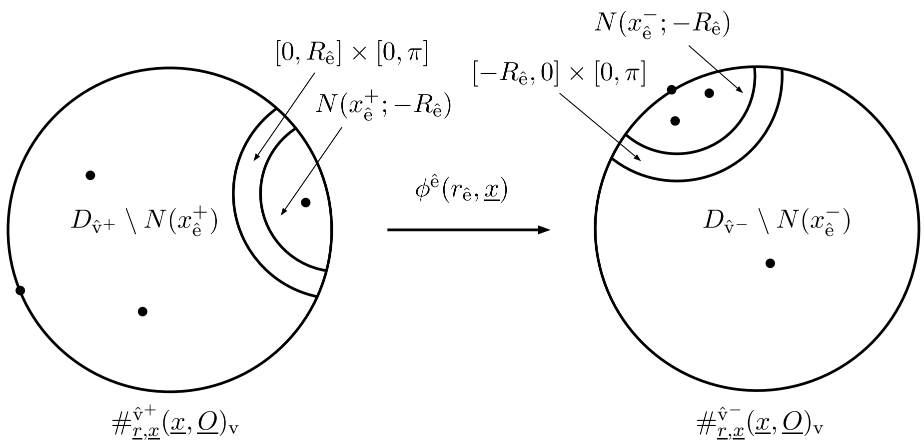


FIGURE 1. Shift map $\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})$ applied to distinguished regions (Lemma 7.3 (2) and (3)), and the standard models of marked points on $D_{\hat{v}^+}$ and $D_{\hat{v}^-}$.

taking parts of disks $D_{\hat{v}}$ for all gluing vertices $\hat{v} \in \hat{V}^{g}_{v}$ and then identifying the strips. The glued disk is a quotient space that is biholomorphic to every standard disk D_v; in other words, each disk $D_{\hat{v}}$ provides a "standard model" for the glued disk. Instead of having a varying domain $\#_{\underline{r},\underline{x}}(\underline{D})_v$ as $(\underline{r}, \underline{x})$ changes, we can use such a standard model to fix the domain and study its marked points and maps.

In this section, we shall abbreviate the boundary and interior marked points together as

$$\#_{\underline{r},\underline{x}}(\underline{x}, \underline{O})_v := (\#_{\underline{r}}(\underline{x})_v, \#_{\underline{r},\underline{x}}(\underline{O})_v) \tag{7.1}$$

Definition 7.2. Given a gluing edge $\hat{e} = (\hat{v}^-, \hat{v}^+) \in \hat{E}^{g}_{\underline{r}}$ with gluing parameter $r_{\hat{e}} > 0$, recall that $L_{R_{\hat{e}}} : [0, R_{\hat{e}}] \times [0, \pi] \to [-R_{\hat{e}}, 0] \times [0, \pi]$ is the left translation on the truncated strips. This left translation extends to a **shift map** from $D_{\hat{v}^+}$ to $D_{\hat{v}^-}$ as follows,

$$\phi^{\hat{e}}(r_{\hat{e}}, \underline{x}) = h^-_{\hat{e}}(x^-_{\hat{e}}, \cdot) \circ L_{R_{\hat{e}}} \circ h^+_{\hat{e}}(x^+_{\hat{e}}, \cdot)^{-1} : D_{\hat{v}^+} \to D_{\hat{v}^-},$$

where $h^-_{\hat{e}}$ and $h^+_{\hat{e}}$ are strip coordinates (see Section 3.2).

As we shall see in Lemma 7.3, the shift map $\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})$ is a disk automorphism (see Section 13.1 for disk automorphisms $\mathrm{Aut}(D)$). We now study the behavior of the shift map and its relationship with shrunk strip neighborhoods (3.3).

Lemma 7.3. *The shift map $\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})$ is a disk automorphism which satisfies the following properties.*

(1) *Suppose the gluing profile R is the logarithm gluing profile given by $\rho(r) = -\ln(r)$. Then the shift map is of the form*

$$\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})(z) = f^{-}_{x^{-}_{\hat{e}}}\left(r_{\hat{e}} \cdot \frac{-1}{\left(f^{+}_{x^{+}_{\hat{e}}}\right)^{-1}(z)} \right),$$

where $f^{-}_{x^{-}_{\hat{e}}}$ and $f^{+}_{x^{+}_{\hat{e}}}$ are smooth families of Möbius transformations in Definition 3.8. For general gluing profile R, the map $(r_{\hat{e}}, \underline{x}) \mapsto \phi^{\hat{e}}(r_{\hat{e}}, \underline{x})$ into $\mathrm{Aut}(D)$ is continuous/smooth if R is continuous/smooth.

(2) $\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})\left(N(x^{+}_{\hat{e}}; -R_{\hat{e}})\right) = \overline{D_{\hat{v}^{-}} \setminus N(x^{-}_{\hat{e}})}$.

(3) $\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})\left(\overline{D_{\hat{v}^{+}} \setminus N(x^{+}_{\hat{e}})}\right) = N(x^{-}_{\hat{e}}; -R_{\hat{e}})$.

Property (2) and (3) are illustrated in Figure 1.

Proof. For the logarithm gluing profile ρ, the formula of the shift map follows from a straightforward computation using the relation $r_{\hat{e}} = e^{-R_{\hat{e}}}$. This formula along with properties of $f^{-}_{x^{-}_{\hat{e}}}$ and $f^{+}_{x^{+}_{\hat{e}}}$ shows that $\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})$ is a disk automorphism, and the map $(r_{\hat{e}}, \underline{x}) \mapsto \phi^{\hat{e}}(r_{\hat{e}}, \underline{x})$ is smooth. For a general gluing profile R, the transition map $\rho^{-1} \circ R$ is continuous/smooth if R is continuous/smooth. This shows (1). Furthermore, (2) and (3) follow from the fact that $\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})$ in strip coordinates is simply a translation from $[0, R_{\hat{e}}] \times [0, \pi]$ to $[-R_{\hat{e}}, 0] \times [0, \pi]$. $\square$

We now examine the following biholomorphism from the standard disk $D_{\hat{v}}$ to the glued disk $\#_{\underline{x},\underline{r}}(\underline{D})_{\mathrm{v}}$ which is the inclusion map outside of strip neighborhoods of $D_{\hat{v}}$. This biholomorphism provides a standard way of identifying each $D_{\hat{v}}$ with the glued disk.

Lemma 7.4. *Let $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$ be a main glued vertex.*

(1) *For each vertex $\hat{v} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$, there is a unique biholomorphic map*

$$\Phi^{\hat{v}}_{\mathrm{v}} : D_{\hat{v}} \to \#_{\underline{x},\underline{r}}(\underline{D})_{\mathrm{v}}$$

whose restriction to the subset $D_{\hat{v}} \setminus \bigsqcup_{\hat{e} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}} N(x_{\hat{v},\hat{e}})$ is the inclusion map (see (4.11) for the construction of the glued disk).

(2) *For each gluing edge $\hat{e} = (\hat{v}^{-}, \hat{v}^{+}) \in \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}$, we have equality of maps $D_{\hat{v}^{+}} \to D_{\hat{v}^{-}}$*

$$(\Phi^{\hat{v}^{-}}_{\mathrm{v}})^{-1} \circ \Phi^{\hat{v}^{+}}_{\mathrm{v}} = \phi^{\hat{e}}(r_{\hat{e}}, \underline{x}).$$

Proof. The uniqueness of such $\Phi^{\hat{v}}_{\mathrm{v}}$ follows from the uniqueness of analytic continuation.

We first construct the biholomorphism while assuming that the set of gluing vertices is given by $\hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}} = \{\hat{v}^{-}, \hat{v}^{+}\}$ and the set of gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}} = \{(\hat{v}^{-}, \hat{v}^{+})\}$. We define $\Phi^{\hat{v}^{+}}_{\mathrm{v}}(z)$ to be

- z for $z \in D_{\hat{v}^{+}} \setminus N(x^{+}_{\hat{e}})$,

- $[z]$ for $z \in [0, R_{\hat{e}}] \times [0, \pi]$, where $[z]$ is the equivalence class of z under the identification $z \sim L_{R_{\hat{e}}}(z)$, and
- $\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})(z)$ for $z \in N(x^{+}_{\hat{e}}; -R_{\hat{e}})$.

This is a well-defined biholomorphic map. We define $\Phi_{\mathrm{v}}^{\hat{\mathrm{v}}^-}$ similarly and we have $(\Phi_{\mathrm{v}}^{\hat{\mathrm{v}}^-})^{-1} \circ \Phi_{\mathrm{v}}^{\hat{\mathrm{v}}^+} = \phi^{\hat{e}}(r_{\hat{e}}, \underline{x})$.

In general, we can construct the biholomorphism $\Phi_{\mathrm{v}}^{\hat{\mathrm{v}}}$ similarly by composing shift map $\phi^{\hat{e}}(r_{\hat{e}}, \underline{x})$ or its inverse for each gluing edge $\hat{e} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}$. This proves the lemma. □

Definition 7.5. Let $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$ be a main glued vertex. For each gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$, we call the biholomorphism $\Phi_{\mathrm{v}}^{\hat{\mathrm{v}}} : D_{\hat{\mathrm{v}}} \to \#_{\underline{r},\underline{x}}(\underline{D})_{\mathrm{v}}$ constructed in Lemma 7.4 the **standard pullback** to $D_{\hat{\mathrm{v}}}$. Moreover we define

$$(\#^{\hat{\mathrm{v}}}_{\underline{r}}(\underline{x})_{\mathrm{v}}, \#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{O})_{\mathrm{v}}) = \#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{x}, \underline{O})_{\mathrm{v}} := (\Phi_{\mathrm{v}}^{\hat{\mathrm{v}}})^{-1}\left(\#_{\underline{r},\underline{x}}(\underline{x}, \underline{O})_{\mathrm{v}}\right)$$

to be the **standard model of the glued marked points** on $D_{\hat{\mathrm{v}}}$.

Figure 1 illustrates the standard models of marked points on $D_{\hat{\mathrm{v}}^+}$ and $D_{\hat{\mathrm{v}}^-}$ in the case $\hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}} = \{\hat{\mathrm{v}}^+, \hat{\mathrm{v}}^-\}$.

We now study the behavior of the standard model of marked points, and their relation with gluing edges.

Lemma 7.6. *For $(\underline{r}, \mu) \in \mathcal{U}_{\varepsilon}(\underline{0}) \times \mathcal{U}_{\varepsilon}(\hat{\mu})$, fix a main vertex $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$.*

(1) *For each gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$ and each gluing edge $\hat{e} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}$ of $\hat{\mathrm{v}}$, the standard model of marked points in the neighborhood $B(\hat{x}_{\hat{\mathrm{v}},\hat{e}})$ (see Remark 5.9) is contained in the shrunk strip neighborhood*

$$\#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{x}, \underline{O})_{\mathrm{v}} \cap B(\hat{x}_{\hat{\mathrm{v}},\hat{e}}) \subset N(x_{\hat{\mathrm{v}},\hat{e}}; -R_{\hat{e}}).$$

(2) *For each gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$ and each edge $\hat{e}$ of $\hat{\mathrm{v}}$, we have $\hat{e} \notin \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}$ iff the numbers of marked points satisfy*

$$n\left(\#^{\hat{\mathrm{v}}}_{\underline{r}}(\underline{x})_{\mathrm{v}} \cap B(\hat{x}_{\hat{\mathrm{v}},\hat{e}})\right) = 1 \quad \textit{and} \quad n\left(\#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{O})_{\mathrm{v}} \cap B(\hat{x}_{\hat{\mathrm{v}},\hat{e}})\right) = 0.$$

Proof. The first statement follows from the definition of $\#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{x}, \underline{O})_{\mathrm{v}}$ and Lemma 7.3. The second statement follows from the stability condition (5.1). □

7.1.2. *Convergence of Biholomorphisms.*

We now prove Proposition 7.1, the convergence result of biholomorphisms. Throughout this section, we fix representatives $\hat{\mu} = (\hat{\mathrm{T}}, \hat{\underline{\ell}}, (\hat{\underline{x}}, \hat{\underline{O}}))$ and $\hat{\mu}' = (\hat{\mathrm{T}}', \hat{\underline{\ell}}', (\hat{\underline{x}}', \hat{\underline{O}}'))$ of elements σ and σ' of $\mathfrak{DM}$. For convenience, we abbreviate

$$\mathcal{U}_{\varepsilon}(\hat{\underline{r}}, \hat{\mu}) := \mathcal{U}_{\varepsilon}(\hat{\underline{r}}) \times \mathcal{U}_{\varepsilon}(\hat{\mu})$$

and let

$$(\underline{\bar{r}}, \bar{\mu}) \in \mathcal{U}_{\varepsilon}(\underline{0}, \hat{\mu}), \quad (\underline{\bar{r}}', \bar{\mu}') \in \mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\mu}')$$

be arbitrarily given.

We first study the relation between the glued trees $\#_{\underline{\bar{r}}}(\hat{\mathrm{T}})$ and $\#_{\underline{r}}(\hat{\mathrm{T}})$ for nearby gluing parameters $\underline{r} \in \mathcal{U}_{\delta}(\underline{\bar{r}})$.

Lemma 7.7. *There exists $\bar{\delta} = \bar{\delta}(\underline{\bar{r}}) > 0$ such that for nearby gluing parameters $\underline{r} \in \mathcal{U}_{\bar{\delta}}(\underline{\bar{r}})$,*

(1) *if a nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$ satisfies $\bar{r}_{\hat{\mathrm{e}}} > 0$, then $r_{\hat{\mathrm{e}}} > 0$; if $\bar{r}_{\hat{\mathrm{e}}} < 0$, then $r_{\hat{\mathrm{e}}} < 0$, and*

(2) *if vertices $\hat{\mathrm{v}}_0, \hat{\mathrm{v}} \in \hat{\mathrm{V}}$ satisfy $[\hat{\mathrm{v}}_0]_{\underline{\bar{r}}} = [\hat{\mathrm{v}}]_{\underline{\bar{r}}}$, then we have $[\hat{\mathrm{v}}_0]_{\underline{r}} = [\hat{\mathrm{v}}]_{\underline{r}}$.*

Proof. We choose $\bar{\delta}$ small enough so that $\bar{\delta} < |\bar{r}_{\hat{\mathrm{e}}}|$ for all edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$ with $\bar{r}_{\hat{\mathrm{e}}} \neq 0$. If $\bar{r}_{\hat{\mathrm{e}}} > 0$, then $r_{\hat{\mathrm{e}}} > \bar{r}_{\hat{\mathrm{e}}} - \delta > 0$. Similarly, if $\bar{r}_{\hat{\mathrm{e}}} < 0$, then $r_{\hat{\mathrm{e}}} < 0$. Hence statement (1) is true. Now if $\hat{\mathrm{v}}_0$ and $\hat{\mathrm{v}}$ satisfy $[\hat{\mathrm{v}}_0]_{\underline{\bar{r}}} = [\hat{\mathrm{v}}]_{\underline{\bar{r}}}$, then by definition they are connected by a sequence of nodal edges, each with $\bar{r}_{\hat{\mathrm{e}}} > 0$. By the choice of $\bar{\delta}$, those sequence of edges satisfy $r_{\hat{\mathrm{e}}} > 0$. This shows $[\hat{\mathrm{v}}_0]_{\underline{r}} = [\hat{\mathrm{v}}]_{\underline{r}}$ and the statement in (2) is true. $\square$

From here on, we always assume $\delta < \bar{\delta}(\underline{\bar{r}})$ and $\delta' < \bar{\delta}'(\underline{\bar{r}}')$.

Definition 7.8. For nearby gluing parameters $\underline{r} \in \mathcal{U}_{\delta}(\underline{\bar{r}})$, we define the **vertex quotient** map

$$\bar{p}_{\underline{r}} : \#_{\underline{\bar{r}}}(\hat{\mathrm{V}}) \to \#_{\underline{r}}(\hat{\mathrm{V}})$$

to be the map which further collapses the glued vertices $\#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$. More precisely, $\bar{p}_{\underline{r}}(\bar{\mathrm{v}}) := [\hat{\mathrm{v}}]_{\underline{r}}$ where $\hat{\mathrm{v}}$ is any gluing vertex in $\hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$. Moreover, define the set of **additional gluing edges** (with respect to gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$) by

$$\hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}} := \{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \,|\, \bar{r}_{\hat{\mathrm{e}}} = 0, r_{\hat{\mathrm{e}}} > 0\}.$$

Remark 7.9. Note that the map $\bar{p}_{\underline{r}}$ is well-defined due to Lemma 7.7 (2), and it is clearly surjective. By definition of the vertex quotient map, we have the following commutative diagram (similar to covering spaces).

$$\begin{tikzcd}
\hat{\mathrm{V}} \arrow[r, "{[\cdot]_{\underline{\bar{r}}}}"] \arrow[d, "{[\cdot]_{\underline{r}}}"'] & \#_{\underline{\bar{r}}}(\hat{\mathrm{V}}) \arrow[dl, "\bar{p}_{\underline{r}}"] \\
\#_{\underline{r}}(\hat{\mathrm{V}}) &
\end{tikzcd}$$

Lemma 7.7 (1) implies inclusion of gluing edges

$$\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}} \subset \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} \subset \{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \,|\, \bar{r}_{\hat{\mathrm{e}}} \geq 0\} \tag{7.2}$$

and decomposition of gluing edges of $\underline{r}$ into gluing edges of $\underline{\bar{r}}$ and additional gluing edges

$$\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} = \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}} \sqcup \hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}}. \tag{7.3}$$

In other words, we obtain $\#_{\underline{r}}(\hat{\mathrm{T}})$ from $\#_{\underline{\bar{r}}}(\hat{\mathrm{T}})$ by further gluing $\hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}}$. Indeed, each additional gluing edge $(\hat{\mathrm{v}}, \hat{\mathrm{w}}) \in \hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}}$ corresponds to two *adjacent* vertices $[\hat{\mathrm{v}}]_{\underline{\bar{r}}}$ and $[\hat{\mathrm{w}}]_{\underline{\bar{r}}}$ in the glued tree $\#_{\underline{\bar{r}}}(\hat{\mathrm{T}})$, which get further collapsed to the same vertex $[\hat{\mathrm{v}}]_{\underline{r}} = [\hat{\mathrm{w}}]_{\underline{r}}$. (See Figure 2 for an illustration of gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$ and additional gluing edges $\hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}_n}$.)

We now study an equivalent pair $\#(\underline{\bar{r}}, \bar{\mu})$ and $\#(\underline{\bar{r}}', \bar{\mu}')$ and their nearby points.

Definition 7.10. Assume there are nearby $(\underline{r}, \mu) \in \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\mu})$ and $(\underline{r}', \mu') \in \mathcal{U}_{\delta'}(\underline{\bar{r}}', \bar{\mu}')$, and biholomorphisms $(\bar{\zeta}, \underline{\bar{\psi}})$ and $(\zeta, \underline{\psi})$ with

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\mu})) = \#(\underline{\bar{r}}', \bar{\mu}'), \quad (\zeta, \underline{\psi})(\#(\underline{r}, \mu)) = \#(\underline{r}', \mu').$$

We say $\boldsymbol{\bar{\zeta}}$ **covers** $\boldsymbol{\zeta}$ if $\bar{p}'_{\underline{r}'} \circ \bar{\zeta} = \zeta \circ \bar{p}_{\underline{r}}$. In other words, the following diagram commutes.

$$\begin{tikzcd} \#_{\underline{\bar{r}}}(\hat{\mathrm{V}}) \arrow[r, "\bar{\zeta}"] \arrow[d, "\bar{p}_{\underline{r}}"'] & \#_{\underline{\bar{r}}'}(\hat{\mathrm{V}}') \arrow[d, "\bar{p}'_{\underline{r}'}"] \\ \#_{\underline{r}}(\hat{\mathrm{V}}) \arrow[r, "\zeta"] & \#_{\underline{r}'}(\hat{\mathrm{V}}') \end{tikzcd} \tag{7.4}$$

We now define a distance between biholomorphisms $\underline{\bar{\psi}}$ and $\underline{\psi}$. Suppose $\bar{\zeta}$ covers ζ. Given a main vertex $\bar{\mathrm{v}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$, let $\bar{\mathrm{v}}' := \bar{\zeta}(\bar{\mathrm{v}})$, $\mathrm{v} := \bar{p}_{\underline{r}}(\bar{\mathrm{v}})$, and $\mathrm{v}' := \zeta(\mathrm{v})$ be the target vertices in each corner of commutative diagram (7.4). Then for gluing vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ and $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$, we define a distance of $\bar{\psi}_{\bar{\mathrm{v}}}$ and ψ_{v} in the standard model from D_{v} to $D_{\mathrm{v}'}$

$$d^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{\mathrm{v}}) := d_{\mathrm{Aut}(D)}\Big(\big(\bar{\Phi}^{\hat{\mathrm{v}}'}_{\bar{\mathrm{v}}'}\big)^{-1} \circ \bar{\psi}_{\bar{\mathrm{v}}} \circ \bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}},\ \big(\Phi^{\hat{\mathrm{v}}'}_{\mathrm{v}'}\big)^{-1} \circ \psi_{\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{\mathrm{v}}\Big), \tag{7.5}$$

where $\bar{\Phi}$ and Φ are the standard pullbacks in Definition 7.5. More precisely, $\bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}$ and $\bar{\Phi}^{\hat{\mathrm{v}}'}_{\bar{\mathrm{v}}'}$ depend on $(\underline{\bar{r}}, \underline{\bar{x}})$ and $(\underline{\bar{r}}', \underline{\bar{x}}')$ respectively, and $\Phi^{\hat{\mathrm{v}}}_{\mathrm{v}}$ and $\Phi^{\hat{\mathrm{v}}'}_{\mathrm{v}'}$ depend on $(\underline{r}, \underline{x})$ and $(\underline{r}', \underline{x}')$ respectively.

With the above notions regarding biholomorphisms, we now state the convergence result which implies the Hausdorff property. ([10] deals with an analogous problem in surfaces without boundary and with arbitrary genus.)

Theorem 7.11. *Assume there are sequences* $(\underline{r}_n, \mu_n) \to (\underline{\bar{r}}, \bar{\mu})$, $(\underline{r}'_n, \mu'_n) \to (\underline{\bar{r}}', \bar{\mu}')$, *and a sequence of biholomorphisms* $(\zeta_n, \underline{\psi}_n)$ *with*

$$(\zeta_n, \underline{\psi}_n)(\#(\underline{r}_n, \mu_n)) = \#(\underline{r}'_n, \mu'_n).$$

Then there exists a biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ *with*

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\mu})) = \#(\underline{\bar{r}}', \bar{\mu}')$$

and a subsequence of $(\zeta_n, \underline{\psi}_n)$ *(also indexed by* n*) such that* $\bar{\zeta}$ *covers* ζ_n *in the sense that (7.4) commutes; moreover for every main vertex* $\bar{\mathrm{v}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$, *we denote* $\bar{\mathrm{v}}' := \bar{\zeta}(\bar{\mathrm{v}})$ *and* $\mathrm{v}_n := \bar{p}_{\underline{r}_n}(\bar{\mathrm{v}})$, *then for all gluing vertices* $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ *and* $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$, *we have* $d^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}_n}) \to 0$ *as defined in (7.5).*

We shall prove this result in Section 7.1.3. Clearly the above theorem implies Proposition 7.1, from which the Hausdorff property followed. Furthermore, the above convergence result also implies the following estimate on the biholomorphism between nearby glued pairs $\#(\underline{r}, \mu)$ and $\#(\underline{r}', \mu')$.

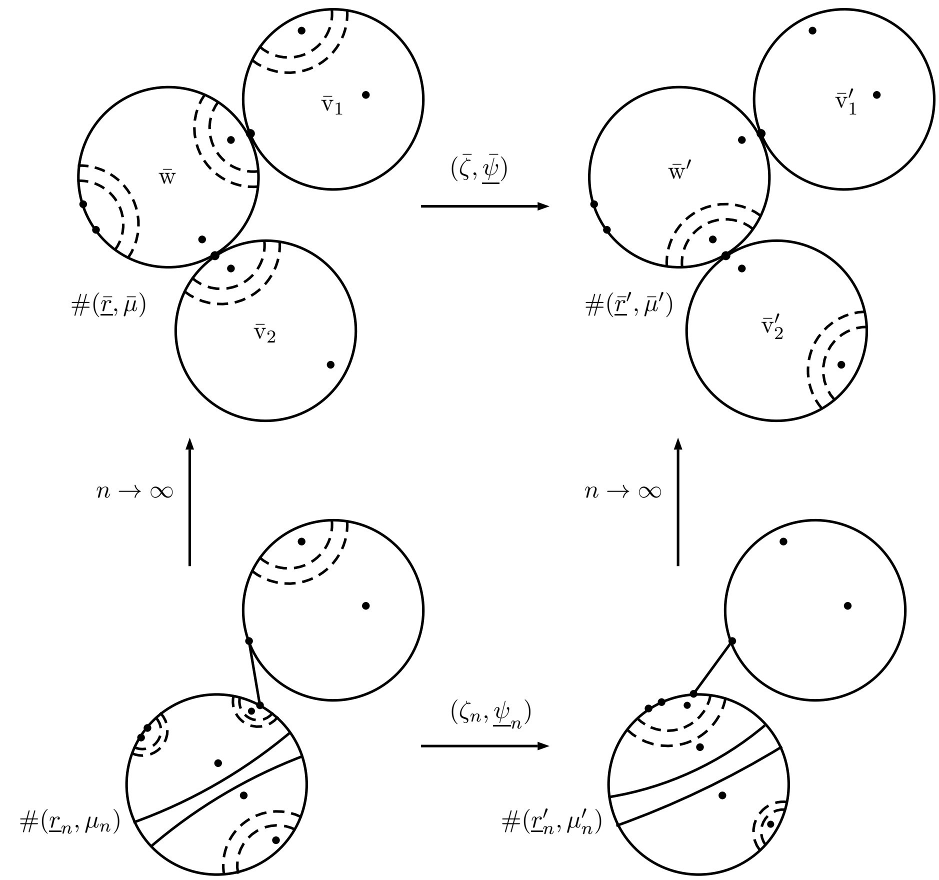


FIGURE 2. $(\zeta_n, \underline{\psi}_n)$ converges to $(\bar{\zeta}, \underline{\bar{\psi}})$ in standard models as in Theorem 7.11. Each *dashed strip* in $\#(\underline{\bar{r}}, \bar{\mu})$ corresponds to a gluing edge in $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$. Those dashed strips are carried over in $\#(\underline{r}_n, \mu_n)$, with each *solid strip* corresponding to an additional gluing edge in $\hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}_n}$. The additional gluing edge as depicted comes from gluing $\bar{\mathrm{v}}_2$ with $\bar{\mathrm{w}}$. Similar symbols are used in $\#(\underline{\bar{r}}', \bar{\mu}')$ and $\#(\underline{r}_n', \mu_n')$.

This estimate is useful in proving the injectivity of charts θ constructed in (5.13).

Theorem 7.12. *Assume there is a biholomorphism* $(\bar{\zeta}, \underline{\bar{\psi}})$ *with*

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\mu})) = \#(\underline{\bar{r}}', \bar{\mu}').$$

Then given $\lambda > 0$*, there are* $\delta, \delta' > 0$ *such that if there are* $(\underline{r}, \mu) \in \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\mu})$ *and* $(\underline{r}', \mu') \in \mathcal{U}_{\delta'}(\underline{\bar{r}}', \bar{\mu}')$ *and a biholomorphism* $(\zeta, \underline{\psi})$ *with*

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mu)) = \#(\underline{r}', \mu'),$$

then $\bar\zeta$ *covers* ζ *in the sense that (7.4) commutes; moreover for every main vertex* $\bar{\mathrm{v}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$, *we denote* $\bar{\mathrm{v}}' := \bar\zeta(\bar{\mathrm{v}})$ *and* $\mathrm{v} := \bar p_{\underline{r}}(\bar{\mathrm{v}})$, *then for all gluing vertices* $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ *and* $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$, *we have* $d^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar\psi_{\bar{\mathrm{v}}}, \psi_{\mathrm{v}}) < \lambda$ *as defined in (7.5).*

Proof. Assuming Theorem 7.11, we prove this theorem by contradiction. Suppose there exists $\lambda > 0$ such that for all small $\delta, \delta' > 0$, $(\zeta, \underline{\psi})$ is *not* λ-close to $(\bar\zeta, \underline{\bar\psi})$ in all standard models. Then there are sequences $(\underline{r}_n, \mu_n) \to (\underline{\bar r}, \bar\mu)$ and $(\underline{r}'_n, \mu'_n) \to (\underline{\bar r}', \bar\mu')$, and a sequence of biholomorphisms $(\zeta_n, \underline{\psi}_n)$ with $(\zeta_n, \underline{\psi}_n)(\#(\underline{r}_n, \mu_n)) = \#(\underline{r}'_n, \mu'_n)$ such that $(\zeta_n, \underline{\psi}_n)$ does *not* converge to $(\bar\zeta, \underline{\bar\psi})$ in all standard models. But Theorem 7.11 extracts a subsequence $(\zeta_n, \underline{\psi}_n)$ and *some* biholomorphism $(\tilde\zeta, \underline{\tilde\psi})$ with $(\tilde\zeta, \underline{\tilde\psi})(\#(\underline{\bar r}, \bar\mu)) = \#(\underline{\bar r}', \bar\mu')$ such that

- $\tilde\zeta$ covers ζ_n, and
- $(\zeta_n, \underline{\psi}_n)$ converges to $(\tilde\zeta, \underline{\tilde\psi})$ in all standard models in the sense that $d^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\tilde\psi_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}_n}) \to 0$ for all gluing vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ and $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$.

By assumption, we also have $(\bar\zeta, \underline{\bar\psi})(\#(\underline{\bar r}, \bar\mu)) = \#(\underline{\bar r}', \bar\mu')$. By the uniqueness of the biholomorphism (Proposition 5.6), we have $(\tilde\zeta, \underline{\tilde\psi}) = (\bar\zeta, \underline{\bar\psi})$. Thus we reached a contradiction, and this proves the desired properties. □

7.1.3. *Proof of Theorem 7.11.*

By picking a subsequence, we assume without loss of generality that gluing parameters $\underline{r}_n$ and $\underline{r}'_n$ give rise to *fixed* gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}_n} = \hat{\mathrm{E}}^{\mathrm{g}}$ and $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}'_n} = \hat{\mathrm{E}}'^{\mathrm{g}}$ for all n. Thus we get fixed trees $\#_{\hat{\mathrm{E}}^{\mathrm{g}}}(\hat{\mathrm{T}})$ and $\#_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\hat{\mathrm{T}}')$; moreover the sequence of tree isomorphisms ζ_n is identically equal to a fixed ζ due to the uniqueness of the ordered tree isomorphism (Theorem 2.1). Consequently, we have fixed vertex quotient maps (Definition 7.8) and we denote them by $\bar p_{\hat{\mathrm{E}}^{\mathrm{g}}} : \#_{\underline{\bar r}}(\hat{\mathrm{V}}) \to \#_{\hat{\mathrm{E}}^{\mathrm{g}}}(\hat{\mathrm{V}})$ and $\bar p'_{\hat{\mathrm{E}}'^{\mathrm{g}}} : \#_{\underline{\bar r}'}(\hat{\mathrm{V}}') \to \#_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\hat{\mathrm{V}}')$.

We now construct the biholomorphism $(\bar\zeta, \underline{\bar\psi})$ iteratively over all vertices of the main glued tree $\#_{\underline{\bar r}}(\hat{\mathrm{T}})^{\mathrm{m}}$: in the base case, we construct the biholomorphism at the root of the main glued tree $\#_{\underline{\bar r}}(\hat{\mathrm{T}})^{\mathrm{m}}$. Assuming there is a biholomorphism at a vertex $\bar{\mathrm{w}}$ in $\#_{\underline{\bar r}}(\hat{\mathrm{V}})^{\mathrm{m}}$, we then construct the biholomorphisms at all children of $\bar{\mathrm{w}}$ (see Section 2.1). This iteration covers all main vertices $\#_{\underline{\bar r}}(\hat{\mathrm{V}})^{\mathrm{m}}$.

Later on, we apply the following claim to the root of the main glued tree $\#_{\underline{\bar r}}(\hat{\mathrm{T}})^{\mathrm{m}}$ to prove the base case.

Claim 7.13. *Let* $\mathrm{v} \in \#_{\hat{\mathrm{E}}^{\mathrm{g}}}(\hat{\mathrm{V}})$ *be a main vertex and* $\mathrm{v}' := \zeta(\mathrm{v})$ *the corresponding vertex, and* v *and* v' *the roots of the gluing trees* $\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}}$ *and* $\hat{\mathrm{T}}'^{\mathrm{g}}_{\mathrm{v}'}$, *respectively. Denoting* $\bar{\mathrm{v}} := [\hat{\mathrm{v}}]_{\underline{\bar r}}$ *and* $\bar{\mathrm{v}}' := [\hat{\mathrm{v}}']_{\underline{\bar r}'}$, *then there exists a biholomorphism* $\bar\psi_{\bar{\mathrm{v}}} : \#_{\underline{\bar r}, \underline{\bar x}}(\underline{D})_{\bar{\mathrm{v}}} \to \#_{\underline{\bar r}', \underline{\bar x}'}(\underline{D})_{\bar{\mathrm{v}}'}$ *with*

$$\bar\psi_{\bar{\mathrm{v}}}\left(\#_{\underline{\bar r}, \underline{\bar x}}(\underline{\bar x}, \underline{\bar O})_{\bar{\mathrm{v}}}\right) = \#_{\underline{\bar r}', \underline{\bar x}'}(\underline{\bar x}', \underline{\bar O}')_{\bar{\mathrm{v}}'}.$$

Moreover, there is a subsequence (also indexed by n) such that for all gluing vertices $\hat{\mathrm{u}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ and $\hat{\mathrm{u}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$ we have the distance convergence

$$d^{\hat{\mathrm{u}},\hat{\mathrm{u}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}}) \to 0.$$

Proof. We denote the biholomorphism $\psi_{n,\mathrm{v}}$ in the standard model on $D_{\hat{\mathrm{v}}}$ and $D_{\hat{\mathrm{v}}'}$ by

$$\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} := \left(\Phi^{\hat{\mathrm{v}}'}_{n,\mathrm{v}'}\right)^{-1} \circ \psi_{n,\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}}.$$

Since the vertices v and v′ are fixed, we shall drop them in the subscripts. It follows from the assumption $(\zeta, \underline{\psi}_n)(\#(\underline{r}_n, \mu_n)) = \#(\underline{r}'_n, \mu'_n)$ that we have

$$\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\left(\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n, \underline{O}_n)\right) = \#^{\hat{\mathrm{v}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n, \underline{O}'_n). \tag{7.6}$$

We shall use Proposition 13.7 to extract a convergent subsequence, and we first make sure its hypothesis is satisfied. Since $\hat{\mathrm{v}}$ and $\hat{\mathrm{v}}'$ are the roots of the gluing trees $\hat{\mathrm{T}}^{\mathrm{g}}_{\hat{\mathrm{v}}}$ and $\hat{\mathrm{T}}'^{\mathrm{g}}_{\hat{\mathrm{v}}'}$, their respective outgoing edges $\hat{\mathrm{f}}$ and $\hat{\mathrm{f}}'$ are *not* glued, i.e., $\hat{\mathrm{f}} \notin \hat{\mathrm{E}}^{\mathrm{g}}$ and $\hat{\mathrm{f}}' \notin \hat{\mathrm{E}}'^{\mathrm{g}}$. Thus $x^-_{n,\hat{\mathrm{f}}}$ and $x'^-_{n,\hat{\mathrm{f}}'}$ are the outgoing boundary marked points in the glued marked points $\#_{\underline{r}_n,\underline{x}_n}(\underline{x}_n, \underline{O}_n)$ and $\#_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n, \underline{O}'_n)$, respectively. Hence $x^-_{n,\hat{\mathrm{f}}}$ corresponds to $x'^-_{n,\hat{\mathrm{f}}'}$ via $\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}$ in equation (7.6)

$$\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(x^-_{n,\hat{\mathrm{f}}}) = x'^-_{n,\hat{\mathrm{f}}'}. \tag{7.7}$$

We denote the rest of the glued marked points by

$$W_n := \#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n, \underline{O}_n) \setminus \{x^-_{n,\hat{\mathrm{f}}}\}, \quad W'_n := \#^{\hat{\mathrm{v}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n, \underline{O}'_n) \setminus \{x'^-_{n,\hat{\mathrm{f}}'}\}.$$

It follows from (7.6) and (7.7) that $\psi_n^{\mathrm{v},\mathrm{v}'}(W_n) = W'_n$. Moreover, Lemma 7.14 implies the convergence $x^-_{n,\hat{\mathrm{f}}} \to \bar{x}^-_{\hat{\mathrm{f}}}$ and $W_n \to \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}}, \underline{\bar{O}})_{\bar{\mathrm{v}}} \setminus \{\bar{x}^-_{\hat{\mathrm{f}}}\}$ in Hausdorff distance (see Definition 13.5), and we have similar convergence for the prime counterpart.

Now apply Proposition 13.7. Then there is a subsequence of $\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}$ that converges to a disk automorphism $\bar{\xi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}$ and it satisfies

$$\bar{\xi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\left(\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}}, \underline{\bar{O}})_{\bar{\mathrm{v}}}\right) = \#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}', \underline{\bar{O}}')_{\bar{\mathrm{v}}'}.$$

Define $\bar{\psi}_{\bar{\mathrm{v}}} := \bar{\Phi}^{\hat{\mathrm{v}}'}_{\bar{\mathrm{v}}'} \circ \bar{\xi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} \circ \left(\bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}\right)^{-1}$, by construction we have $\bar{\psi}_{\bar{\mathrm{v}}}\left(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}}, \underline{\bar{O}})_{\bar{\mathrm{v}}}\right) = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}', \underline{\bar{O}}')_{\bar{\mathrm{v}}'}$ and the distance convergence $d^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}}) \to 0$. We apply Lemma 7.15 to conclude $d^{\hat{\mathrm{u}},\hat{\mathrm{u}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}}) \to 0$ for all gluing vertices $\hat{\mathrm{u}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ and $\hat{\mathrm{u}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$. This proves the claim. □

We now establish the iterative process. As a visual guide to the following iterative step, we refer the readers to Figure 2, where we assume the biholomorphism $\bar{\psi}_{\bar{\mathrm{w}}}$ and the convergence to it, and we construct the biholomorphisms $\bar{\psi}_{\bar{\mathrm{v}}_1}$ and $\bar{\psi}_{\bar{\mathrm{v}}_2}$ and conclude the convergence to them.

Iteration hypothesis. *Let $\bar{\mathrm{w}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$ be a main vertex and we assigned a target vertex $\bar{\zeta}(\bar{\mathrm{w}}) = \bar{\mathrm{w}}'$, such that by denoting the vertices* $\mathrm{w} := \bar{p}_{\hat{\mathrm{E}}^{\mathrm{g}}}(\bar{\mathrm{w}})$ *and*

$\mathrm{w}' := \bar{p}'_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\bar{\mathrm{w}}')$*, we have* $\zeta(\mathrm{w}) = \mathrm{w}'$*. Moreover, there is a biholomorphism* $\bar{\psi}_{\bar{\mathrm{w}}} : \#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{D})_{\bar{\mathrm{w}}} \to \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{D})_{\bar{\mathrm{w}}'}$ *with*

$$\bar{\psi}_{\bar{\mathrm{w}}}\left(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{O}})_{\bar{\mathrm{w}}}\right) = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}',\underline{\bar{O}}')_{\bar{\mathrm{w}}'}. \tag{7.8}$$

Furthermore, for all gluing vertices $\hat{\mathrm{w}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{w}}}$ *and* $\hat{\mathrm{w}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{w}}'}$*, we have the distance convergence*

$$d^{\hat{\mathrm{w}},\hat{\mathrm{w}}'}(\bar{\psi}_{\bar{\mathrm{w}}}, \psi_{n,\mathrm{w}}) \to 0.$$

Iteration conclusion. *For each edge* $\bar{\mathrm{e}} = (\bar{\mathrm{v}}, \bar{\mathrm{w}})$*, let* $\bar{\mathrm{e}}' = (\bar{\mathrm{v}}', \bar{\mathrm{w}}')$ *be the edge given by the one-to-one correspondence of all boundary marked points* $\bar{\psi}_{\bar{\mathrm{w}}}(\#_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{w}},\bar{\mathrm{e}}}) = \#_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{w}}',\bar{\mathrm{e}}'}$ *as a part of equation (7.8). We denote the vertices* $\mathrm{v} := \bar{p}_{\hat{\mathrm{E}}^{\mathrm{g}}}(\bar{\mathrm{v}})$ *and* $\mathrm{v}' := \bar{p}'_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\bar{\mathrm{v}}')$*. For each pair* $\bar{\mathrm{v}}$ *and* $\bar{\mathrm{v}}'$ *as above, we have* $\zeta(\mathrm{v}) = \mathrm{v}'$*. Moreover, there exists a biholomorphism* $\bar{\psi}_{\bar{\mathrm{v}}} : \#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{D})_{\bar{\mathrm{v}}} \to \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{D})_{\bar{\mathrm{v}}'}$ *with*

$$\bar{\psi}_{\bar{\mathrm{v}}}\left(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{O}})_{\bar{\mathrm{v}}}\right) = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}',\underline{\bar{O}}')_{\bar{\mathrm{v}}'}.$$

Furthermore, there is a subsequence (also indexed by n) such that for all gluing vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ *and* $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$*, we have the distance convergence*

$$d^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}}) \to 0.$$

Lastly, we have equality in displaced edge lengths

$$\#_{\underline{\bar{r}}}(\underline{\bar{\ell}})_{\bar{\mathrm{e}}} = \#_{\underline{\bar{r}}'}(\underline{\bar{\ell}}')_{\bar{\mathrm{e}}'}.$$

Later on we shall show that the iteration conclusion can be derived from the iteration hypothesis. First assume this, we now construct the biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ iteratively over all main glued vertices $\#_{\underline{\bar{r}}}(\hat{\mathrm{V}})^{\mathrm{m}}$.

We first handle the base case. Let vertices $\bar{\mathrm{v}}$ and $\bar{\mathrm{v}}'$ be the roots of the main glued trees $\#_{\underline{\bar{r}}}(\hat{\mathrm{T}})^{\mathrm{m}}$ and $\#_{\underline{\bar{r}}'}(\hat{\mathrm{T}}')^{\mathrm{m}}$, respectively. Also, let v be the root of the main glued tree $\#_{\hat{\mathrm{E}}^{\mathrm{g}}}(\hat{\mathrm{T}})^{\mathrm{m}}$, and then the vertex $\mathrm{v}' := \zeta(\mathrm{v})$ is the root of the main glued tree $\#_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\hat{\mathrm{T}}')^{\mathrm{m}}$. Then $\mathrm{v}, \mathrm{v}', \bar{\mathrm{v}}'$, and $\bar{\mathrm{v}}'$ satisfy the hypothesis in Claim 7.13. Thus we get a biholomorphism $\bar{\psi}_{\bar{\mathrm{v}}} : \#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{D})_{\bar{\mathrm{v}}} \to \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{D})_{\bar{\mathrm{v}}'}$ with $\bar{\psi}_{\bar{\mathrm{v}}}\left(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{O}})_{\bar{\mathrm{v}}}\right) = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}',\underline{\bar{O}}')_{\bar{\mathrm{v}}'}$. We also get a subsequence with the distance convergence $d^{\hat{\mathrm{u}},\hat{\mathrm{u}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}}) \to 0$ for all gluing vertices $\hat{\mathrm{u}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ and $\hat{\mathrm{u}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$. We then define $\bar{\zeta}(\bar{\mathrm{v}}) := \bar{\mathrm{v}}'$, and it follows from $\zeta(\mathrm{v}) = \mathrm{v}'$ that $\bar{\zeta}$ covers ζ at $\bar{\mathrm{v}}$. This finishes the construction for the base case; the vertices $\bar{\mathrm{v}}$ and $\bar{\mathrm{v}}'$ satisfy the iteration hypothesis.

Now suppose the vertices $\bar{\mathrm{w}}$ and $\bar{\mathrm{w}}'$ satisfy the iteration hypothesis and in the previous iteration step we defined $\bar{\zeta}(\bar{\mathrm{w}}) = \bar{\mathrm{w}}'$. Then the iteration conclusion shows that each pair $\bar{\mathrm{v}}$ and $\bar{\mathrm{v}}'$ satisfy the iteration hypothesis. We then define $\bar{\zeta}(\bar{\mathrm{v}}) := \bar{\mathrm{v}}'$, and it follows from $\zeta(\mathrm{v}) = \mathrm{v}'$ that $\bar{\zeta}$ covers ζ at $\bar{\mathrm{v}}$. This finishes the iterative construction of the biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ and the proof of $(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\mu})) = \#(\underline{\bar{r}}', \bar{\mu}')$, the covering property, and the convergence. Since at each iteration step the vertex $\bar{\mathrm{v}}$ and $\bar{\mathrm{v}}'$ are in one-to-one correspondence, the map $\bar{\zeta}$ is indeed a tree isomorphism.

Proof of iteration conclusion. Assume the iteration hypothesis. Via the edge identifying map (4.6), each pair of glued edges $\bar{\mathrm{e}} = (\bar{\mathrm{v}}, \bar{\mathrm{w}})$ and $\bar{\mathrm{e}}' = (\bar{\mathrm{v}}', \bar{\mathrm{w}}')$ are identified with $\hat{\mathrm{e}} = (\hat{\mathrm{v}}, \hat{\mathrm{w}}) \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$ and $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}', \hat{\mathrm{w}}') \in \hat{\mathrm{E}}' \setminus \hat{\mathrm{E}}'^{\mathrm{g}}_{\underline{\bar{r}}'}$, respectively. And by definition of glued marked points, we have $\#_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{w}},\bar{\mathrm{e}}} = \bar{x}_{\hat{\mathrm{w}},\hat{\mathrm{e}}}$ and $\#_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{w}}',\bar{\mathrm{e}}'} = \bar{x}_{\hat{\mathrm{w}}',\hat{\mathrm{e}}'}$. Thus by the iteration hypothesis we have

$$\bar{\psi}_{\bar{\mathrm{w}}}(\bar{x}_{\hat{\mathrm{w}},\hat{\mathrm{e}}}) = \bar{x}'_{\hat{\mathrm{w}}',\hat{\mathrm{e}}'}. \tag{7.9}$$

For each pair $\hat{\mathrm{e}} = (\hat{\mathrm{v}}, \hat{\mathrm{w}})$ and $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}', \hat{\mathrm{w}}')$, we denote the biholomorphism $\psi_{n,\mathrm{w}}$ in the standard model on $D_{\hat{\mathrm{w}}}$ and $D_{\hat{\mathrm{w}}'}$ by

$$\psi_n^{\hat{\mathrm{w}},\hat{\mathrm{w}}'} := \left(\Phi^{\hat{\mathrm{w}}'}_{n,\mathrm{w}'}\right)^{-1} \circ \psi_{n,\mathrm{w}} \circ \Phi^{\hat{\mathrm{w}}}_{n,\mathrm{w}}.$$

Since the vertices w and w' are fixed, we shall drop them in the subscripts. Similarly, denote $\bar{\psi}_{\bar{\mathrm{w}}}$ in the standard model on $D_{\hat{\mathrm{w}}}$ and $D_{\hat{\mathrm{w}}'}$ by

$$\bar{\psi}^{\hat{\mathrm{w}},\hat{\mathrm{w}}'} := \left(\bar{\Phi}^{\hat{\mathrm{w}}'}_{\bar{\mathrm{w}}'}\right)^{-1} \circ \bar{\psi}_{\bar{\mathrm{w}}} \circ \bar{\Phi}^{\hat{\mathrm{w}}}_{\bar{\mathrm{w}}}.$$

It follows from the assumption $(\zeta, \underline{\psi}_n)(\#(\underline{r}_n, \mu_n)) = \#(\underline{r}'_n, \mu'_n)$ that we have

$$\psi_n^{\hat{\mathrm{w}},\hat{\mathrm{w}}'}\left(\#^{\hat{\mathrm{w}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n, \underline{O}_n)\right) = \#^{\hat{\mathrm{w}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n, \underline{O}'_n). \tag{7.10}$$

By the distance convergence assumption, we have $\psi_n^{\hat{\mathrm{w}},\hat{\mathrm{w}}'} \to \bar{\psi}^{\hat{\mathrm{w}},\hat{\mathrm{w}}'}$ uniformly as disk maps. Moreover, by (7.9) we have $\bar{\psi}^{\hat{\mathrm{w}},\hat{\mathrm{w}}'}(\bar{x}_{\hat{\mathrm{w}},\hat{\mathrm{e}}}) = \bar{x}'_{\hat{\mathrm{w}}',\hat{\mathrm{e}}'}$. Lastly, recall that $B(\hat{x}_{\hat{\mathrm{w}},\hat{\mathrm{e}}})$ and $B(\hat{x}'_{\hat{\mathrm{w}}',\hat{\mathrm{e}}'})$ are open neighborhoods of $\hat{x}_{\hat{\mathrm{w}},\hat{\mathrm{e}}}$ and $\hat{x}'_{\hat{\mathrm{w}}',\hat{\mathrm{e}}'}$ in $D_{\hat{\mathrm{w}}}$ and $D_{\hat{\mathrm{w}}'}$, respectively. Lemma 7.14 implies the convergence $\#^{\hat{\mathrm{w}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n, \underline{O}_n) \cap B(\hat{x}_{\hat{\mathrm{w}},\hat{\mathrm{e}}}) \to \bar{x}_{\hat{\mathrm{w}},\hat{\mathrm{e}}}$, and similar convergence holds for the prime counterpart. Combining these observations we conclude that for n large enough, we have

$$\psi_n^{\hat{\mathrm{w}},\hat{\mathrm{w}}'}\left(\#^{\hat{\mathrm{w}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n, \underline{O}_n) \cap B(\hat{x}_{\hat{\mathrm{w}},\hat{\mathrm{e}}})\right) = \#^{\hat{\mathrm{w}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n, \underline{O}'_n) \cap B(\hat{x}'_{\hat{\mathrm{w}}',\hat{\mathrm{e}}'}). \tag{7.11}$$

Then the criterion in Lemma 7.6 (2) shows that the edges $\hat{\mathrm{e}} = (\hat{\mathrm{v}}, \hat{\mathrm{w}})$ and $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}', \hat{\mathrm{w}}')$ satisfy either

(1) $\hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}$ and $\hat{\mathrm{e}}' \notin \hat{\mathrm{E}}'^{\mathrm{g}}$, i.e., the glued vertices satisfy $\mathrm{v} \neq \mathrm{w}$ and $\mathrm{v}' \neq \mathrm{w}'$, or

(2) $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}$ and $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{g}}$, i.e., the glued vertices satisfy $\mathrm{v} = \mathrm{w}$ and $\mathrm{v}' = \mathrm{w}'$.

This dichotomy and the assumption $\zeta(\mathrm{w}) = \mathrm{w}'$ imply that $\zeta(\mathrm{v}) = \mathrm{v}'$ for all pairs v and v' as above.

Now fix a pair of edges $\hat{\mathrm{e}} = (\hat{\mathrm{v}}, \hat{\mathrm{w}})$ and $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}', \hat{\mathrm{w}}')$. Suppose this pair is in case (1), then $\hat{\mathrm{v}}$ and $\hat{\mathrm{v}}'$ are the respective roots of the gluing trees $\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}}$ and $\hat{\mathrm{T}}'^{\mathrm{g}}_{\mathrm{v}'}$. Since $\zeta(\mathrm{v}) = \mathrm{v}'$ as shown above, we apply Claim 7.13 to get the biholomorphism $\bar{\psi}_{\bar{\mathrm{v}}}$ and a subsequence with the distance convergence $d^{\hat{\mathrm{u}},\hat{\mathrm{u}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}}) \to 0$ for all gluing vertices $\hat{\mathrm{u}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$ and $\hat{\mathrm{u}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\mathrm{v}'}$. Lastly, we denote the edges $\mathrm{e} = (\mathrm{v}, \mathrm{w})$ and $\mathrm{e}' = (\mathrm{v}', \mathrm{w}')$. The convergence assumptions $(\underline{r}_n, \mu_n) \to (\underline{\bar{r}}, \bar{\mu})$ and $(\underline{r}'_n, \mu'_n) \to (\underline{\bar{r}}', \bar{\mu}')$ imply that the displaced edge lengths $\#_{\underline{r}_n}(\underline{\ell}_n)_{\mathrm{e}}$ and $\#_{\underline{r}'_n}(\underline{\ell}'_n)_{\mathrm{e}'}$ converge to $\#_{\underline{\bar{r}}}(\underline{\bar{\ell}})_{\bar{\mathrm{e}}}$ and $\#_{\underline{\bar{r}}'}(\underline{\bar{\ell}}')_{\bar{\mathrm{e}}'}$, respectively. It follows from the assumption $(\zeta, \underline{\psi}_n)(\#(\underline{r}_n, \mu_n)) = \#(\underline{r}'_n, \mu'_n)$ that we have $\#_{\underline{r}_n}(\underline{\ell}_n)_{\mathrm{e}} = \#_{\underline{r}'_n}(\underline{\ell}'_n)_{\mathrm{e}'}$. Therefore we have the desired equality in displaced edge lengths $\#_{\bar{\underline{r}}}(\bar{\underline{\ell}})_{\bar{\mathrm{e}}} = \#_{\bar{\underline{r}}'}(\bar{\underline{\ell}}')_{\bar{\mathrm{e}}'}$.

It suffices to prove the result for $\hat{\mathrm{e}} = (\hat{\mathrm{v}}, \hat{\mathrm{w}})$ and $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}', \hat{\mathrm{w}}')$ in case (2) when $\mathrm{v} = \mathrm{w}$ and $\mathrm{v}' = \mathrm{w}'$. First of all, since we have $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\bar{\underline{r}}}$ and $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{g}} \setminus \hat{\mathrm{E}}'^{\mathrm{g}}_{\bar{\underline{r}}'}$, both $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$ are additional gluing edges. Hence we have $\bar{r}_{\hat{\mathrm{e}}} = 0$ and $\bar{r}'_{\hat{\mathrm{e}}'} = 0$, and it follows $\#_{\bar{\underline{r}}}(\bar{\underline{\ell}})_{\bar{\mathrm{e}}} = 0 = \#_{\bar{\underline{r}}'}(\bar{\underline{\ell}}')_{\bar{\mathrm{e}}'}$.

We now construct the biholomorphism $\bar{\psi}_{\bar{\mathrm{v}}}$ and show the desired convergence. Similar as before, we denote the biholomorphism $\psi_{n,\mathrm{w}}$ in the standard model on $D_{\hat{\mathrm{v}}}$ and $D_{\hat{\mathrm{v}}'}$ by

$$\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} := \left(\Phi^{\hat{\mathrm{v}}'}_{n,\mathrm{w}'}\right)^{-1} \circ \psi_{n,\mathrm{w}} \circ \Phi^{\hat{\mathrm{v}}}_{n,\mathrm{w}}.$$

We shall use Proposition 13.7 to extract a convergent subsequence of $\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}$. Applying Lemma 7.6 (1) to equation (7.11), we get a correspondence of glued marked points in the shrunk strip neighborhoods

(7.12)

$$\psi_n^{\hat{\mathrm{w}},\hat{\mathrm{w}}'}\left(\#^{\hat{\mathrm{w}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n,\underline{O}_n) \cap N(x^+_{n,\hat{\mathrm{e}}}; -R_{n,\hat{\mathrm{e}}})\right) = \#^{\hat{\mathrm{w}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n,\underline{O}'_n) \cap N(x'^+_{n,\hat{\mathrm{e}}'}; -R'_{n,\hat{\mathrm{e}}'}).$$

Lemma 7.4 (2) implies that $\psi_n^{\hat{\mathrm{w}},\hat{\mathrm{w}}'}$ and $\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}$ are related by the shift maps

$$\psi_n^{\hat{\mathrm{w}},\hat{\mathrm{w}}'} = \phi^{\hat{\mathrm{e}}'}(r'_{n,\hat{\mathrm{e}}'}, \underline{x}'_n)^{-1} \circ \psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} \circ \phi^{\hat{\mathrm{e}}}(r_{n,\hat{\mathrm{e}}}, \underline{x}_n). \tag{7.13}$$

Moreover, Lemma 7.3 (2) shows $\phi^{\hat{\mathrm{e}}}(r_{n,\hat{\mathrm{e}}}, \underline{x}_n)(N(x^+_{n,\hat{\mathrm{e}}}; -R_{n,\hat{\mathrm{e}}})) = \overline{D_{\hat{\mathrm{v}}} \setminus N(x^-_{n,\hat{\mathrm{e}}})}$ and there is a similar relation for the prime counterpart. Combining (7.12) and (7.13) we conclude

$$\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\left(\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n,\underline{O}_n) \setminus N(x^-_{n,\hat{\mathrm{e}}})\right) = \#^{\hat{\mathrm{v}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n,\underline{O}'_n) \setminus N(x'^-_{n,\hat{\mathrm{e}}'}).$$

Apply Lemma 7.6 (1) again we have

$$\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\left(\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n,\underline{O}_n) \setminus B(\hat{x}^-_{\hat{\mathrm{e}}})\right) = \#^{\hat{\mathrm{v}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n,\underline{O}'_n) \setminus B(\hat{x}'^-_{\hat{\mathrm{e}}'}). \tag{7.14}$$

We now denote the glued marked points outside of $B(\hat{x}^-_{\hat{\mathrm{e}}})$ and $B(\hat{x}'^-_{\hat{\mathrm{e}}'})$ by

$$W_n := \#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n,\underline{O}_n) \setminus B(\hat{x}^-_{\hat{\mathrm{e}}}), \quad W'_n := \#^{\hat{\mathrm{v}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n,\underline{O}'_n) \setminus B(\hat{x}'^-_{\hat{\mathrm{e}}'}).$$

Hence (7.14) shows $\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(W_n) = W'_n$.

In order to apply Proposition 13.7, we still need to establish the convergence of W_n and W'_n, and that of a pair of boundary marked points. Lemma 7.14 shows Hausdorff convergence $W_n \to \#^{\hat{\mathrm{v}}}_{\bar{\underline{r}},\bar{\underline{x}}}(\bar{\underline{x}},\bar{\underline{O}}) \setminus B(\hat{x}^-_{\hat{\mathrm{e}}}) = \#^{\hat{\mathrm{v}}}_{\bar{\underline{r}},\bar{\underline{x}}}(\bar{\underline{x}},\bar{\underline{O}}) \setminus \{\bar{x}^-_{\hat{\mathrm{e}}}\}$, and similar convergence holds for the prime counterpart. Now let $\hat{\mathrm{f}}$ and $\hat{\mathrm{f}}'$ be the outgoing edges of the roots of the gluing trees $\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}}$ and $\hat{\mathrm{T}}'^{\mathrm{g}}_{\mathrm{v}'}$, respectively. Then $x^-_{n,\hat{\mathrm{f}}}$ and $x'^-_{n,\hat{\mathrm{f}}'}$ are the outgoing boundary marked points in the glued marked points $\#_{\underline{r}_n,\underline{x}_n}(\underline{x}_n,\underline{O}_n)$ and $\#_{\underline{r}'_n,\underline{x}'_n}(\underline{x}'_n,\underline{O}'_n)$, respectively. Thus in the standard model on $D_{\hat{\mathrm{v}}}$ and $D_{\hat{\mathrm{v}}'}$, we have

$$\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\left(\left(\Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}}\right)^{-1}(x^-_{n,\hat{\mathrm{f}}})\right) = \left(\Phi^{\hat{\mathrm{v}}'}_{n,\mathrm{v}'}\right)^{-1}(x'^-_{n,\hat{\mathrm{f}}'}).$$

Furthermore, it follows from Lemma 7.14 that we have $\left(\Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}}\right)^{-1}(x^-_{n,\hat{\mathrm{f}}}) \to \left(\bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}\right)^{-1}(\bar{x}^-_{\hat{\mathrm{f}}})$ and similar convergence holds for the prime counterpart.

We now apply Proposition 13.7 to conclude that there is a subsequence of $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}_n$ that converges to a disk automorphism $\bar{\xi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}$ and it satisfies

$$\bar{\xi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{O}})_{\bar{\mathrm{v}}}) = \#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}',\underline{\bar{O}}')_{\bar{\mathrm{v}}'}.$$

Define $\bar{\psi}_{\bar{\mathrm{v}}} := \bar{\Phi}^{\hat{\mathrm{v}}'}_{\bar{\mathrm{v}}'} \circ \bar{\xi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} \circ \left(\bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}\right)^{-1}$, by construction we have $\bar{\psi}_{\bar{\mathrm{v}}}\left(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{O}})_{\bar{\mathrm{v}}}\right) = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}',\underline{\bar{O}}')_{\bar{\mathrm{v}}'}$ and the distance convergence $d^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}},\psi_{n,\mathrm{v}}) \to 0$. We apply Lemma 7.15 to conclude $d^{\hat{\mathrm{u}},\hat{\mathrm{u}}'}(\bar{\psi}_{\bar{\mathrm{v}}},\psi_{n,\mathrm{v}}) \to 0$ for all gluing vertices $\hat{\mathrm{u}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ and $\hat{\mathrm{u}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$. This proves the iteration conclusion. ☐

This finishes the proof of the main theorem, and we now prove the supporting lemmas used in the above proof.

Lemma 7.14. *For every gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$, we have convergence in Hausdorff distance*

$$\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n,\underline{O}_n)_{\mathrm{v}} \to \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{O}})_{\bar{\mathrm{v}}}.$$

Proof. Fix a gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$. We now show the convergence in Hausdorff distance. Note that every boundary marked point in $\#_{\underline{r}_n,\underline{x}_n}(\underline{x}_n,\underline{O}_n)_{\mathrm{v}}$ is of the form

$$x_{n,\hat{\mathrm{w}},\hat{\mathrm{f}}} \text{ for vertex } \hat{\mathrm{w}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}} \text{ and edge } \hat{\mathrm{f}} \notin \hat{\mathrm{E}}^{\mathrm{g}}_{\bar{\mathrm{v}}}.$$

Fix such a vertex $\hat{\mathrm{w}}$ and an edge $\hat{\mathrm{f}}$. In the gluing tree $\hat{\mathrm{T}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$, there is a unique sequence of distinct vertices $\hat{\mathrm{v}} = \hat{\mathrm{v}}_0, \hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_k = \hat{\mathrm{w}}$ such that each consecutive pair is adjacent. We denote the glued vertex of $\hat{\mathrm{w}}$ by $\bar{\mathrm{w}} := [\hat{\mathrm{w}}]_{\underline{\bar{r}}}$, and discuss the following two cases regarding glued vertices: $\bar{\mathrm{v}} = \bar{\mathrm{w}}$ or $\bar{\mathrm{v}} \neq \bar{\mathrm{w}}$. (As a visual aid, see $\#(\underline{r}_n,\mu_n)$ and $\#(\underline{r},\mu)$ in Figure 2.)

Suppose we have $\bar{\mathrm{v}} = \bar{\mathrm{w}}$. Then the marked point $x_{n,\hat{\mathrm{w}},\hat{\mathrm{f}}}$ in the standard model on $D_{\hat{\mathrm{v}}}$ is given by

$$\left(\Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}}\right)^{-1} \circ \Phi^{\hat{\mathrm{w}}}_{n,\mathrm{v}}\,(x_{n,\hat{\mathrm{w}},\hat{\mathrm{f}}}). \tag{7.15}$$

By Lemma 7.4 (2), the disk automorphism $\left(\Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}}\right)^{-1} \circ \Phi^{\hat{\mathrm{w}}}_{n,\mathrm{v}}$ can be written as composition of shift maps $\phi^{\hat{\mathrm{e}}_i}(r_{n,\hat{\mathrm{e}}_i},\underline{x}_n)$ or their inverse, where each edge $\hat{\mathrm{e}}_i$ is formed by vertices $\hat{\mathrm{v}}_{i-1}$ and $\hat{\mathrm{v}}_i$. Moreover we have $r_{n,\hat{\mathrm{e}}_i} \to \bar{r}_{\hat{\mathrm{e}}_i}$, and the assumption on glued vertices $\bar{\mathrm{v}} = \bar{\mathrm{w}}$ implies that the gluing parameter $\bar{r}_{\hat{\mathrm{e}}_i} > 0$ for each i. Hence shift map converges $\phi^{\hat{\mathrm{e}}_i}(r_{n,\hat{\mathrm{e}}_i},\underline{x}_n) \to \phi^{\hat{\mathrm{e}}_i}(\bar{r}_{\hat{\mathrm{e}}_i},\underline{\bar{x}})$. Then it follows the continuity in Lemma 7.3 (1) that we have disk automorphism convergence

$$\left(\Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}}\right)^{-1} \circ \Phi^{\hat{\mathrm{w}}}_{n,\mathrm{v}} \to \left(\bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}\right)^{-1} \circ \bar{\Phi}^{\hat{\mathrm{w}}}_{\bar{\mathrm{v}}}. \tag{7.16}$$

In particular, the sequence converges uniformly on the disk (see Lemma 13.3). By assumption, we have the convergence of the boundary marked point $x_{n,\hat{w},\hat{f}} \to \bar{x}_{\hat{w},\hat{f}}$. Therefore the sequence of marked point in (7.15) converges to $\left(\bar{\Phi}^{\hat{v}}_{\bar{v}}\right)^{-1} \circ \bar{\Phi}^{\hat{w}}_{\bar{v}}\,(\bar{x}_{\hat{w},\hat{f}})$, which is a marked point in $\#^{\hat{v}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{O}})_{\bar{v}}$.

Suppose we have $\bar{v} \neq \bar{w}$. Then in the sequence of vertices $\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_k$, there is a consecutive pair $\hat{v}'$ and $\hat{w}'$ such that

- $[\hat{v}']_{\underline{\bar{r}}} = \bar{v}$ and $[\hat{w}']_{\underline{\bar{r}}} \neq \bar{v}$, and
- $\hat{v}'$ and $\hat{w}'$ form a gluing edge $\hat{e} \in \hat{E}^{g}$.

In this case, the marked point $x_{n,\hat{w},\hat{f}}$ in the standard model on $D_{\hat{v}}$ is given by

$$\begin{aligned}(7.17)\qquad &\left(\Phi^{\hat{v}}_{n,v}\right)^{-1} \circ \Phi^{\hat{w}}_{n,v}\,(x_{n,\hat{w},\hat{f}}) \\ &= \left(\left(\Phi^{\hat{v}}_{n,v}\right)^{-1} \circ \Phi^{\hat{v}'}_{n,v}\right) \circ \left(\left(\Phi^{\hat{v}'}_{n,v}\right)^{-1} \circ \Phi^{\hat{w}}_{n,v}\right)(x_{n,\hat{w},\hat{f}}).\end{aligned}$$

Since we have $[\hat{v}']_{\underline{\bar{r}}} = \bar{v}$, by the same argument as the first case, the sequence $\left(\Phi^{\hat{v}}_{n,v}\right)^{-1} \circ \Phi^{\hat{v}'}_{n,v}$ converges to $\left(\bar{\Phi}^{\hat{v}}_{\bar{v}}\right)^{-1} \circ \bar{\Phi}^{\hat{v}'}_{\bar{v}}$ uniformly on the disk. On the other hand, Lemma 7.6 (1) implies that the marked point $\left(\Phi^{\hat{v}'}_{n,v}\right)^{-1} \circ \Phi^{\hat{w}}_{n,v}\,(x_{n,\hat{w},\hat{f}})$ in the standard model on $D_{\hat{v}'}$ is contained in the shrunk strip neighborhood $N(x_{n,\hat{v}',\hat{e}}; -R_{n,\hat{e}})$. Hence it converges to $\bar{x}_{\hat{v}',\hat{e}}$. Combining the above observations, the sequence of marked point in (7.15) converges to $\left(\bar{\Phi}^{\hat{v}}_{\bar{v}}\right)^{-1} \circ \bar{\Phi}^{\hat{v}'}_{\bar{v}}\,(\bar{x}_{\hat{v}',\hat{e}})$, which is a marked point in $\#^{\hat{v}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{O}})_{\bar{v}}$.

This proves that the boundary marked points $\#^{\hat{v}}_{\underline{r}_n,\underline{x}_n}(\underline{x}_n,\underline{O}_n)_{v}$ have the desired convergence. By the same argument, the interior marked points converge as well (to interior marked points in the first case, and to boundary marked points in the second). This proves the desired convergence. □

Lemma 7.15. *Let* $v \in \#_{\hat{E}^{g}}(\hat{V})$ *be a main vertex, and* $v' := \zeta(v)$ *the corresponding vertex. Suppose there are vertices* $\bar{v} \in \bar{p}^{-1}_{\hat{E}^{g}}(v)$ *and* $\bar{v}' \in \bar{p}'^{-1}_{\hat{E}'^{g}}(v')$ *and a biholomorphism* $\bar{\psi}_{\bar{v}} : \#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{D})_{\bar{v}} \to \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{D})_{\bar{v}'}$. *If there is one pair of gluing vertices* $\hat{v}_0 \in \hat{V}^{g}_{\bar{v}}$ *and* $\hat{v}'_0 \in \hat{V}'^{g}_{\bar{v}'}$ *with the distance convergence* $d^{\hat{v}_0,\hat{v}'_0}(\bar{\psi}_{\bar{v}}, \psi_{n,v}) \to 0$*, then for every pair of gluing vertices* $\hat{v} \in \hat{V}^{g}_{\bar{v}}$ *and* $\hat{v}' \in \hat{V}'^{g}_{\bar{v}'}$ *we have the distance convergence* $d^{\hat{v},\hat{v}'}(\bar{\psi}_{\bar{v}}, \psi_{n,v}) \to 0$.

Proof. Fix a pair of gluing vertices $\hat{v} \in \hat{V}^{g}_{\bar{v}}$ and $\hat{v}' \in \hat{V}'^{g}_{\bar{v}'}$. Then the biholomorphism $\psi_{n,v}$ in the standard model from $D_{\hat{v}}$ to $D_{\hat{v}'}$ can be written as

$$\begin{aligned}&\left(\Phi^{\hat{v}'}_{n,v'}\right)^{-1} \circ \psi_{n,v} \circ \Phi^{\hat{v}}_{n,v} \\ (7.18)\qquad & \\ &= \left(\left(\Phi^{\hat{v}'}_{n,v'}\right)^{-1} \circ \Phi^{\hat{v}'_0}_{n,v'}\right) \circ \left(\left(\Phi^{\hat{v}'_0}_{n,v'}\right)^{-1} \circ \psi_{n,v} \circ \Phi^{\hat{v}_0}_{n,v}\right) \circ \left(\left(\Phi^{\hat{v}_0}_{n,v}\right)^{-1} \circ \Phi^{\hat{v}}_{n,v}\right).\end{aligned}$$

We first analyze the last factor. There is a unique sequence of distinct vertices $\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_k = \hat{v}$ such that each consecutive pair is adjacent.

The disk automorphism $\left(\Phi^{\hat{v}_0}_{n,v}\right)^{-1} \circ \Phi^{\hat{v}}_{n,v}$ can be written as composition of shift maps $\phi^{\hat{e}_i}(r_{n,\hat{e}_i}, \underline{x}_n)$ or their inverse, where each edge $\hat{e}_i$ is formed by vertices $\hat{\mathrm{v}}_{i-1}$ and $\hat{\mathrm{v}}_i$. Since both $\hat{\mathrm{v}}_0$ and $\hat{\mathrm{v}}$ are glued to $\bar{\mathrm{v}}$, similar to the proof of Lemma 7.14 we have disk automorphism convergence

$$\left(\Phi^{\hat{\mathrm{v}}_0}_{n,\mathrm{v}}\right)^{-1} \circ \Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}} \to \left(\bar{\Phi}^{\hat{\mathrm{v}}_0}_{\bar{\mathrm{v}}}\right)^{-1} \circ \bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}.$$

By the same argument, the first factor in (7.18) has $\left(\Phi^{\hat{\mathrm{v}}'}_{n,\mathrm{v}'}\right)^{-1} \circ \Phi^{\hat{\mathrm{v}}'_0}_{n,\mathrm{v}'} \to \left(\bar{\Phi}^{\hat{\mathrm{v}}'}_{\bar{\mathrm{v}}'}\right)^{-1} \circ \bar{\Phi}^{\hat{\mathrm{v}}'_0}_{\bar{\mathrm{v}}'}$. Combining with the convergence assumption in the standard model on $D_{\hat{\mathrm{v}}_0}$ and $D_{\hat{\mathrm{v}}'_0}$, we have

$$\left(\Phi^{\hat{\mathrm{v}}'}_{n,\mathrm{v}'}\right)^{-1} \circ \psi_{n,\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}} \to \left(\bar{\Phi}^{\hat{\mathrm{v}}'}_{\bar{\mathrm{v}}'}\right)^{-1} \circ \bar{\psi}_{\bar{\mathrm{v}}} \circ \bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}.$$

Thus we have distance convergence $d^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}}) \to 0$. □

7.1.4. *Additional Gluing Edges Correspondence.*

Let $(\underline{\bar{r}}, \bar{\mu}) \in \mathcal{U}_\varepsilon(\underline{0}, \hat{\mu})$ and $(\underline{\bar{r}}', \bar{\mu}') \in \mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\mu}')$ be an equivalent pair, i.e., there is a biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ with

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\mu})) = \#(\underline{\bar{r}}', \bar{\mu}').$$

In this section, we study nearby pairs $(\underline{r}, \mu)$ and $(\underline{r}', \mu')$ whose glued pairs are equivalent via a biholomorphism. We shall show that this biholomorphism yields a correspondence between additional gluing edges on both sides, and moreover under this correspondence, the biholomorphism maps a uniform portion of the glued strip into another. We shall use this result in proving the topology and atlas of the quotient space of disk trees, by applying the splicing core analysis on the said glued strips.

Firstly, the ordered tree isomorphism $\bar{\zeta} : \#_{\underline{\bar{r}}}(\hat{\mathrm{T}}) \to \#_{\underline{\bar{r}}'}(\hat{\mathrm{T}}')$ induces a bijection of non-gluing edges

$$\bar{\chi} : \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}} \xrightarrow{\sim} \hat{\mathrm{E}}' \setminus \hat{\mathrm{E}}'^{\mathrm{g}}_{\underline{\bar{r}}'} \tag{7.19}$$

by composing with the edge identifying maps (4.6) with the induced bijection of glued edges $\bar{\zeta} : \#_{\underline{\bar{r}}}(\hat{\mathrm{E}}) \xrightarrow{\sim} \#_{\underline{\bar{r}}'}(\hat{\mathrm{E}}')$.

We now show that the bijection $\bar{\chi}$ gives a one-to-one correspondence between edges with vanishing gluing parameters.

Lemma 7.16. *We have* $\bar{\chi}(\{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \,|\, \bar{r}_{\hat{\mathrm{e}}} = 0\}) = \{\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{nd}} \,|\, \bar{r}'_{\hat{\mathrm{e}}'} = 0\}$.

Proof. Let $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$ be a nodal edge with gluing parameter $\bar{r}_{\hat{\mathrm{e}}} = 0$. We denote its corresponding edge by $\hat{\mathrm{e}}' := \bar{\chi}(\hat{\mathrm{e}})$, and we shall show that $\hat{\mathrm{e}}'$ is a nodal edge with gluing parameter $\bar{r}'_{\hat{\mathrm{e}}'} = 0$. Denote the targets of $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$ via map (4.6) by $\bar{\mathrm{e}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{E}})$ and $\bar{\mathrm{e}}' \in \#_{\underline{\bar{r}}'}(\hat{\mathrm{E}}')$. By construction (5.7) we have $\#_{\underline{\bar{r}}}(\underline{\bar{\ell}})_{\bar{\mathrm{e}}} = 0$. Since $\#_{\underline{\bar{r}}}(\underline{\bar{\ell}})_{\bar{\mathrm{e}}} = \#_{\underline{\bar{r}}'}(\underline{\bar{\ell}}')_{\bar{\mathrm{e}}'}$, we have $\#_{\underline{\bar{r}}'}(\underline{\bar{\ell}}')_{\bar{\mathrm{e}}'} = 0$. If $\hat{\mathrm{e}}'$ is not a nodal edge, then by construction $\#_{\underline{\bar{r}}'}(\underline{\bar{\ell}}')_{\bar{\mathrm{e}}'} = \bar{\ell}'_{\hat{\mathrm{e}}'}$, which is *not* zero by the choice of $\hat{\varepsilon}'$ in Remark 5.9 (2). Thus we have $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{nd}}$. By (5.7) again, we have $\bar{r}'_{\hat{\mathrm{e}}'} = 0$. The implication the other way works similarly. This proves the result. □

We state the following relation (without proof) between the tree isomorphism of nearby glued trees and their additional gluing edges.

Lemma 7.17. *For nearby gluing parameters $\underline{r} \in \mathcal{U}_\delta(\underline{\bar{r}})$ and $\underline{r}' \in \mathcal{U}_{\delta'}(\underline{\bar{r}}')$, we have $\bar{\chi}(\hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}}) = \hat{\mathrm{E}}'^{\mathrm{ag}}_{\underline{r}'}$ if and only there exists an ordered tree isomorphism $\zeta : \#_{\underline{r}}(\hat{\mathrm{T}}) \to \#_{\underline{r}'}(\hat{\mathrm{T}}')$ such that $\bar{\zeta}$ covers ζ in the sense that diagram (7.4) commutes.*

In Figure 2, each solid strip corresponds to an additional gluing edge in $\hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}_n}$ and $\hat{\mathrm{E}}'^{\mathrm{ag}}_{\underline{r}'_n}$, note that there is a bijection between additional gluing edges.

The following proposition shows that for a nearby equivalent pair, the image of an additional glued strip under the biholomorphism contains a uniform portion of the corresponding glued strip. Thus in the quotient space of disk trees, we can apply the splicing core analysis on these glued strips.

Proposition 7.18. *There exist small $\delta, \delta' > 0$ and a large $k' > 0$ such that if there is a pair $(\underline{r}, \mu) \in \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\mu})$ and $(\underline{r}', \mu') \in \mathcal{U}_{\delta'}(\underline{\bar{r}}', \bar{\mu}')$ and a biholomorphism $(\zeta, \underline{\psi})$ with*

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mu)) = \#(\underline{r}', \mu'),$$

then each additional gluing edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$ in $\hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}}$ bijectively corresponds to an additional gluing edge $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}'^-, \hat{\mathrm{v}}'^+) := \bar{\chi}(\hat{\mathrm{e}})$ in $\hat{\mathrm{E}}'^{\mathrm{ag}}_{\underline{r}'}$. Moreover, we have inclusion of the glued strips of edges $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$

$$\psi_{\mathrm{v}}([0, R_{\hat{\mathrm{e}}}] \times [0, \pi]) \supset [k', R'_{\hat{\mathrm{e}}'} - k'] \times [0, \pi]$$

with $\mathrm{v} := [\hat{\mathrm{v}}^-]_{\underline{r}} = [\hat{\mathrm{v}}^+]_{\underline{r}}$. On the other hand, each edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$ with $\bar{r}_{\hat{\mathrm{e}}} = 0$ and $r_{\hat{\mathrm{e}}} \le 0$ bijectively corresponds to an edge $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}'^-, \hat{\mathrm{v}}'^+) := \bar{\chi}(\hat{\mathrm{e}})$ with $\bar{r}'_{\hat{\mathrm{e}}'} = 0$ and $r'_{\hat{\mathrm{e}}'} \le 0$. Moreover, we have inclusion of shrunk strip neighborhoods of edges $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$

$$\psi_{\mathrm{v}^\pm}(N(x^\pm_{\hat{\mathrm{e}}})) \supset N(x'^\pm_{\hat{\mathrm{e}}'}; -k')$$

with $\mathrm{v}^\pm := [\hat{\mathrm{v}}^\pm]_{\underline{r}}$.

Here $[0, R_{\hat{\mathrm{e}}}] \times [0, \pi]$ is the strip obtained by gluing $h^+_{\hat{\mathrm{e}}}(x^+_{\hat{\mathrm{e}}}, [0, R_{\hat{\mathrm{e}}}] \times [0, \pi])$ with $h^-_{\hat{\mathrm{e}}}(x^-_{\hat{\mathrm{e}}}, [-R_{\hat{\mathrm{e}}}, 0] \times [0, \pi])$, and $[0, R'_{\hat{\mathrm{e}}'}] \times [0, \pi]$ is obtained similarly. Also see (3.3) for shrunk strip neighborhoods.

Proof. It follows from Theorem 7.12 that for δ, δ' small enough the tree isomorphism $\bar{\zeta}$ covers ζ. Then by Lemma 7.17, we have $\bar{\chi}(\hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}}) = \hat{\mathrm{E}}'^{\mathrm{ag}}_{\underline{r}'}$. Thus for each additional gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}}$, its corresponding edge $\bar{\chi}(\hat{\mathrm{e}})$ is an additional gluing edge. On the other hand, it follows from Lemma 7.16 that for an edge $\hat{\mathrm{e}}$ with $\bar{r}_{\hat{\mathrm{e}}} = 0$ and $r_{\hat{\mathrm{e}}} \le 0$, its corresponding edge $\hat{\mathrm{e}}' = \bar{\chi}(\hat{\mathrm{e}})$ has $\bar{r}'_{\hat{\mathrm{e}}'} = 0$ and $r'_{\hat{\mathrm{e}}'} \le 0$.

We shall prove the first inclusion by contradiction. Suppose there does not exist such δ, δ' and k'. Then there are sequences $(\underline{r}_n, \mu_n) \to (\underline{\bar{r}}, \bar{\mu})$,

$(\underline{r}'_n, \mu'_n) \to (\underline{\bar{r}}', \bar{\mu}')$, and $k'_n \to \infty$ such that we get fixed gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}_n} = \hat{\mathrm{E}}^{\mathrm{g}}$ and $\hat{\mathrm{E}}'^{\mathrm{g}}_{\underline{r}'_n} = \hat{\mathrm{E}}'^{\mathrm{g}}$ and moreover, there are

- an additional gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$ with corresponding additional gluing edge $\hat{\mathrm{e}}' = \bar{\chi}(\hat{\mathrm{e}}) \in \hat{\mathrm{E}}'^{\mathrm{ag}}$, and
- a sequence of points z_n in the glued disk $\#_{\underline{r}_n,\underline{x}_n}(D)_{\mathrm{v}}$ which lies *outside* of the glued strip $[0, R_{n,\hat{\mathrm{e}}}] \times [0,\pi]$ with image $\psi_{n,\mathrm{v}}(z_n) \in [k'_n, R'_{n,\hat{\mathrm{e}}'} - k'_n] \times [0,\pi]$.

The above additional gluing edges are of the form $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$ and $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}'^-, \hat{\mathrm{v}}'^+)$, which descend to edges $(\bar{\mathrm{v}}^-, \bar{\mathrm{v}}^+) := ([\hat{\mathrm{v}}^-]_{\underline{\bar{r}}}, [\hat{\mathrm{v}}^+]_{\underline{\bar{r}}})$ and $(\bar{\mathrm{v}}'^-, \bar{\mathrm{v}}'^+) := ([\hat{\mathrm{v}}'^-]_{\underline{\bar{r}}'}, [\hat{\mathrm{v}}'^+]_{\underline{\bar{r}}'})$; the vertices $\hat{\mathrm{v}}^-$ and $\hat{\mathrm{v}}^+$ are glued to the same vertex v in the tree $\#_{\hat{\mathrm{E}}^{\mathrm{g}}}(\hat{\mathrm{T}})$, and $\hat{\mathrm{v}}'^-$ and $\hat{\mathrm{v}}'^+$ are glued to v' in $\#_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\hat{\mathrm{T}}')$.

Since the point z_n lies outside of the glued strip $[0, R_{n,\hat{\mathrm{e}}}] \times [0,\pi]$, it either (1) lies outside of the strip neighborhood $N(x^-_{n,\hat{\mathrm{e}}})$ in the standard model on $D_{\hat{\mathrm{v}}^-}$, or (2) lies outside of the strip neighborhood $N(x^+_{n,\hat{\mathrm{e}}})$ in the standard model on $D_{\hat{\mathrm{v}}^+}$ (see Figure 1). Thus there exists a subsequence with either

(1) $(\Phi^{\hat{\mathrm{v}}^-}_{n,\mathrm{v}})^{-1}(z_n) \in D_{\hat{\mathrm{v}}^-} \setminus N(x'^-_{n,\hat{\mathrm{e}}'})$, or
(2) $(\Phi^{\hat{\mathrm{v}}^+}_{n,\mathrm{v}})^{-1}(z_n) \in D_{\hat{\mathrm{v}}^+} \setminus N(x'^+_{n,\hat{\mathrm{e}}'})$.

Suppose we have a subsequence in case (1). We denote the biholomorphism $\psi_{n,\mathrm{v}}$ in the standard model on $D_{\hat{\mathrm{v}}^-}$ and $D_{\hat{\mathrm{v}}'^-}$ by

$$\psi_n^{\hat{\mathrm{v}}^-,\hat{\mathrm{v}}'^-} := \left(\Phi^{\hat{\mathrm{v}}'^-}_{n,\mathrm{v}'}\right)^{-1} \circ \psi_{n,\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}^-}_{n,\mathrm{v}}.$$

The image $\psi_{n,\mathrm{v}}(z_n)$ lies in $[k'_n, R'_{n,\hat{\mathrm{e}}'} - k'_n] \times [0,\pi]$, so in the standard model on $D_{\hat{\mathrm{v}}'^-}$ the image $\psi_n^{\hat{\mathrm{v}}^-,\hat{\mathrm{v}}'^-}\left((\Phi^{\hat{\mathrm{v}}^-}_{n,\mathrm{v}})^{-1}(z_n)\right)$ lies in the shrunk neighborhood $N(x'^-_{n,\hat{\mathrm{e}}'}; -k'_n)$. Because we have convergence of boundary marked point $x'^-_{n,\hat{\mathrm{e}}'} \to \bar{x}'^-_{\hat{\mathrm{e}}'}$ and $k'_n \to \infty$, we have

$$\psi_n^{\hat{\mathrm{v}}^-,\hat{\mathrm{v}}'^-}\left((\Phi^{\hat{\mathrm{v}}^-}_{n,\mathrm{v}})^{-1}(z_n)\right) \to \bar{x}^-_{\hat{\mathrm{e}}'}. \tag{7.20}$$

On the other hand, we denote the biholomorphism $\bar{\psi}_{\bar{\mathrm{v}}^-}$ in the standard model on $D_{\hat{\mathrm{v}}^-}$ and $D_{\hat{\mathrm{v}}'^-}$ by

$$\bar{\psi}^{\hat{\mathrm{v}}^-,\hat{\mathrm{v}}'^-} := \left(\bar{\Phi}^{\hat{\mathrm{v}}'^-}_{\bar{\mathrm{v}}'^-}\right)^{-1} \circ \bar{\psi}_{\bar{\mathrm{v}}^-} \circ \bar{\Phi}^{\hat{\mathrm{v}}^-}_{\bar{\mathrm{v}}^-}.$$

By assumption, we have $\bar{\psi}_{\bar{\mathrm{v}}^-}\left(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}}, \underline{\bar{O}})_{\bar{\mathrm{v}}^-}\right) = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}', \underline{\bar{O}}')_{\bar{\mathrm{v}}'^-}$. It follows

$$\bar{\psi}^{\hat{\mathrm{v}}^-,\hat{\mathrm{v}}'^-}(\bar{x}^-_{\hat{\mathrm{e}}}) = \bar{x}'^-_{\hat{\mathrm{e}}'}. \tag{7.21}$$

Moreover by Theorem 7.12, we have a subsequence (also indexed by n) with convergence $d^{\hat{\mathrm{v}}^-,\hat{\mathrm{v}}'^-}(\bar{\psi}_{\bar{\mathrm{v}}^-}, \psi_{n,\mathrm{v}}) \to 0$. Therefore the following automorphisms converge uniformly on the disk

$$\psi_n^{\hat{\mathrm{v}}^-,\hat{\mathrm{v}}'^-} \to \bar{\psi}^{\hat{\mathrm{v}}^-,\hat{\mathrm{v}}'^-}. \tag{7.22}$$

Thus by (7.22), (7.20), and (7.21), the sequence $(\Phi_{n,\mathrm{v}}^{\hat{\mathrm{v}}^-})^{-1}(z_n)$ must converge to $\bar{x}^-_{\hat{\mathrm{e}}}$. However, this contradicts with condition (1), which implies the sequence is bounded away from $\bar{x}^-_{\hat{\mathrm{e}}}$.

Suppose we have a sequence in case (2), we derive a contradiction similarly. This proves that there exists δ, δ' and k' that give the first inclusion. We can prove the second inclusion by a similar argument by contradiction. ☐

7.2. **Openness of the Biholomorphic Equivalence in $\mathfrak{DM}$.**

In this section, we show that the collection of neighborhoods $\mathfrak{U}_\varepsilon(\sigma; \hat{\mu}; R)$ in (5.10) forms a basis for the Deligne-Mumford space $\mathfrak{DM}$. Combining it with the Hausdorff property in Section 7.1 we prove Theorem 5.11. In addition, we also prove Theorem 5.17: the collection of maps θ in (5.13) forms a smooth atlas for $\mathfrak{DM}$.

We shall prove the theorems above by using the following results of the openness of the biholomorphic equivalence. More precisely, we show that if we have gluing $\#(\bar{\underline{r}}, \bar{\mu})$ equivalent to gluing $\#(\bar{\underline{r}}', \bar{\mu}')$, then gluing near $\#(\bar{\underline{r}}, \bar{\mu})$ is equivalent to gluing near $\#(\bar{\underline{r}}', \bar{\mu}')$. For the following results, we fix representatives $\hat{\mu} = (\hat{\mathrm{T}}, \hat{\underline{\ell}}, (\hat{\underline{x}}, \hat{\underline{o}}))$ and $\hat{\mu}' = (\hat{\mathrm{T}}', \hat{\underline{\ell}}', (\hat{\underline{x}}', \hat{\underline{o}}'))$ of elements σ and σ' of $\mathfrak{DM}$. For convenience, we abbreviate

$$\mathcal{U}_\varepsilon(\hat{\underline{r}}, \hat{\mu}) := \mathcal{U}_\varepsilon(\hat{\underline{r}}) \times \mathcal{U}_\varepsilon(\hat{\mu}).$$

We shall use the following result to prove the collection of neighborhoods in (5.10) forms a basis for the Deligne-Mumford space; we also use it to show the continuity of each chart θ in (5.13) and the continuity of its inverse. As we shall see, this result is a consequence of Theorem 7.22.

Theorem 7.19. *Let R be any gluing profile. Assume there is a pair $(\bar{\underline{r}}, \bar{\mu}) \in \mathcal{U}_\varepsilon(\underline{0}, \hat{\mu})$ and $(\bar{\underline{r}}', \bar{\mu}') \in \mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\mu}')$ with*

$$[\#](\bar{\underline{r}}, \bar{\mu}) = [\#](\bar{\underline{r}}', \bar{\mu}').$$

Then given $\delta' > 0$, there is $\delta > 0$ with inclusion

$$[\#](\mathcal{U}_\delta(\bar{\underline{r}}, \bar{\mu})) \subset \theta'(\mathcal{U}^{\mathrm{slc}}_{\delta'}(\bar{\underline{r}}', \bar{\mu}')).$$

We shall use the following result in showing the smoothness of the transition maps of charts. This result is a consequence of Theorem 7.22.

Theorem 7.20. *Let R be given by the exponential gluing profile $\varphi(r) = e^{1/r} - e$. Assume there is a pair $(\bar{\underline{r}}, \bar{\mu}) \in \mathcal{U}^{\mathrm{slc}}_\varepsilon(\underline{0}, \hat{\mu})$ and $(\bar{\underline{r}}', \bar{\mu}') \in \mathcal{U}^{\mathrm{slc}}_{\varepsilon'}(\underline{0}, \hat{\mu}')$ with*

$$\theta(\bar{\underline{r}}, \bar{\mu}) = \theta'(\bar{\underline{r}}', \bar{\mu}').$$

Then there are $\delta, \delta' > 0$ such that for neighborhoods $B_\delta(\bar{\underline{r}}, \bar{\mu}) \subset \mathcal{U}^{\mathrm{slc}}_\varepsilon(\underline{0}, \hat{\mu})$ and $B_{\delta'}(\bar{\underline{r}}', \bar{\mu}') \subset \mathcal{U}^{\mathrm{slc}}_{\varepsilon'}(\underline{0}, \hat{\mu}')$ in neighborhood slices, there exists a smooth map

$$\lambda : B_\delta(\bar{\underline{r}}, \bar{\mu}) \to B_{\delta'}(\bar{\underline{r}}', \bar{\mu}')$$

with $\lambda(\underline{\bar{r}}, \bar{\mu}) = (\underline{\bar{r}}', \bar{\mu}')$ *such that for* $(\underline{r}, \mu) \in B_\delta(\underline{\bar{r}}, \bar{\mu})$ *we have*

$$\theta(\underline{r}, \mu) = \theta'(\lambda(\underline{r}, \mu)).$$

We shall use the following result to show the injectivity of each chart θ in (5.13). We shall prove it in Section 7.2.2 by using the estimate given by Theorem 7.12.

Theorem 7.21. *Let R be any gluing profile. There exists an injectivity radius $\hat{\varepsilon} = \hat{\varepsilon}_{\text{inj}}(\hat{\mu}) > 0$ such that if $(\underline{r}, \mu), (\underline{r}', \mu') \in \mathcal{U}^{\text{slc}}_{\hat{\varepsilon}}(\underline{0}, \hat{\mu})$ satisfy*

$$\theta(\underline{r}, \mu) = \theta(\underline{r}', \mu'),$$

then we have $(\underline{r}, \mu) = (\underline{r}', \mu')$.

We now show that the collection of neighborhoods $\mathfrak{U}_\varepsilon(\sigma; \hat{\mu}; R)$ in (5.10) forms a basis for the Deligne-Mumford space.

Proof of Theorem 5.11. Fix a gluing profile R. We now show that the collection $\{\mathfrak{U}_\varepsilon(\sigma; \hat{\mu}; R)\}$ forms a basis. For convenience, we drop the gluing profile R in our notation. Clearly, this collection covers the Deligne-Mumford space $\mathfrak{DM}$. Suppose there is an element σ in the intersection $\mathfrak{U}_{\varepsilon'}(\sigma'; \hat{\mu}') \cap \mathfrak{U}_{\varepsilon''}(\sigma''; \hat{\mu}'')$. Let $\bar{\mu}$ be a representative of σ. We claim that there exists $\delta > 0$ with

$$\mathfrak{U}_\delta(\sigma; \bar{\mu}) \subset \mathfrak{U}_{\varepsilon'}(\sigma'; \hat{\mu}'), \tag{7.23}$$

and analogously there is $\delta > 0$ with $\mathfrak{U}_\delta(\sigma; \bar{\mu}) \subset \mathfrak{U}_{\varepsilon''}(\sigma''; \hat{\mu}'')$. Thus the neighborhood $\mathfrak{U}_\delta(\sigma; \bar{\mu})$ lies in the intersection $\mathfrak{U}_{\varepsilon'}(\sigma'; \hat{\mu}') \cap \mathfrak{U}_{\varepsilon''}(\sigma''; \hat{\mu}'')$, proving the collection forms a basis. We now find $\delta > 0$ satisfying (7.23). Firstly, since the element σ lies in $\mathfrak{U}_{\varepsilon'}(\sigma'; \hat{\mu}')$, there is $(\underline{\bar{r}}', \bar{\mu}') \in \mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\mu}')$ with $[\#](\underline{0}, \bar{\mu}) = \sigma = [\#](\underline{\bar{r}}', \bar{\mu}')$. We choose $\delta' > 0$ with $\mathcal{U}_{\delta'}(\underline{\bar{r}}', \bar{\mu}') \subset \mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\mu}')$. Then by Theorem 7.19, there exists $\delta > 0$ with

$$[\#](\mathcal{U}_\delta(\underline{0}, \bar{\mu})) \subset \theta'(\mathcal{U}^{\text{slc}}_{\delta'}(\underline{\bar{r}}', \bar{\mu}')).$$

Since the neighborhood slice is contained in the neighborhood, we have

$$[\#](\mathcal{U}_\delta(\underline{0}, \bar{\mu})) \subset \theta'(\mathcal{U}^{\text{slc}}_{\delta'}(\underline{\bar{r}}', \bar{\mu}')) \subset [\#](\mathcal{U}_{\delta'}(\underline{\bar{r}}', \bar{\mu}')) \subset [\#](\mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\mu}')),$$

where by definition the first set is the neighborhood $\mathfrak{U}_\delta(\sigma; \bar{\mu})$ and the last set the neighborhood $\mathfrak{U}_{\varepsilon'}(\sigma'; \hat{\mu}')$. This proves (7.23).

The Hausdorff property of this topology is proved after Proposition 7.1. Furthermore, the topology is independent of the choices of gluing profiles R because for two gluing profiles R and R', the transition $R'^{-1} \circ R$ is a homeomorphism. This finishes the proof of the theorem. □

We now show that the collection of maps θ in (5.13) forms a smooth atlas for $\mathfrak{DM}$.

Proof of Theorem 5.17. We show that for a given representative $\hat{\mu}$ of an element of $\mathfrak{DM}$, there exists $\varepsilon > 0$ such that the map

$$\theta : \mathcal{U}^{\text{slc}}_\varepsilon(\underline{0}, \hat{\mu}) \to \mathfrak{DM}, \quad (\underline{r}, \mu) \mapsto [\#](\underline{r}, \mu).$$

is a homeomorphism onto its image. First of all, it follows from Theorem 7.21 that for $\varepsilon < \hat{\varepsilon}_{\text{inj}}(\hat{\mu})$, the map θ is injective.

We now prove the continuity of θ. Let $\mathfrak{V}$ be an arbitrary open set in $\mathfrak{DM}$, we now show that $\theta^{-1}(\mathfrak{V})$ is open. It suffices to show that for a given $(\underline{\bar{r}}, \bar{\mu}) \in \theta^{-1}(\mathfrak{V})$, there exists a neighborhood $B_\delta(\underline{\bar{r}}, \bar{\mu}) \subset \mathcal{U}_\varepsilon^{\text{slc}}(\underline{0}, \hat{\mu})$ in the neighborhood slice with

$$B_\delta(\underline{\bar{r}}, \bar{\mu}) \subset \theta^{-1}(\mathfrak{V}). \tag{7.24}$$

Note that each neighborhood $B_\delta(\underline{\bar{r}}, \bar{\mu})$ could be thought of as a neighborhood slice $\mathcal{U}_\delta^{\text{slc}}(\underline{\bar{r}}, \bar{\mu})$ with the same fixed automorphism components as $\mathcal{U}_\varepsilon^{\text{slc}}(\underline{0}, \hat{\mu})$. Denote $\sigma := \theta(\underline{\bar{r}}, \bar{\mu})$. Since $\mathfrak{V}$ is open, there is a basis element $\mathfrak{U}_\delta(\sigma; \bar{\mu}')$ with $\mathfrak{U}_\delta(\sigma; \bar{\mu}') \subset \mathfrak{V}$. Note that by definition we have $\theta(\underline{\bar{r}}, \bar{\mu}) = \sigma = [\#](\underline{0}, \bar{\mu}')$. Applying Theorem 7.19, there exists $\delta > 0$ with

$$[\#](\mathcal{U}_\delta(\underline{\bar{r}}, \bar{\mu})) \subset \theta'(\mathcal{U}_{\delta'}^{\text{slc}}(\underline{0}, \bar{\mu}')).$$

Since the neighborhood slice is contained in the neighborhood, we have $\theta(B_\delta(\underline{\bar{r}}, \bar{\mu})) \subset [\#](\mathcal{U}_\delta(\underline{\bar{r}}, \bar{\mu})) \subset \theta'(\mathcal{U}_{\delta'}^{\text{slc}}(\underline{0}, \bar{\mu}')) \subset [\#](\mathcal{U}_{\delta'}(\underline{0}, \bar{\mu}'))$, where the last set is the neighborhood $\mathfrak{U}_{\delta'}(\sigma; \bar{\mu}')$ contained in $\mathfrak{V}$. This proves (7.24).

We now prove the continuity of θ^{-1}. Let V be an arbitrary open subset of the domain $\mathcal{U}_\varepsilon^{\text{slc}}(\underline{0}, \hat{\mu})$, we now show that the image $\theta(V)$ is open. It suffices to prove that for a given $\sigma \in \theta(V)$ with a representative $\bar{\mu}'$, there exists $\delta > 0$ with

$$\mathfrak{U}_\delta(\sigma; \bar{\mu}') \subset \theta(V). \tag{7.25}$$

Firstly, since the element σ lies in $\theta(V)$, there is $(\underline{\bar{r}}, \bar{\mu}) \in V$ with $[\#](\underline{0}, \bar{\mu}') = \sigma = \theta(\underline{\bar{r}}, \bar{\mu})$. We choose a neighborhood $B_\delta(\underline{\bar{r}}, \bar{\mu}) \subset \mathcal{U}_\varepsilon^{\text{slc}}(\underline{0}, \hat{\mu})$ in the neighborhood slice with $B_\delta(\underline{\bar{r}}, \bar{\mu}) \subset V$. As before, $B_\delta(\underline{\bar{r}}, \bar{\mu})$ could be thought of as a neighborhood slice $\mathcal{U}_\delta^{\text{slc}}(\underline{\bar{r}}, \bar{\mu})$. Then by Theorem 7.19 (with the primed and the un-primed swapped), there exists $\delta' > 0$ with

$$\mathfrak{U}_{\delta'}(\sigma; \bar{\mu}') = [\#](\mathcal{U}_{\delta'}(\underline{0}, \bar{\mu}')) \subset \theta(\mathcal{U}_\delta^{\text{slc}}(\underline{\bar{r}}, \bar{\mu})).$$

The last set is contained in the image $\theta(V)$. This proves (7.25).

Lastly, we prove the transition map between two charts is smooth. Suppose there are two charts $\theta : \mathcal{U}_\varepsilon^{\text{slc}}(\underline{0}, \hat{\mu}) \to \mathfrak{DM}$ and $\theta' : \mathcal{U}_{\varepsilon'}^{\text{slc}}(\underline{0}, \hat{\mu}') \to \mathfrak{DM}$ whose images have a non-empty overlap $\text{im}(\theta) \cap \text{im}(\theta') \neq \emptyset$. Then the smoothness of the transition map $\theta'^{-1} \circ \theta$ follows directly from the smoothness of λ in Theorem 7.20. $\square$

We now state the openness result which implies Theorem 7.19 and Theorem 7.20. We shall prove this theorem in Section 7.2.1.

Theorem 7.22. *Let R be any gluing profile. Assume there is a pair $(\underline{\bar{r}}, \bar{\mu}) \in \mathcal{U}_\varepsilon(\underline{0}, \hat{\mu})$ and $(\underline{\bar{r}}', \bar{\mu}') \in \mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\mu}')$, and a biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ with*

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\mu})) = \#(\underline{\bar{r}}', \bar{\mu}').$$

Then there exist $\delta, \delta' > 0$ and a continuous map

$$\lambda : \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\mu}) \to \mathcal{U}_{\delta'}^{\text{slc}}(\underline{\bar{r}}', \bar{\mu}')$$

with $\lambda(\underline{\bar{r}},\bar{\mu}) = (\underline{\bar{r}}',\bar{\mu}')$*, and for each* $(\underline{r},\mu) \in \mathcal{U}_\delta(\underline{\bar{r}},\bar{\mu})$ *there is a biholomorphism* $\left(\zeta_{\underline{r}}, \underline{\psi}_{(\underline{r},\underline{x},\underline{o})}\right)$ *with*

$$\left(\zeta_{\underline{r}}, \underline{\psi}_{(\underline{r},\underline{x},\underline{o})}\right)(\#(\underline{r},\mu)) = \#(\lambda(\underline{r},\mu)).$$

This family of biholomorphisms satisfies $\left(\zeta_{\underline{\bar{r}}}, \underline{\psi}_{(\underline{\bar{r}},\underline{\bar{x}},\underline{\bar{o}})}\right) = (\bar{\zeta}, \underline{\bar{\psi}})$*, with* $\zeta_{\underline{\bar{r}}}$ *covering each* $\zeta_{\underline{r}}$ *in the sense that (7.4) commutes. Moreover, for vertices* $\hat{\mathrm{v}}$ *and* $\hat{\mathrm{v}}'$ *satisfying* $\bar{\zeta}([\hat{\mathrm{v}}]_{\underline{\bar{r}}}) = [\hat{\mathrm{v}}']_{\underline{\bar{r}}'}$*, we denote* $(\underline{r}',\mu') := \lambda(\underline{r},\mu)$*,* $\mathrm{v} := [\hat{\mathrm{v}}]_{\underline{r}}$*, and* $\mathrm{v}' := [\hat{\mathrm{v}}]_{\underline{r}'}$*, then the biholomorphism in standard model*

$$\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}_{(\underline{r},\underline{x},\underline{o})} := (\Phi^{\hat{\mathrm{v}}'}_{\mathrm{v}'})^{-1} \circ \psi_{(\underline{r},\underline{x},\underline{o}),\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{\mathrm{v}},$$

depends continuously on $(\underline{r},\underline{x},\underline{o})$*.*

Let R *be given by the exponential gluing profile* $\varphi(r) = e^{\frac{1}{r}} - e$*. Then the map* $\lambda : \mathcal{U}_\delta(\underline{\bar{r}},\bar{\mu}) \to \mathcal{U}^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}',\bar{\mu}')$ *is smooth, and the biholomorphism in standard model* $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}_{(\underline{r},\underline{x},\underline{o})}$ *depends smoothly on* $(\underline{r},\underline{x},\underline{o})$*.*

Proof of Theorem 7.19. The result follows directly from the map λ in Theorem 7.22 and its continuity. □

Proof of Theorem 7.20. Define the neighborhood slice $\mathcal{U}^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}',\bar{\mu}')$ by fixing the same automorphism components as the neighborhood slice $\mathcal{U}^{\mathrm{slc}}_{\varepsilon'}(\underline{0},\hat{\mu}')$ (see Definition 5.14). By Theorem 7.22, there is a smooth map $\lambda : \mathcal{U}_\delta(\underline{\bar{r}},\bar{\mu}) \to \mathcal{U}^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}',\bar{\mu}')$ such that for each $(\underline{r},\mu) \in \mathcal{U}_\delta(\underline{\bar{r}},\bar{\mu})$, we have $\theta(\underline{r},\mu) = \theta'(\lambda(\underline{r},\mu))$. This smooth map induces our desired map as follows. The neighborhood $B_\delta(\underline{\bar{r}},\bar{\mu}) \subset \mathcal{U}^{\mathrm{slc}}_{\varepsilon}(\underline{0},\hat{\mu})$ is a submanifold of the neighborhood $\mathcal{U}_\delta(\underline{\bar{r}},\bar{\mu})$; by our choice of the neighborhood slice, $\mathcal{U}^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}',\bar{\mu}')$ is the same as the neighborhood $B_{\delta'}(\underline{\bar{r}}',\bar{\mu}')$. Therefore the map obtained from Theorem 7.22 induces a smooth map $\lambda : B_\delta(\underline{\bar{r}},\bar{\mu}) \to B_{\delta'}(\underline{\bar{r}}',\bar{\mu}')$. This proves the theorem. □

7.2.1. *Proof of Theorem 7.22.*

We first assume R is given by the logarithm gluing profile $R(r) = -\ln(r)$.

By assumption, for each main vertex $\bar{\mathrm{v}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$ and $\bar{\mathrm{v}}' := \bar{\zeta}(\bar{\mathrm{v}})$, we have a biholomorphism $\bar{\psi}_{\bar{\mathrm{v}}} : \#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{D})_{\bar{\mathrm{v}}} \to \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{D})_{\bar{\mathrm{v}}'}$ with

$$\bar{\psi}_{\bar{\mathrm{v}}}(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{o}})_{\bar{\mathrm{v}}}) = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}',\underline{\bar{o}}')_{\bar{\mathrm{v}}'}. \tag{7.26}$$

For every pair of corresponding vertices $\bar{\mathrm{v}}$ and $\bar{\mathrm{v}}'$, we fix an arbitrary pair of factorization $\bar{\tau}_{\bar{\mathrm{v}}} : \#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{D})_{\bar{\mathrm{v}}} \to D$ and $\bar{\tau}'_{\bar{\mathrm{v}}'} : \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{D})_{\bar{\mathrm{v}}'} \to D$ with

$$\bar{\tau}'^{-1}_{\bar{\mathrm{v}}'} \circ \bar{\tau}_{\bar{\mathrm{v}}} = \bar{\psi}_{\bar{\mathrm{v}}}.$$

Thus we have

$$\bar{\tau}_{\bar{\mathrm{v}}}(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{x}},\underline{\bar{o}})_{\bar{\mathrm{v}}}) = \bar{\tau}'_{\bar{\mathrm{v}}'}(\#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{x}}',\underline{\bar{o}}')_{\bar{\mathrm{v}}'}). \tag{7.27}$$

For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$, we denote the biholomorphism $\bar{\tau}_{[\hat{\mathrm{v}}]_{\underline{\bar{r}}}}$ in the standard model on $D_{\hat{\mathrm{v}}}$ by

$$\bar{\xi}^{\hat{\mathrm{v}}} := \bar{\tau}_{[\hat{\mathrm{v}}]_{\underline{\bar{r}}}} \circ \bar{\Phi}^{\hat{\mathrm{v}}}_{[\hat{\mathrm{v}}]_{\underline{\bar{r}}}}, \tag{7.28}$$

where $\bar{\Phi}$ is the standard pullback in Definition 7.4. We denote this tuple of disk automorphisms by $\underline{\bar{\xi}} := (\bar{\xi}^{\hat{\mathrm{v}}})_{\hat{\mathrm{v}}\in\hat{\mathrm{V}}^{\mathrm{m}}}$, and the δ-neighborhood of $\underline{\bar{\xi}}$ by $\mathcal{U}_\delta(\underline{\bar{\xi}})$. Similarly, we denote its prime counterpart by $\underline{\bar{\xi}}' := (\bar{\xi}'^{\hat{\mathrm{v}}'})_{\hat{\mathrm{v}}'\in\hat{\mathrm{V}}'^{\mathrm{m}}}$.

We now construct a map α which will induce the desired map λ. We first specify the domain and the target of α. We define sets $M_\delta(\underline{\bar{r}},\underline{\bar{\ell}},\underline{\bar{x}},\underline{\bar{o}},\underline{\bar{\xi}})$ and $M^{\mathrm{slc}}_\delta(\underline{\bar{r}},\underline{\bar{\ell}},\underline{\bar{x}},\underline{\bar{o}},\underline{\bar{\xi}})$ around the point $(\underline{\bar{r}},\underline{\bar{\ell}},\underline{\bar{x}},\underline{\bar{o}},\underline{\bar{\xi}})$. They are similar to the neighborhood $\mathcal{U}_\delta(\underline{\bar{r}},\bar{\mu})$ and neighborhood slice $\mathcal{U}^{\mathrm{slc}}_\delta(\underline{\bar{r}},\bar{\mu})$, along with a neighborhood of disk automorphisms $\underline{\bar{\xi}}$ which satisfy certain constraints.

$$M_\delta(\underline{\bar{r}},\underline{\bar{\ell}},\underline{\bar{x}},\underline{\bar{o}},\underline{\bar{\xi}}) := \left\{(\underline{r},\underline{\ell},\underline{x},\underline{o},\underline{\xi}) \,|\, (1),(2)\right\}. \tag{7.29}$$

1. We have $(\underline{r},(\hat{\mathrm{T}},\underline{\ell},\underline{x},\underline{o})) \in \mathcal{U}_\delta(\underline{\bar{r}},\bar{\mu})$, and $\underline{\xi}\in\mathcal{U}_\delta(\underline{\bar{\xi}})$.
2. We impose the constraint $\xi^{\hat{\mathrm{v}}^-}\circ\phi^{\hat{\mathrm{e}}}(r_{\hat{\mathrm{e}}},\underline{x})\circ(\xi^{\hat{\mathrm{v}}^+})^{-1} = \mathrm{Id}$ for all edges $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-,\hat{\mathrm{v}}^+)\in\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$, where $\phi^{\hat{\mathrm{e}}}$ is the shift map in Definition 7.2.

Define the set $M^{\mathrm{slc}}_\delta(\underline{\bar{r}},\underline{\bar{\ell}},\underline{\bar{x}},\underline{\bar{o}},\underline{\bar{\xi}})$ the same as above except replacing the neighborhood $\mathcal{U}_\delta(\underline{\bar{r}},\bar{\mu})$ by the neighborhood slice $\mathcal{U}^{\mathrm{slc}}_\delta(\underline{\bar{r}},\bar{\mu})$. Note that the point $(\underline{\bar{r}},\underline{\bar{\ell}},\underline{\bar{x}},\underline{\bar{o}},\underline{\bar{\xi}})$ indeed belongs to the set, because by construction (7.28) the constraint $\bar{\xi}^{\hat{\mathrm{v}}^-}\circ\phi^{\hat{\mathrm{e}}}(\bar{r}_{\hat{\mathrm{e}}},\underline{\bar{x}})\circ(\bar{\xi}^{\hat{\mathrm{v}}^+})^{-1} = \mathrm{Id}$ is satisfied for all edges $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-,\hat{\mathrm{v}}^+)\in\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$.

We show that there exists a map α, from which we shall derive the map λ in the statement of the theorem.

Claim 7.23. *There exist $\delta,\delta'>0$, and a continuous map*

$$\begin{aligned}\alpha : M_\delta(\underline{\bar{r}},\underline{\bar{\ell}},\underline{\bar{x}},\underline{\bar{o}},\underline{\bar{\xi}}) &\to M^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}',\underline{\bar{\ell}}',\underline{\bar{x}}',\underline{\bar{o}}',\underline{\bar{\xi}}')\\ (\underline{r},\underline{\ell},\underline{x},\underline{o},\underline{\xi}) &\mapsto (\underline{r}',\underline{\ell}',\underline{x}',\underline{o}',\underline{\xi}')\end{aligned}$$

such that the domain $M_\delta(\underline{\bar{r}},\underline{\bar{\ell}},\underline{\bar{x}},\underline{\bar{o}},\underline{\bar{\xi}})$ and the target $M^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}',\underline{\bar{\ell}}',\underline{\bar{x}}',\underline{\bar{o}}',\underline{\bar{\xi}}')$ are manifolds; for $(\underline{r}',\underline{\ell}',\underline{x}',\underline{o}',\underline{\xi}') := \alpha(\underline{r},\underline{\ell},\underline{x},\underline{o},\underline{\xi})$, there is a biholomorphism $(\zeta,\underline{\psi})$ with

$$(\zeta,\underline{\psi})(\#(\underline{r},(\hat{\mathrm{T}},\underline{\ell},\underline{x},\underline{o})) = \#(\underline{r}',(\hat{\mathrm{T}}',\underline{\ell}',\underline{x}',\underline{o}')).$$

Moreover, the biholomorphism in standard model is given by

$$\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} = (\xi'^{\hat{\mathrm{v}}'})^{-1}\circ\xi^{\hat{\mathrm{v}}}.$$

In order to construct the map α, we first fix a set of gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}$. We then construct a *smooth* map $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$ on the subset of $M_\delta(\underline{\bar{r}},\underline{\bar{\ell}},\underline{\bar{x}},\underline{\bar{o}},\underline{\bar{\xi}})$ whose gluing parameters $\underline{r}$ belong to the quadrant

$$\{\underline{r}\,|\, r_{\hat{\mathrm{e}}}\geq 0 \text{ for } \hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{g}} \text{ and } r_{\hat{\mathrm{e}}}\leq 0 \text{ for } \hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{nd}}\setminus\hat{\mathrm{E}}^{\mathrm{g}}\}. \tag{7.30}$$

Note that (7.30) is the closure of the set $\{\underline{r}\,|\,\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} = \hat{\mathrm{E}}^{\mathrm{g}}\}$, where in (7.30) the parameter $r_{\hat{\mathrm{e}}}$ is allowed to vanish for a gluing edge $\hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{g}}$. To prove that this defines a continuous map α, we then verify that the set of maps $\{\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}\}$ matches pairwise at the intersection of their domains.

Fix a set of gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}$ with $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}\subset\hat{\mathrm{E}}^{\mathrm{g}}\subset\{\hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{nd}}\,|\,\bar{r}_{\hat{\mathrm{e}}}\geq 0\}$ (see (7.2)) and define the additional gluing edges by $\hat{\mathrm{E}}^{\mathrm{ag}} := \hat{\mathrm{E}}^{\mathrm{g}}\setminus\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$ (see (7.3)).

Applying the bijection $\bar{\chi}$ in (7.19) we define the corresponding additional gluing edges by

$$\hat{\mathrm{E}}'^{\mathrm{ag}} := \bar{\chi}(\hat{\mathrm{E}}^{\mathrm{ag}}) \tag{7.31}$$

and the corresponding gluing edges by $\hat{\mathrm{E}}'^{\mathrm{g}} := \hat{\mathrm{E}}'^{\mathrm{g}}_{\underline{\bar{r}}'} \sqcup \hat{\mathrm{E}}'^{\mathrm{ag}}$.

We decompose the gluing parameters $\underline{\bar{r}}$ and $\underline{\bar{r}}'$ into non-gluing and gluing components

$$\underline{\bar{r}} = \left((\bar{s}_{\hat{\mathrm{e}}})_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}}, (\bar{t}_{\hat{\mathrm{e}}})_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}} \right), \quad \underline{\bar{r}}' = \left((\bar{s}'_{\hat{\mathrm{e}}'})_{\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{nd}} \setminus \hat{\mathrm{E}}'^{\mathrm{g}}}, (\bar{t}'_{\hat{\mathrm{e}}'})_{\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{g}}} \right). \tag{7.32}$$

By construction we have $\bar{s}_{\hat{\mathrm{e}}} \leq 0$ and $\bar{t}_{\hat{\mathrm{e}}} \geq 0$, and $\bar{s}'_{\hat{\mathrm{e}}'} \leq 0$ and $\bar{t}'_{\hat{\mathrm{e}}'} \geq 0$. Writing $\underline{\bar{r}} = (\underline{\bar{s}}, \underline{\bar{t}})$ and $\underline{\bar{r}}' = (\underline{\bar{s}}', \underline{\bar{t}}')$, we shall construct a *smooth* map

$$\begin{aligned} \alpha_{\hat{\mathrm{E}}^{\mathrm{g}}} : M_\delta(\underline{\bar{s}}, \underline{\bar{\ell}}) \times M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) &\to M_{\delta'}(\underline{\bar{s}}', \underline{\bar{\ell}}') \times M^{\mathrm{slc}}_{\delta'}(\underline{\bar{t}}', \underline{\bar{x}}', \underline{\bar{o}}', \underline{\bar{\xi}}') \\ ((\underline{s}, \underline{\ell}), (\underline{t}, \underline{x}, \underline{o}, \underline{\xi})) &\mapsto ((\underline{s}', \underline{\ell}'), (\underline{t}', \underline{x}', \underline{o}', \underline{\xi}')), \end{aligned} \tag{7.33}$$

such that all parameters $\underline{t}$ are positive if and only if all parameters $\underline{t}'$ are positive, and in that case there is a biholomorphism $(\zeta, \underline{\psi})$ with

$$(\zeta, \underline{\psi})(\#((\underline{s}, \underline{t}), (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{o})) = \#((\underline{s}', \underline{t}'), (\hat{\mathrm{T}}', \underline{\ell}', \underline{x}', \underline{o}')). \tag{7.34}$$

Moreover, the biholomorphism in standard model is given by

$$\psi^{\hat{\mathrm{v}}, \hat{\mathrm{v}}'} = (\xi'^{\hat{\mathrm{v}}'})^{-1} \circ \xi^{\hat{\mathrm{v}}}. \tag{7.35}$$

We first construct this smooth map $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$, and then verify that the set of maps $\{\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}\}$ matches pairwise at the intersection of their domains. By construction we have the one-to-one correspondence of additional gluing edges $\bar{\chi}(\hat{\mathrm{E}}^{\mathrm{ag}}) = \hat{\mathrm{E}}'^{\mathrm{ag}}$. Then it follows from Lemma 7.17 that there is an ordered tree isomorphism

$$\zeta : \#_{\hat{\mathrm{E}}^{\mathrm{g}}}(\hat{\mathrm{T}}) \to \#_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\hat{\mathrm{T}}'), \tag{7.36}$$

such that $\bar{\zeta}$ covers ζ in the sense that (7.4) commutes. We construct the *non-gluing components* $M_\delta(\underline{\bar{s}}, \underline{\bar{\ell}}) \to M_{\delta'}(\underline{\bar{s}}', \underline{\bar{\ell}}')$ of the map $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$, and then the *gluing components* $M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) \to M^{\mathrm{slc}}_{\delta'}(\underline{\bar{t}}', \underline{\bar{x}}', \underline{\bar{o}}', \underline{\bar{\xi}}')$ of the map $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$.

We first construct the non-gluing components of $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$. Similar to (7.29), we define a set $M_\delta(\underline{\bar{s}}, \underline{\bar{\ell}})$ around the point $(\underline{\bar{s}}, \underline{\bar{\ell}})$.

$$M_\delta(\underline{\bar{s}}, \underline{\bar{\ell}}) := \{(\underline{s}, \underline{\ell}) \,|\, (1), (2)\}\,. \tag{7.37}$$

(1) We have $\underline{s} \in \mathcal{U}_\delta(\underline{\bar{s}})$, and $s_{\hat{\mathrm{e}}} \leq 0$ for each non-gluing nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}$.

(2) We have $\ell_{\hat{\mathrm{e}}} \in (\hat{\ell}_{\hat{\mathrm{e}}} - \varepsilon, \hat{\ell}_{\hat{\mathrm{e}}} + \varepsilon) \cap [0,1]$ for each $\hat{\mathrm{e}} \in \hat{\mathrm{E}}$. Moreover, we fix $\ell_{\hat{\mathrm{e}}} = 0$ for all nodal edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}}$, and fix $\ell_{\hat{\mathrm{e}}} = 1$ for edges $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$ with either $\hat{\mathrm{v}}^-$ or $\hat{\mathrm{v}}^+$ being a critical vertex (see Definition 5.8).

We now construct the non-gluing components of $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$

$$M_\delta(\underline{\bar{s}}, \underline{\bar{\ell}}) \to M_{\delta'}(\underline{\bar{s}}', \underline{\bar{\ell}}'), \quad (\underline{s}, \underline{\ell}) \mapsto (\underline{s}', \underline{\ell}') \tag{7.38}$$

such that equation (7.34) holds. Since the bijection $\bar{\chi}$ maps $\hat{\mathrm{E}}^{\mathrm{ag}}$ to $\hat{\mathrm{E}}'^{\mathrm{ag}}$ by construction, we have a one-to-one correspondence of non-gluing edges $\bar{\chi}(\hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}) = \bar{\chi}((\hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}) \setminus \hat{\mathrm{E}}^{\mathrm{ag}}) = (\hat{\mathrm{E}}' \setminus \hat{\mathrm{E}}'^{\mathrm{g}}_{\underline{\bar{r}}'}) \setminus \hat{\mathrm{E}}'^{\mathrm{ag}} = \hat{\mathrm{E}}' \setminus \hat{\mathrm{E}}'^{\mathrm{g}}$. For corresponding edges $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$, we denote their glued edges by e and e' in (4.6). In order for equation (7.34) to hold, we need to solve the equation

$$\#_{\underline{s}}(\underline{\ell})_{\mathrm{e}} = \#_{\underline{s}'}(\underline{\ell}')_{\mathrm{e}'}. \tag{7.39}$$

Decompose the equation $\bar{\chi}(\hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}) = \hat{\mathrm{E}}' \setminus \hat{\mathrm{E}}'^{\mathrm{g}}$ as

$$\bar{\chi}((\hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{nd}}) \sqcup (\hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}})) = (\hat{\mathrm{E}}' \setminus \hat{\mathrm{E}}'^{\mathrm{nd}}) \sqcup (\hat{\mathrm{E}}'^{\mathrm{nd}} \setminus \hat{\mathrm{E}}'^{\mathrm{g}}).$$

Thus equation (7.39) gives four possible forms of equations. Depending on where $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$ lie in the above partition, by the construction of displaced length in (5.6) and (5.7) we can have

$$\ell_{\hat{\mathrm{e}}} = \ell'_{\hat{\mathrm{e}}'}, \ \ell_{\hat{\mathrm{e}}} = \frac{-\iota(s'_{\hat{\mathrm{e}}'})}{1 - \iota(s'_{\hat{\mathrm{e}}'})}, \ \frac{-\iota(s_{\hat{\mathrm{e}}})}{1 - \iota(s_{\hat{\mathrm{e}}})} = \ell'_{\hat{\mathrm{e}}'}, \text{ or } s_{\hat{\mathrm{e}}} = s'_{\hat{\mathrm{e}}'}, \tag{7.40}$$

where $\iota : (-\infty, 0] \to (-\infty, 0]$ is a smooth increasing function chosen before Remark 4.2. There exist $\delta, \delta' > 0$ such that for every $(\underline{s}, \underline{\ell}) \in M_\delta(\underline{\bar{s}}, \underline{\bar{\ell}})$ we can find $(\underline{s}', \underline{\ell}') \in M_{\delta'}(\underline{\bar{s}}', \underline{\bar{\ell}}')$ that solve the above equation. Moreover, the solution $(\underline{s}', \underline{\ell}')$ depends smoothly on $(\underline{s}, \underline{\ell})$. In particular, for a non-gluing nodal edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}$ with $\bar{s}_{\hat{\mathrm{e}}} = 0$, by Lemma 7.16 we have $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{nd}} \setminus \hat{\mathrm{E}}'^{\mathrm{g}}$ with $\bar{s}'_{\hat{\mathrm{e}}'} = 0$. In this case, (7.39) gives us

$$s_{\hat{\mathrm{e}}} = s'_{\hat{\mathrm{e}}'}. \tag{7.41}$$

We now construct the gluing components of $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$. Similar to (7.29), we define sets $M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$ and $M^{\mathrm{slc}}_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$ around the point $(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$.

$$M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) := \left\{ (\underline{t}, \underline{x}, \underline{o}, \underline{\xi}) \,|\, (1), (2), (3) \right\}. \tag{7.42}$$

(1) We have $\underline{t} \in \mathcal{U}_\delta(\underline{\bar{t}})$, and $t_{\hat{\mathrm{e}}} \geq 0$ for each gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}$ (in particular $t_{\hat{\mathrm{e}}} \in [0, \delta)$ for each additional gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$).
(2) We have $(\underline{x}, \underline{o}) \in \mathcal{U}_\delta(\underline{\bar{x}}, \underline{\bar{o}})$ and $\underline{\xi} \in \mathcal{U}_\delta(\underline{\bar{\xi}})$.
(3) We impose the constraint $\xi^{\hat{\mathrm{v}}^-} \circ \phi^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}) \circ (\xi^{\hat{\mathrm{v}}^+})^{-1} = \mathrm{Id}$ for all edges $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$, where $\phi^{\hat{\mathrm{e}}}$ is the shift map in Definition 7.2.

Define the set $M^{\mathrm{slc}}_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$ the same as a above except replacing the neighborhood $\mathcal{U}_\delta(\underline{\bar{x}}, \underline{\bar{o}})$ by the neighborhood slice $\mathcal{U}^{\mathrm{slc}}_\delta(\underline{\bar{x}}, \underline{\bar{o}})$. We now construct the gluing components of the map $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$

$$M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) \to M^{\mathrm{slc}}_{\delta'}(\underline{\bar{t}}', \underline{\bar{x}}', \underline{\bar{o}}', \underline{\bar{\xi}}'), \quad (\underline{t}, \underline{x}, \underline{o}, \underline{\xi}) \mapsto (\underline{t}', \underline{x}', \underline{o}', \underline{\xi}') \tag{7.43}$$

along with the biholomorphism $\underline{\psi}$ such that equation (7.34) holds. Towards that end, we construct a map Ω in the next lemma. Certain components of the map Ω take value in the compactified disk automorphism group $\overline{\mathrm{Aut}}(D)$

(see Remark 13.2); such a component results from a potentially zero gluing parameter $t_{\hat{\mathrm{e}}}$ for an additional gluing edge $\hat{\mathrm{e}}$.

Lemma 7.24. *Define* $\Omega : M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) \to \overline{\mathrm{Aut}}(D)^{\hat{\mathrm{E}}^{\mathrm{ag}}} \times \underline{MP}(\partial D) \times \underline{MP}(D^\circ)$ *as follows.*

$$\begin{aligned}&\Omega(\underline{t}, \underline{x}, \underline{o}, \underline{\xi})\\ &= \Big(\big(\xi^{\hat{\mathrm{v}}^-} \circ \phi^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}) \circ (\xi^{\hat{\mathrm{v}}^+})^{-1}\big)_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}}, \big(\xi^{\hat{\mathrm{v}}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})\big)_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}, \hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}}, \big(\xi^{\hat{\mathrm{v}}}(\underline{o}_{\hat{\mathrm{v}}})\big)_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}} \Big).\end{aligned}$$

Then there exists $\delta > 0$ with the following properties.

(1) $M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$ *is a manifold with boundary, and the map Ω is a submersion onto its image.*

(2) $M_\delta^{\mathrm{slc}}(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$ *is a manifold with boundary, and the map Ω restricted to $M_\delta^{\mathrm{slc}}(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$ is a diffeomorphism onto its image.*

(3) *For each additional gluing edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+) \in \hat{\mathrm{E}}^{\mathrm{ag}}$ with vanishing parameter $t_{\hat{\mathrm{e}}} = 0$, the first component $\xi^{\hat{\mathrm{v}}^-} \circ \phi^{\hat{\mathrm{e}}}(0, \underline{x}) \circ (\xi^{\hat{\mathrm{v}}^+})^{-1}$ lies on the boundary of the compactification $\overline{\mathrm{Aut}}(D)$, with (η, a) coordinates (see Remark 13.2) given by*

$$(\eta, a) = \Big(\xi^{\hat{\mathrm{v}}^-}(x_{\hat{\mathrm{e}}}^-) \cdot \big(\xi^{\hat{\mathrm{v}}^+}(x_{\hat{\mathrm{e}}}^+)\big)^{-1},\ \xi^{\hat{\mathrm{v}}^+}(x_{\hat{\mathrm{e}}}^+) \Big).$$

We shall prove this lemma later in this section. We claim that with the correct ordering, we have

$$\Omega(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) = \Omega'(\underline{\bar{t}}', \underline{\bar{x}}', \underline{\bar{o}}', \underline{\bar{\xi}}'). \tag{7.44}$$

Indeed, for each additional gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$, we denote its corresponding additional gluing edge by $\hat{\mathrm{e}}' := \bar{\chi}(\hat{\mathrm{e}})$. It follows from equation (7.27) and definition (7.28) that we have $\bar{\xi}^{\hat{\mathrm{v}}^-}(\bar{x}_{\hat{\mathrm{e}}}^-) = \bar{\xi}'^{\hat{\mathrm{v}}'^-}(\bar{x}'^-_{\hat{\mathrm{e}}'})$ and $\bar{\xi}^{\hat{\mathrm{v}}^+}(\bar{x}_{\hat{\mathrm{e}}}^+) = \bar{\xi}'^{\hat{\mathrm{v}}'^+}(\bar{x}'^+_{\hat{\mathrm{e}}'})$. Then the (η, a) coordinates given in Lemma 7.24 (3) imply

$$\bar{\xi}^{\hat{\mathrm{v}}^-} \circ \phi^{\hat{\mathrm{e}}}(0, \underline{\bar{x}}) \circ (\bar{\xi}^{\hat{\mathrm{v}}^+})^{-1} = \bar{\xi}'^{\hat{\mathrm{v}}'^-} \circ \phi^{\hat{\mathrm{e}}'}(0, \underline{\bar{x}}') \circ (\bar{\xi}'^{\hat{\mathrm{v}}'^+})^{-1}.$$

Now let $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$ be an arbitrary main vertex and $\mathrm{v}' := \zeta(\mathrm{v})$ its corresponding vertex. For each gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$ with edge $\hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}_{\mathrm{v}}$, equation (7.27) shows that there is a gluing vertex $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\mathrm{v}'}$ with edge $\hat{\mathrm{e}}' \notin \hat{\mathrm{E}}'^{\mathrm{g}}_{\mathrm{v}'}$ whose boundary marked point corresponds $\bar{\tau}_{\bar{\mathrm{v}}}(\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) = \bar{\tau}'_{\bar{\mathrm{v}}'}(\bar{x}'_{\hat{\mathrm{v}}',\hat{\mathrm{e}}'})$ with $\bar{\zeta}([\hat{\mathrm{v}}]_{\underline{\bar{r}}}) = [\hat{\mathrm{v}}']_{\underline{\bar{r}}'}$. Then by definition (7.28) we have

$$\bar{\xi}^{\hat{\mathrm{v}}}(\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) = \bar{\xi}'^{\hat{\mathrm{v}}'}(\bar{x}'_{\hat{\mathrm{v}}',\hat{\mathrm{e}}'}). \tag{7.45}$$

Similarly, for each gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$ with index $j \leq n(\underline{\bar{o}}_{\hat{\mathrm{v}}})$, equation (7.27) shows that there is a gluing vertex $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\mathrm{v}'}$ with index $j' \leq n(\underline{\bar{o}}'_{\hat{\mathrm{v}}'})$ with

$$\bar{\xi}^{\hat{\mathrm{v}}}(\bar{o}_{\hat{\mathrm{v}},j}) = \bar{\xi}'^{\hat{\mathrm{v}}'}(\bar{o}'_{\hat{\mathrm{v}}',j'}). \tag{7.46}$$

This proves equation (7.44).

Applying Lemma 7.24, we pick $\delta' > 0$ such that the map Ω' with domain $M^{\mathrm{slc}}_{\delta'}(\underline{\bar{t}}', \underline{\bar{x}}', \underline{\bar{o}}', \underline{\bar{\xi}}')$ is a diffeomorphism onto its image, and we pick $\delta > 0$ with inclusion $\Omega(M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})) \subset \Omega'(M^{\mathrm{slc}}_{\delta'}(\underline{\bar{t}}', \underline{\bar{x}}', \underline{\bar{o}}', \underline{\bar{\xi}}'))$. Keeping the same ordering as in equation (7.44), the above process defines a smooth map

$$\begin{aligned} \Omega'^{-1} \circ \Omega : M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) &\to M^{\mathrm{slc}}_{\delta'}(\underline{\bar{t}}', \underline{\bar{x}}', \underline{\bar{o}}', \underline{\bar{\xi}}') \\ (\underline{t}, \underline{x}, \underline{o}, \underline{\xi}) &\mapsto (\underline{t}', \underline{x}', \underline{o}', \underline{\xi}'). \end{aligned} \tag{7.47}$$

This defines the gluing components of the map $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$ in (7.43).

Note that for corresponding additional gluing edges $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$, we have $t_{\hat{\mathrm{e}}} = 0$ if and only if $t'_{\hat{\mathrm{e}}'} = 0$, because by Lemma 7.24 (3) this is the case when the element $\xi^{\hat{\mathrm{v}}^-} \circ \phi^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}) \circ (\xi^{\hat{\mathrm{v}}^+})^{-1} = \xi'^{\hat{\mathrm{v}}'^-} \circ \phi^{\hat{\mathrm{e}}'}(t'_{\hat{\mathrm{e}}'}, \underline{x}') \circ (\xi'^{\hat{\mathrm{v}}'^+})^{-1}$ lies on the boundary of the compactification $\overline{\mathrm{Aut}}(D)$. Hence all parameters $\underline{t}$ are positive if and only if all parameters $\underline{t}'$ are positive. In the case when all gluing parameters are positive, we show that there are biholomorphisms $\underline{\psi}$ satisfying equation (7.34), which translates to

$$\underline{\psi}(\#_{\underline{t},\underline{x}}(\underline{x}, \underline{o})) = \#_{\underline{t}',\underline{x}'}(\underline{x}', \underline{o}'), \tag{7.48}$$

with standard model $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} = (\xi'^{\hat{\mathrm{v}}'})^{-1} \circ \xi^{\hat{\mathrm{v}}}$. For a fixed tuple $(\underline{t}, \underline{x}, \underline{o}, \underline{\xi})$ and its image $(\underline{t}', \underline{x}', \underline{o}', \underline{\xi}')$ in (7.47) and each main vertex $\mathrm{v} \in \#_{\hat{\mathrm{E}}^{\mathrm{g}}}(\hat{\mathrm{V}})$ and $\mathrm{v}' := \zeta(\mathrm{v})$, we now construct a pair of biholomorphisms $\tau_{\mathrm{v}} : \#_{\underline{t},\underline{x}}(\underline{D})_{\mathrm{v}} \to D$ and $\tau'_{\mathrm{v}'} : \#_{\underline{t}',\underline{x}'}(\underline{D})_{\mathrm{v}'} \to D$ with

$$\tau_{\mathrm{v}}(\#_{\underline{t},\underline{x}}(\underline{x}, \underline{o})_{\mathrm{v}}) = \tau'_{\mathrm{v}'}(\#_{\underline{t}',\underline{x}'}(\underline{x}', \underline{o}')_{\mathrm{v}'}). \tag{7.49}$$

Let $\hat{\mathrm{rt}}_{\mathrm{v}}$ and $\hat{\mathrm{rt}}_{\mathrm{v}'}$ be the respective roots of the gluing trees $\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}}$ and $\hat{\mathrm{T}}'^{\mathrm{g}}_{\mathrm{v}'}$. We define biholomorphisms

$$\tau_{\mathrm{v}} := \xi^{\hat{\mathrm{rt}}_{\mathrm{v}}} \circ \left(\Phi^{\hat{\mathrm{rt}}_{\mathrm{v}}}_{\mathrm{v}}\right)^{-1}, \quad \tau'_{\mathrm{v}'} := \xi'^{\hat{\mathrm{rt}}_{\mathrm{v}'}} \circ \left(\Phi^{\hat{\mathrm{rt}}_{\mathrm{v}'}}_{\mathrm{v}'}\right)^{-1}. \tag{7.50}$$

We now verify equation (7.49) for the pair τ_{v} and $\tau'_{\mathrm{v}'}$. Denoting their expressions in standard models by $\tau^{\hat{\mathrm{v}}} := \tau_{\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{\mathrm{v}}$ and $\tau'^{\hat{\mathrm{v}}'} := \tau_{\mathrm{v}'} \circ \Phi^{\hat{\mathrm{v}}'}_{\mathrm{v}'}$. It suffices to check $\tau^{\hat{\mathrm{v}}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) = \tau'^{\hat{\mathrm{v}}'}(x'_{\hat{\mathrm{v}}',\hat{\mathrm{e}}'})$ for corresponding $(\hat{\mathrm{v}}, \hat{\mathrm{e}})$ and $(\hat{\mathrm{v}}', \hat{\mathrm{e}}')$ same as in (7.45), and $\tau^{\hat{\mathrm{v}}}(o_{\hat{\mathrm{v}},j}) = \tau'^{\hat{\mathrm{v}}'}(o'_{\hat{\mathrm{v}}',j'})$ for corresponding $(\hat{\mathrm{v}}, j)$ and $(\hat{\mathrm{v}}', j')$ same as in (7.46). Recall that in the above correspondence, vertices $\hat{\mathrm{v}}$ and $\hat{\mathrm{v}}'$ satisfy $\bar{\zeta}([\hat{\mathrm{v}}]_{\underline{\bar{r}}}) = [\hat{\mathrm{v}}']_{\underline{\bar{r}}'}$. Moreover, by the construction of map (7.47), we have $\xi^{\hat{\mathrm{v}}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) = \xi'^{\hat{\mathrm{v}}'}(x'_{\hat{\mathrm{v}}',\hat{\mathrm{e}}'})$ and $\xi^{\hat{\mathrm{v}}}(o_{\hat{\mathrm{v}},j}) = \xi'^{\hat{\mathrm{v}}'}(o'_{\hat{\mathrm{v}}',j'})$ for $(\hat{\mathrm{v}}, \hat{\mathrm{e}})$, $(\hat{\mathrm{v}}', \hat{\mathrm{e}}')$, $(\hat{\mathrm{v}}, j)$, and $(\hat{\mathrm{v}}', j')$ as above. Thus in order to show equation (7.49), it suffices to verify that for vertices $\hat{\mathrm{v}}$ and $\hat{\mathrm{v}}'$ satisfying $\bar{\zeta}([\hat{\mathrm{v}}]_{\underline{\bar{r}}}) = [\hat{\mathrm{v}}']_{\underline{\bar{r}}'}$, we have

$$\tau^{\hat{\mathrm{v}}} \circ (\xi^{\hat{\mathrm{v}}})^{-1} = \tau'^{\hat{\mathrm{v}}'} \circ (\xi'^{\hat{\mathrm{v}}'})^{-1}. \tag{7.51}$$

Conceptually, we need to show that disk automorphisms $\xi^{\hat{\mathrm{v}}}$ and $\xi'^{\hat{\mathrm{v}}'}$ are obtained from rescaling $\tau^{\hat{\mathrm{v}}}$ and $\tau'^{\hat{\mathrm{v}}'}$ by the same factor. We first examine how to rescale $\tau^{\hat{\mathrm{v}}}$ to get $\xi^{\hat{\mathrm{v}}}$. Let $\hat{\mathrm{rt}}_{\mathrm{v}} = \hat{\mathrm{v}}_0, \hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_k = \hat{\mathrm{v}}$ be consecutive vertices, and let $\hat{\mathrm{e}}_i = (\hat{\mathrm{v}}_{i-1}, \hat{\mathrm{v}}_i)$ be their connecting edges. Thus we write $\tau^{\hat{\mathrm{v}}} = \tau^{\hat{\mathrm{rt}}_{\mathrm{v}}} \circ \left(\Phi_{\mathrm{v}}^{\hat{\mathrm{rt}}_{\mathrm{v}}}\right)^{-1} \circ \Phi_{\mathrm{v}}^{\hat{\mathrm{v}}}$ as follows.

$$\begin{aligned}
\tau^{\hat{\mathrm{v}}} &= \tau^{\hat{\mathrm{rt}}_{\bar{\mathrm{v}}}} \circ \phi^{\hat{\mathrm{e}}_1}(r_{\hat{\mathrm{e}}_1}, \underline{x})^{-1} \circ \cdots \circ \phi^{\hat{\mathrm{e}}_k}(r_{\hat{\mathrm{e}}_k}, \underline{x})^{-1} \\
&= \left(\xi^{\hat{\mathrm{v}}_0} \circ \phi^{\hat{\mathrm{e}}_1}(r_{\hat{\mathrm{e}}_1}, \underline{x})^{-1} \circ (\xi^{\hat{\mathrm{v}}_1})^{-1}\right) \circ \left(\xi^{\hat{\mathrm{v}}_1} \circ \phi^{\hat{\mathrm{e}}_2}(r_{\hat{\mathrm{e}}_2}, \underline{x})^{-1} \circ (\xi^{\hat{\mathrm{v}}_2})^{-1}\right) \circ \\
&\quad \cdots \circ \left(\xi^{\hat{\mathrm{v}}_{k-1}} \circ \phi^{\hat{\mathrm{e}}_k}(r_{\hat{\mathrm{e}}_k}, \underline{x})^{-1} \circ (\xi^{\hat{\mathrm{v}}_k})^{-1}\right) \circ \xi^{\hat{\mathrm{v}}}
\end{aligned}$$

The first equality is by Lemma 7.4 (2), and the second equality is due to the construction (7.50) $\tau^{\hat{\mathrm{rt}}_{\mathrm{v}}} = \xi^{\hat{\mathrm{rt}}_{\mathrm{v}}}$. Thus we write

$$\tau^{\hat{\mathrm{v}}} = \Psi \circ \xi^{\hat{\mathrm{v}}}.$$

In other words, $\xi^{\hat{\mathrm{v}}}$ is a rescaling of $\tau^{\hat{\mathrm{v}}}$ by a disk automorphism Ψ. Furthermore, each rescaling factor

$$\xi^{\hat{\mathrm{v}}_{i-1}} \circ \phi^{\hat{\mathrm{e}}_i}(r_{\hat{\mathrm{e}}_i}, \underline{x})^{-1} \circ (\hat{\xi}^{\mathrm{v}_i})^{-1}$$

is either equal to the identity map for $\hat{\mathrm{e}}_i \in \hat{\mathrm{E}}^{\mathrm{g}}_{\bar{\underline{r}}}$, or equal to the inverse of a component of Ω for $\hat{\mathrm{e}}_i \in \hat{\mathrm{E}}^{\mathrm{ag}}$ (see Lemma 7.24). Similarly, we can express $\xi'^{\hat{\mathrm{v}}'}$ as a sequence of rescaling $\tau'^{\hat{\mathrm{v}}'} = \Psi' \circ \xi'^{\hat{\mathrm{v}}'}$. Since $\hat{\mathrm{v}}$ and $\hat{\mathrm{v}}'$ give corresponding glued vertices $\bar{\zeta}([\hat{\mathrm{v}}]_{\bar{\underline{r}}}) = [\hat{\mathrm{v}}']_{\bar{\underline{r}}'}$, the rescaling Ψ and Ψ' go through the same sequence of non-identity rescaling factor

$$\xi^{\hat{\mathrm{v}}^+} \circ \phi^{\hat{\mathrm{e}}}(r_{\hat{\mathrm{e}}}, \underline{x})^{-1} \circ (\xi^{\hat{\mathrm{v}}^-})^{-1} = \xi'^{\hat{\mathrm{v}}'^+} \circ \phi^{\hat{\mathrm{e}}'}(r'_{\hat{\mathrm{e}}'}, \underline{x}')^{-1} \circ (\xi'^{\hat{\mathrm{v}}'^-})^{-1},$$

where additional gluing edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$ and $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{ag}}$ are such that $\bar{\chi}(\hat{\mathrm{e}}) = \hat{\mathrm{e}}'$. This shows the total rescalings are the same $\Psi = \Psi'$. This proves equation (7.51) and also (7.49). Lastly, define the biholomorphism $\psi_{\mathrm{v}} := \tau'^{-1}_{\mathrm{v}'} \circ \tau_{\mathrm{v}}$, and equivalence $\psi_{\mathrm{v}}(\#_{\underline{r},\underline{x}}(\underline{x}, \underline{o})_{\mathrm{v}}) = \#_{\underline{r}',\underline{x}'}(\underline{x}', \underline{o}')_{\mathrm{v}'}$ follows from (7.49). Moreover, for vertices $\hat{\mathrm{v}}$ and $\hat{\mathrm{v}}'$ satisfying $\bar{\zeta}([\hat{\mathrm{v}}]_{\bar{\underline{r}}}) = [\hat{\mathrm{v}}']_{\bar{\underline{r}}'}$, the standard model of ψ_{v} is given by

$$\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} = (\tau'^{\hat{\mathrm{v}}'})^{-1} \circ \tau^{\hat{\mathrm{v}}} = (\xi'^{\hat{\mathrm{v}}'})^{-1} \circ \xi^{\hat{\mathrm{v}}},$$

where the second equality follows from (7.51). This along with the equality of displaced edge lengths (7.39) verify the biholomorphic equivalence (7.34) and the form of the biholomorphism (7.35). This finishes the construction of the map $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$.

We now verify that the set of maps $\{\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}\}$ matches pairwise at the intersection of their domains. Let $\hat{\mathrm{E}}^{\mathrm{g}}_1$ and $\hat{\mathrm{E}}^{\mathrm{g}}_2$ be two arbitrary sets of gluing edges with $\hat{\mathrm{E}}^{\mathrm{g}}_{\bar{\underline{r}}} \subset \hat{\mathrm{E}}^{\mathrm{g}}_1, \hat{\mathrm{E}}^{\mathrm{g}}_2 \subset \{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \,|\, \bar{r}_{\hat{\mathrm{e}}} \geq 0\}$ and we define their corresponding gluing edges $\hat{\mathrm{E}}'^{\mathrm{g}}_1$ and $\hat{\mathrm{E}}'^{\mathrm{g}}_2$ by (7.31) as before. For each point $(\underline{r}, \underline{\ell}, \underline{x}, \underline{o}, \underline{\xi}) \in M_\delta(\bar{\underline{r}}, \bar{\underline{\ell}}, \bar{\underline{x}}, \bar{\underline{o}}, \bar{\underline{\xi}})$ in the intersection of two quadrants, satisfying

$$\begin{aligned}
&r_{\hat{\mathrm{e}}} \geq 0 \text{ for } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_1 \text{ and } r_{\hat{\mathrm{e}}} \leq 0 \text{ for } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_1, \text{ and} \\
&r_{\hat{\mathrm{e}}} \geq 0 \text{ for } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_2 \text{ and } r_{\hat{\mathrm{e}}} \leq 0 \text{ for } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_2,
\end{aligned}$$

we prove

$$\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_1}(\underline{r},\underline{\ell},\underline{x},\underline{o},\underline{\xi}) = \alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_2}(\underline{r},\underline{\ell},\underline{x},\underline{o},\underline{\xi}). \tag{7.52}$$

By construction, the defining equations for two sides are the same except at edges $\hat{\mathrm{e}}$ with vanishing gluing parameter $r_{\hat{\mathrm{e}}} = 0$ and either $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}_1$ and $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_2$, or $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}_2$ and $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_1$. In other words, the defining equations differ when an edge is treated as a gluing edge on one side and as a non-gluing edge on the other. By symmetry, we only consider the former case and fix an edge $\hat{\mathrm{e}}$ with $r_{\hat{\mathrm{e}}} = 0$, $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}_1$, and $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_2$. By construction (7.31), the corresponding edge $\hat{\mathrm{e}}'$ satisfies $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{ag}}_1$ and $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{nd}} \setminus \hat{\mathrm{E}}'^{\mathrm{g}}_2$. The defining equation of $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_1}$ which involves the edge $\hat{\mathrm{e}}$ is given by (7.47)

$$\xi^{\hat{\mathrm{v}}^-} \circ \phi^{\hat{\mathrm{e}}}(r_{\hat{\mathrm{e}}}, \underline{x}) \circ (\xi^{\hat{\mathrm{v}}^+})^{-1} = \xi'^{\hat{\mathrm{v}}'^-} \circ \phi^{\hat{\mathrm{e}}'}(r'_{\hat{\mathrm{e}}'}, \underline{x}') \circ (\xi'^{\hat{\mathrm{v}}'^+})^{-1}. \tag{7.53}$$

On the other hand, the defining equation of $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_2}$ which involves the edge $\hat{\mathrm{e}}$ is given by (7.41) and (7.47)

$$r_{\hat{\mathrm{e}}} = r'_{\hat{\mathrm{e}}'}, \quad \xi^{\hat{\mathrm{v}}^\pm}(x^\pm_{\hat{\mathrm{e}}}) = \xi'^{\hat{\mathrm{v}}'^\pm}(x'^\pm_{\hat{\mathrm{e}}'}). \tag{7.54}$$

By the (η, a) coordinates in Lemma 7.24 (3), at $r_{\hat{\mathrm{e}}} = 0$ both (7.53) and (7.54) give the same equation $r'_{\hat{\mathrm{e}}'} = 0$ and $\xi^{\hat{\mathrm{v}}^\pm}(x^\pm_{\hat{\mathrm{e}}}) = \xi'^{\hat{\mathrm{v}}^\pm}(x'^\pm_{\hat{\mathrm{e}}'})$. This proves (7.52), in other words, the maps $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_1}$ and $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_2}$ agree. This shows that α is a continuous map, and we finish proving Claim 7.23.

As we shall see later, we can fix a family of disk automorphisms $\underline{\xi}$, plug in the map α in Claim 7.23, and obtain the desired map $\lambda : \mathcal{U}_\delta(\underline{\bar{r}}, \bar{u}) \to \mathcal{U}^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}', \bar{u}')$. But before that, we explore the continuity/smoothness property further. Recall that in the construction of the map α in Claim 7.23, we use $\rho(r) = -\ln(r)$ as the the gluing profile. We now study this map under general gluing profiles R. Define a map given by α with gluing profile R

$$\begin{aligned} \alpha^R : M_\delta(\underline{\bar{r}}, \underline{\bar{\ell}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) &\to M^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}', \underline{\bar{\ell}}', \underline{\bar{x}}', \underline{\bar{o}}', \underline{\bar{\xi}}'), \\ (\underline{r},\underline{\ell},\underline{x},\underline{o},\underline{\xi}) &\mapsto (R^{-1} \circ \rho(\underline{r}'), \underline{\ell}', \underline{x}', \underline{o}', \underline{\xi}') \end{aligned} \tag{7.55}$$

with $(\underline{r}', \underline{\ell}', \underline{x}', \underline{o}', \underline{\xi}') := \alpha(\rho^{-1} \circ R(\underline{r}), \underline{\ell}, \underline{x}, \underline{o}, \underline{\xi})$.

We now prove its continuity/smoothness property.

Claim 7.25. *For any continuous gluing profile R, the map α^R is continuous. Let $\varphi(r) = e^{1/r} - e$ be the exponential gluing profile. Then the map α^φ is smooth.*

By Remark 5.9 (1), the gluing profile $R : (0, 1] \to [0, \infty)$ is a homeomorphism. Hence the map α^R is continuous. It suffices to show that the map α^φ under the exponential gluing profile is smooth.

Let $\Omega'^{-1} \circ \Omega : M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) \to M^{\mathrm{slc}}_{\delta'}(\underline{\bar{t}}', \underline{\bar{x}}', \underline{\bar{o}}', \underline{\bar{\xi}}')$ be the map in (7.47), where the submersion Ω and diffeomorphism Ω' are given in Lemma 7.24.

Lemma 7.26. *We view* $(\underline{t}', \underline{x}', \underline{o}', \underline{\xi}') := \Omega'^{-1} \circ \Omega(\underline{t}, \underline{x}, \underline{o}, \underline{\xi})$ *as a function. Fix additional gluing edges* $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$ *and* $\hat{\mathrm{e}}' = \bar{\chi}(\hat{\mathrm{e}}) \in \hat{\mathrm{E}}'^{\mathrm{ag}}$. *Then at a point* $(\check{\underline{t}}, \check{\underline{x}}, \check{\underline{o}}, \check{\underline{\xi}}) \in M_\delta(\bar{\underline{t}}, \bar{\underline{x}}, \bar{\underline{o}}, \bar{\underline{\xi}})$ *with* $\check{t}_{\hat{\mathrm{e}}} = 0$, *the partial derivative satisfies*

$$\partial_{t_{\hat{\mathrm{e}}}} t'_{\hat{\mathrm{e}}'} > 0.$$

We shall prove this lemma later in this section. In order to show that α^φ is smooth, we first examine the smoothness of α. Recall that the map α is constructed from a set of smooth maps $\{\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}\}$ (see (7.33)). Let $\hat{\mathrm{E}}^{\mathrm{g}}_1$ and $\hat{\mathrm{E}}^{\mathrm{g}}_2$ be two arbitrary sets of gluing edges with $\hat{\mathrm{E}}^{\mathrm{g}}_{\bar{\underline{r}}} \subset \hat{\mathrm{E}}^{\mathrm{g}}_1, \hat{\mathrm{E}}^{\mathrm{g}}_2 \subset \{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \,|\, \bar{r}_{\hat{\mathrm{e}}} \geq 0\}$, and $\hat{\mathrm{E}}'^{\mathrm{g}}_1$ and $\hat{\mathrm{E}}'^{\mathrm{g}}_2$ their corresponding gluing edges. Fix a point $(\check{\underline{r}}, \check{\underline{\ell}}, \check{\underline{x}}, \check{\underline{o}}, \check{\underline{\xi}}) \in M_\delta(\bar{\underline{r}}, \bar{\underline{\ell}}, \bar{\underline{x}}, \bar{\underline{o}}, \bar{\underline{\xi}})$ in the intersection of two quadrants, satisfying

$$\check{r}_{\hat{\mathrm{e}}} \geq 0 \text{ for } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_1 \text{ and } \check{r}_{\hat{\mathrm{e}}} \leq 0 \text{ for } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_1, \text{ and}$$
$$\check{r}_{\hat{\mathrm{e}}} \geq 0 \text{ for } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_2 \text{ and } \check{r}_{\hat{\mathrm{e}}} \leq 0 \text{ for } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_2,$$

We compare the derivatives $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_1}$ and $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_2}$ at the point $(\check{\underline{r}}, \check{\underline{\ell}}, \check{\underline{x}}, \check{\underline{o}}, \check{\underline{\xi}})$. In the same logic as verifying equation (7.52), we only focus on terms involving $r_{\hat{\mathrm{e}}}$, where $\hat{\mathrm{e}}$ is an edge with

$$\check{r}_{\hat{\mathrm{e}}} = 0,\ \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}_1, \text{ and } \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_2.$$

We first study $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_1}$. Viewing $(\underline{r}', \underline{\ell}', \underline{x}', \underline{o}', \underline{\xi}') := \alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_1}(\underline{r}, \underline{\ell}, \underline{x}, \underline{o}, \underline{\xi})$ as a function, at the point $(\check{\underline{r}}, \check{\underline{\ell}}, \check{\underline{x}}, \check{\underline{o}}, \check{\underline{\xi}})$ we have partial derivatives

$$\partial_{r_{\hat{\mathrm{e}}}} r'_{\hat{\mathrm{e}}'} > 0, \text{ and } \partial^k_{r_{\hat{\mathrm{e}}}} y' \text{ not necessarily } 0, \tag{7.56}$$

where y' is any component in $(\underline{r}', \underline{\ell}', \underline{x}', \underline{o}', \underline{\xi}')$ other than $r'_{\hat{\mathrm{e}}'}$. The above result follows from Lemma 7.26 since $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}_1$.

We now study $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_2}$. Viewing $(\underline{r}', \underline{\ell}', \underline{x}', \underline{o}', \underline{\xi}') := \alpha_{\hat{\mathrm{E}}^{\mathrm{g}}_2}(\underline{r}, \underline{\ell}, \underline{x}, \underline{o}, \underline{\xi})$ as a function, nearby the point $(\check{\underline{r}}, \check{\underline{\ell}}, \check{\underline{x}}, \check{\underline{o}}, \check{\underline{\xi}})$ we have

$$r'_{\hat{\mathrm{e}}'} = r_{\hat{\mathrm{e}}}, \text{ and } y' \text{ independent of } r_{\hat{\mathrm{e}}}, \tag{7.57}$$

where y' is any component in $(\underline{r}', \underline{\ell}', \underline{x}', \underline{o}', \underline{\xi}')$ other than $r'_{\hat{\mathrm{e}}'}$. The above result follows from equation (7.41) since $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{nd}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_2$.

Cross-examining (7.56) and (7.57) we notice that the partial derivatives do not match at the point $(\check{\underline{r}}, \check{\underline{\ell}}, \check{\underline{x}}, \check{\underline{o}}, \check{\underline{\xi}})$. Therefore the map α is not smooth. However by changing to the exponential gluing profile φ, Theorem 13.9 implies that $r'_{\hat{\mathrm{e}}'}$ is a smooth function around $(\check{\underline{r}}, \check{\underline{\ell}}, \check{\underline{x}}, \check{\underline{o}}, \check{\underline{\xi}})$. Moreover, Lemma 13.10 (3) implies that each y' as above is a smooth function around $(\check{\underline{r}}, \check{\underline{\ell}}, \check{\underline{x}}, \check{\underline{o}}, \check{\underline{\xi}})$. This finishes proving the smoothness result in Claim 7.25.

Finally, we establish the desired map $\lambda : \mathcal{U}_\delta(\bar{\underline{r}}, \bar{\mu}) \to \mathcal{U}^{\mathrm{slc}}_{\delta'}(\bar{\underline{r}}', \bar{\mu}')$ from the map α in Lemma 7.23. We claim that we can choose a family of disk automorphisms $\underline{\xi}(\underline{r}, \underline{x})$ such that for each element $(\underline{r}, (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{o})) \in \mathcal{U}_\delta(\bar{\underline{r}}, \bar{\mu})$, the point $(\underline{r}, \underline{\ell}, \underline{x}, \underline{o}, \underline{\xi}(\underline{r}, \underline{x}))$ lies in the manifold $M_\delta(\bar{\underline{r}}, \bar{\underline{\ell}}, \bar{\underline{x}}, \bar{\underline{o}}, \bar{\underline{\xi}})$ defined in

(7.29). Indeed, for each main glued vertex $\bar{\mathrm{v}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$, we pick a gluing vertex $\hat{\mathrm{v}}_0 \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$. Then choose an arbitrary disk automorphism $\xi^{\hat{\mathrm{v}}_0} \in \mathcal{U}_\delta(\bar{\xi}^{\hat{\mathrm{v}}_0})$, and for other gluing vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ we define $\xi^{\hat{\mathrm{v}}}$ using the standard pullbacks $\xi^{\hat{\mathrm{v}}} := \xi^{\hat{\mathrm{v}}_0} \circ (\Phi^{\hat{\mathrm{v}}_0}_{\mathrm{v}})^{-1} \circ \Phi^{\hat{\mathrm{v}}}_{\mathrm{v}}$. Here the map $(\Phi^{\hat{\mathrm{v}}_0}_{\mathrm{v}})^{-1} \circ \Phi^{\hat{\mathrm{v}}}_{\mathrm{v}}$ can be expressed as a composition of shift maps depending on the variables $(\underline{r}, \underline{x})$. By construction, this family satisfies the constraint in condition (2) of (7.29).

Given $(\underline{r}, (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{o})) \in \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\mu})$, we plug the family of automorphisms into the map α and define

$$(\underline{r}', \underline{\ell}', \underline{x}', \underline{o}', \underline{\xi}') := \alpha(\underline{r}, \underline{\ell}, \underline{x}, \underline{o}, \underline{\xi}(\underline{r}, \underline{x})). \tag{7.58}$$

Lastly, we define $\lambda(\underline{r}, (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{o})) = (\underline{r}', (\hat{\mathrm{T}}', \underline{\ell}', \underline{x}', \underline{o}'))$. The continuity of the map λ follows from the continuity of the map α in Lemma 7.23. Moreover, by construction (7.47), the disk automorphisms $\underline{\xi}'$ are of the form $\underline{\xi}'(\underline{r}, \underline{x}, \underline{o}, \underline{\xi}(\underline{r}, \underline{x}))$. Thus Lemma 7.23 shows that the biholomorphism in standard model is of the form

$$\psi^{\hat{\mathrm{v}}, \hat{\mathrm{v}}'}_{(\underline{r}, \underline{x}, \underline{o})} = \xi'^{\hat{\mathrm{v}}'}(\underline{r}, \underline{x}, \underline{o})^{-1} \circ \xi^{\hat{\mathrm{v}}}(\underline{r}, \underline{x})$$

and it depends on $(\underline{r}, \underline{x}, \underline{o})$ continuously. (In fact, one can show that the map λ is independent of the choices of the family of disk automorphisms $\underline{\xi}(\underline{r}, \underline{x})$.)

Under the exponential gluing profile φ, the smoothness of λ and the smooth dependence of the biholomorphism $\psi^{\hat{\mathrm{v}}, \hat{\mathrm{v}}'}_{(\underline{r}, \underline{x}, \underline{o})}$ follow directly from the smoothness of the map α^φ in Claim 7.25.

We now gear towards proving Lemma 7.24. Firstly we study the shift map between two families of strip coordinates around a gluing parameter $\check{r}_{\hat{\mathrm{e}}} > 0$.

Lemma 7.27. *Given an edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$ and a pair of families of strip coordinates $h^\pm_{\hat{\mathrm{e}}}(x^\pm_{\hat{\mathrm{e}}}, \cdot) : \mathbb{R}^\pm \times [0, \pi] \to D_{\mathrm{v}^\pm}$ near $\check{x}^\pm_{\hat{\mathrm{e}}}$. Recall that the shift map is the disk automorphism given by*

$$\phi^{\hat{\mathrm{e}}}(r_{\hat{\mathrm{e}}}, \underline{x}) = h^-_{\hat{\mathrm{e}}}(x^-_{\hat{\mathrm{e}}}, \cdot) \circ L_{R_{\hat{\mathrm{e}}}} \circ h^+_{\hat{\mathrm{e}}}(x^+_{\hat{\mathrm{e}}}, \cdot)^{-1} : D_{\hat{\mathrm{v}}^+} \to D_{\hat{\mathrm{v}}^-}.$$

Let $\check{r}_{\hat{\mathrm{e}}}$ be a positive gluing parameter. Then viewing $(r_{\hat{\mathrm{e}}}, \underline{x}) \mapsto \phi^{\hat{\mathrm{e}}}(r_{\hat{\mathrm{e}}}, \underline{x})$ as a map into $\mathrm{Aut}(D)$, *the derivative*

$$D_{(r_{\hat{\mathrm{e}}}, x^-_{\hat{\mathrm{e}}}, x^+_{\hat{\mathrm{e}}})} \phi^{\hat{\mathrm{e}}}(\check{r}_{\hat{\mathrm{e}}}, \underline{\check{x}})$$

is invertible.

Proof. Since the edge ê is fixed, we shall drop it from the notation. We now choose a convenient set of coordinates to compute this derivative. Recall from Definition 3.8 that $h^\pm(x^\pm, \cdot)$ can be written as $h^\pm(x^\pm, \cdot) = f^\pm_{x^\pm} \circ p^\pm$, where each $f^\pm_{x^\pm}$ is a Möbius transformation that maps the extended upper half plane to the disk, and we have

$$f^\pm_{x^\pm}(0) = x^\pm, \quad f^\pm_{x^\pm}(\infty) = f^\pm_{\check{x}^\pm}(\infty). \tag{7.59}$$

Note that $f^+_{\check{x}^+}(\infty)$, $f^+_{\check{x}^+}(0)$, and $f^+_{\check{x}^+}(1)$ are three fixed boundary marked points on the disk. It follows from Proposition 13.8 that the tuple

$$\Lambda(x^-, x^+, r) := \big(\phi(r,\underline{x})(f^+_{\check{x}^+}(\infty)),\ \phi(r,\underline{x})(f^+_{\check{x}^+}(0)),\ \phi(r,\underline{x})(f^+_{\check{x}^+}(1)\big) \tag{7.60}$$

provides a set of coordinates for the point $\phi(r,\underline{x}) \in \mathrm{Aut}(D)$. We claim that the derivative $D_{(x^-,x^+,r)}\phi(\check{r},\underline{\check{x}})$ under this system of coordinates is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ * & * & d \end{bmatrix},$$

with $c, d \neq 0$. The columns of the above derivative are written in the order ∂_{x^-}, ∂_{x^+}, and ∂_r, and the rows in the same order as appeared in (7.60). We now verify this claim.

It follows from 7.3 (1) we have the the expression

$$\phi(r,\underline{x})(z) = f^-_{x^-}\left(r \cdot \frac{-1}{(f^+_{x^+})^{-1}(z)}\right). \tag{7.61}$$

Therefore we have

$$\Lambda(x^-, \check{x}^+, \check{r}) = (f^-_{x^-}(0), f^-_{x^-}(\infty), f^-_{x^-}(-\check{r})) = (x^-, f^-_{\check{x}^-}(\infty), f^-_{x^-}(-\check{r})),$$

where the second equality follows from (7.59). This explains the first column of the derivative. Similarly, we have

$$\Lambda(\check{x}^-, x^+, \check{r}) = \left(\check{x}^-, f^-_{\check{x}^-}\left(\check{r} \cdot \frac{-1}{(f^+_{x^+})^{-1}(\check{x}^+)}\right), f^-_{\check{x}^-}\left(\check{r} \cdot \frac{-1}{(f^+_{x^+})^{-1} \circ f^+_{\check{x}^+}(1)}\right)\right).$$

We have $\partial_{x^+}(f^+_{x^+})^{-1}(\check{x}^+) \neq 0$. This explains the second column. Lastly, we have

$$\Lambda(\check{x}^-, \check{x}^+, r) = (f^-_{\check{x}^-}(0), f^-_{\check{x}^-}(\infty), f^-_{\check{x}^-}(-r)).$$

The last column is self-explanatory. This proves that the derivative is invertible. □

We now study the shift map between two families of coordinates around a gluing parameter $\check{r}_{\hat{\mathrm{e}}} = 0$.

Lemma 7.28. *Given a gluing edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^+)$, and a pair of families of strip coordinates $h^\pm_{\hat{\mathrm{e}}}(x^\pm_{\hat{\mathrm{e}}}, \cdot) : \mathbb{R}^\pm \times [0,\pi] \to D_{\hat{\mathrm{v}}^\pm}$ near $\check{x}^\pm_{\hat{\mathrm{e}}}$. Recall that the shift map is the disk automorphism given by*

$$\phi^{\hat{\mathrm{e}}}(r_{\hat{\mathrm{e}}}, \underline{x}) = h^-_{\hat{\mathrm{e}}}(\underline{x}^-_{\hat{\mathrm{e}}}, \cdot) \circ L_{R_{\hat{\mathrm{e}}}} \circ h^+_{\hat{\mathrm{e}}}(x^+_{\hat{\mathrm{e}}}, \cdot)^{-1} : D_{\hat{\mathrm{v}}^+} \to D_{\hat{\mathrm{v}}^-}.$$

Then $\phi^{\hat{\mathrm{e}}}(0, \underline{x})$ lies in the compactified disk automorphism group $\overline{\mathrm{Aut}}(D)$ (see Remark 13.2). Moreover, the (η, a) coordinates (see (13.1)) of $\phi^{\hat{\mathrm{e}}}(0,\underline{x})$ is given by

$$\eta(0,\underline{x}) = x^-_{\hat{\mathrm{e}}} \cdot (x^+_{\hat{\mathrm{e}}})^{-1}, \quad a(0,\underline{x}) = x^+_{\hat{\mathrm{e}}}.$$

Viewing $(r_{\hat{\mathrm{e}}}, \underline{x}) \mapsto \phi^{\hat{\mathrm{e}}}(r_{\hat{\mathrm{e}}}, \underline{x})$ as a map into the compactification $\overline{\mathrm{Aut}}(D)$, the derivative

$$D_{(r_{\hat{\mathrm{e}}}, x^-_{\hat{\mathrm{e}}}, x^+_{\hat{\mathrm{e}}})}\phi^{\hat{\mathrm{e}}}(0, \underline{\check{x}})$$

is invertible.

Proof. Since the edge ê is fixed, we shall drop it from the notation. By a computation analogous to Lemma 7.3 (1) using the generalized family of strip neighborhoods, we have

$$\phi(r, \underline{x})(z) = f^-_{x^-}\left(r \cdot \frac{-1}{\left(f^+_{x^+}\right)^{-1}(z)}\right), \tag{7.62}$$

and its inverse is given by

$$\phi(r, \underline{x}, \underline{o})^{-1}(z) = f^+_{x^+}\left(r \cdot \frac{-1}{\left(f^-_{x^-}\right)^{-1}(z)}\right). \tag{7.63}$$

Recall from (13.1) that we parametrize $\mathrm{Aut}(D)$ by $\psi_{\eta,a}(z) = \eta\frac{z-a}{\bar{a}z-1}$. Note that $\psi^{-1}_{\eta,a}(0) = a$ and $\psi_{\eta,a}(0) = \eta a$. Therefore the (η, a) coordinates of $\phi(r, \underline{x})$ is given by

$$\eta(r, \underline{x}) = f^-_{x^-}\left(r \cdot \frac{-1}{\left(f^+_{x^+}\right)^{-1}(0)}\right) \cdot \left[f^+_{x^+}\left(r \cdot \frac{-1}{\left(f^-_{x^-}\right)^{-1}(0)}\right)\right]^{-1}, \tag{7.64}$$

$$a(r, \underline{x}) = f^+_{x^+}\left(r \cdot \frac{-1}{\left(f^-_{x^-}\right)^{-1}(0)}\right). \tag{7.65}$$

We claim that the derivative $D_{(x^-, x^+, r)}\phi(0, \underline{\check{x}})$ in (η, a) coordinates is of the form

$$\begin{bmatrix} 1 & -1 & * \\ 0 & 1 & * \\ 0 & 0 & c \end{bmatrix},$$

with $c \neq 0$. The columns of the above derivative are written in the order ∂_{x^-}, ∂_{x^+}, and ∂_r, and the rows in the order η and a. We now verify this claim.

Recall that by definition, we have $f^{\pm}_{x^{\pm}}(0) = x^{\pm}_{\mathrm{e}}$. It follows from (7.64) and (7.65) that

$$\eta(0, \underline{x}) = x^- \cdot (x^+)^{-1}, \quad a(0, \underline{x}) = x^+. \tag{7.66}$$

Using charts around $\eta(0, \underline{\check{x}})$ and $a(0, \underline{\check{x}})$, the above observation explains the first two columns of the matrix.

We now show the last column is of the desired form by use $\left(f^+_{\check{x}^+}\right)^{-1}$ as a chart around $a(0, \underline{\check{x}}) = \check{x}^+$. The partial of a with respect to r is

$$\partial_r\left(\left(f^+_{\check{x}^+}\right)^{-1} \circ a\right)(0, \underline{\check{x}}) = \frac{-1}{\left(f^-_{\check{x}^-}\right)^{-1}(0)}, \tag{7.67}$$

whose imaginary part equals to some $c > 0$. □

By using the automorphism components (see Remark 5.13), the following lemma establishes a one-to-one correspondence between the neighborhood slice and the neighborhood. Since the correspondence is local in nature, we shall not keep track of the size of the neighborhoods.

Lemma 7.29. *Let $(\underline{\bar{x}}, \underline{\bar{o}})$ be boundary and interior marked points on a single disk and $\mathcal{U}^{\mathrm{slc}}_\delta(\underline{\bar{x}}, \underline{\bar{o}})$ a neighborhood slice (Definition 5.14). Let $\bar{\xi} \in \mathrm{Aut}(D)$ be a disk automorphism and $\mathcal{U}_\delta(\bar{\xi})$ a neighborhood in $\mathrm{Aut}(D)$. Then the inverse of the map*

$$\begin{aligned} \Psi : \mathcal{U}^{\mathrm{slc}}_\delta(\underline{\bar{x}}, \underline{\bar{o}}) \times \mathcal{U}_\delta(\bar{\xi}) &\to \mathcal{U}_\lambda(\bar{\xi}(\underline{\bar{x}}, \underline{\bar{o}})) \\ (\underline{x}, \underline{o}, \xi) &\mapsto \xi(\underline{x}, \underline{o}) \end{aligned}$$

is of the form

$$\begin{aligned} \Gamma : \mathcal{U}_\delta(\bar{\xi}(\underline{\bar{x}}, \underline{\bar{o}})) &\to \mathcal{U}^{\mathrm{slc}}_\lambda(\underline{\bar{x}}, \underline{\bar{o}}) \times \mathcal{U}_\lambda(\bar{\xi}) \\ (\underline{x}, \underline{o}) &\mapsto \left(\psi^{-1}_{\pi^{\mathrm{a}}(\underline{x}, \underline{o})}(\underline{x}, \underline{o}), \psi_{\pi^{\mathrm{a}}(\underline{x}, \underline{o})}\right), \end{aligned}$$

where $\psi_{\pi^{\mathrm{a}}(\underline{x}, \underline{o})}$ is defined in Remark 5.13.

Proof. This proof is mostly tautology. By definition, we have $\Psi \circ \Gamma(\underline{x}, \underline{o}) = (\underline{x}, \underline{o})$. It suffices to verify the other composition.

Given $(\underline{x}, \underline{o}) \in \mathcal{U}^{\mathrm{slc}}_\lambda(\underline{\bar{x}}, \underline{\bar{o}})$ and $\xi \in \mathcal{U}_\lambda(\bar{\xi})$, by definition of the neighborhood slice, we have

$$\pi^{\mathrm{a}}(\xi(\underline{x}, \underline{o})) = \xi(\pi^{\mathrm{a}}(\underline{\hat{x}}, \underline{\hat{o}})). \tag{7.68}$$

By Proposition 13.8, ξ is the unique disk automorphism which maps $\pi^{\mathrm{a}}(\underline{\hat{x}}, \underline{\hat{o}})$ to $\pi^{\mathrm{a}}(\xi(\underline{x}, \underline{o}))$. By definition, we have $\psi_{\pi^{\mathrm{a}}(\xi(\underline{x}, \underline{o}))} = \xi$. Then it immediately follows $\Gamma \circ \Psi(\underline{x}, \underline{o}, \xi) = (\underline{x}, \underline{o}, \xi)$. ☐

We now prove Lemma 7.24.

Proof of Lemma 7.24. This result is local in nature, so we will not keep track of the constant $\delta > 0$ by abuse of notation.

(2) We first prove (2), the case restricting to the domain $M^{\mathrm{slc}}_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$. We prove this result by first pre-composing the map Ω with Γ from Lemma 7.29. Since the map Ω involves the map Ψ in Lemma 7.29, by a change of coordinates, $\Omega \circ \Gamma$ takes on a simpler form. Next we use the implicit function theorem to eliminate the dependent variables defined by condition (3) in the definition of $M^{\mathrm{slc}}_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$ in (7.42). And then we take the derivative and show that it is invertible.

Recall from Lemma 7.29 that the map Γ identify points in a neighborhood with pairs of points in a neighborhood slice and disk automorphisms. We denote the following marked points by

$$(\underline{\check{x}}, \underline{\check{o}}) := \underline{\bar{\xi}}(\underline{\bar{x}}, \underline{\bar{o}}).$$

The set $\Gamma^{-1}(M_\delta^{\mathrm{slc}}(\bar{\underline{t}}, \bar{\underline{x}}, \bar{\underline{o}}, \bar{\underline{\xi}}))$ consists of $(\underline{t}, \underline{x}, \underline{o})$ where the marked points $(\underline{x}, \underline{o}) \in \mathcal{U}_\delta(\check{\underline{x}}, \check{\underline{o}})$ are such that for each gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$, the map

$$\psi_{\pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}-}, \underline{o}_{\hat{\mathrm{v}}-})} \circ \phi^{\hat{\mathrm{e}}}\left(t_{\hat{\mathrm{e}}}, \underline{\psi}^{-1}_{\pi^{\mathrm{a}}(\underline{x}, \underline{o})}(\underline{x})\right) \circ \psi^{-1}_{\pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}+}, \underline{o}_{\hat{\mathrm{v}}+})} \tag{7.69}$$

is equal to the identity map (see the constraint in condition (3) of (7.42)). Furthermore, the expression of the first component of $\Omega \circ \Gamma$ is the same as (7.69) for additional gluing edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$. The second component of $\Omega \circ \Gamma$ is simply given by $(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}, \hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}}$, and the last component $(\underline{o}_{\hat{\mathrm{v}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}}$.

We now define a set of generalized family of strip coordinates near $\check{x}^{\pm}_{\hat{\mathrm{e}}}$ for each gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}$ by

$$g^{\pm}_{\hat{\mathrm{e}}}(\underline{x}_{\hat{\mathrm{v}}\pm}, \underline{o}_{\hat{\mathrm{v}}\pm}, \cdot) := \psi_{\pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}\pm}, \underline{o}_{\hat{\mathrm{v}}\pm})} \circ h^{\pm}_{\hat{\mathrm{e}}}\left(\psi^{-1}_{\pi^{\mathrm{a}}(\underline{x}_{\hat{\mathrm{v}}\pm}, \underline{o}_{\hat{\mathrm{v}}\pm})}(x^{\pm}_{\hat{\mathrm{e}}}), \cdot\right). \tag{7.70}$$

The map in (7.69) is the generalized shift map

$$\phi^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}, \underline{o}) = g^{-}_{\hat{\mathrm{e}}}(\underline{x}_{\hat{\mathrm{v}}-}, \underline{o}_{\hat{\mathrm{v}}-}, \cdot) \circ L_{R(t_{\hat{\mathrm{e}}})} \circ g^{+}_{\hat{\mathrm{e}}}(\underline{x}_{\hat{\mathrm{v}}+}, \underline{o}_{\hat{\mathrm{v}}+}, \cdot)^{-1}. \tag{7.71}$$

In summary, the composite map $\Xi := \Omega \circ \Gamma$ of the form

$$\Xi(\underline{t}, \underline{x}, \underline{o}) = \left((\phi^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}, \underline{o}))_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}}, (x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}, \hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}}, (\underline{o}_{\hat{\mathrm{v}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}}\right),$$

whose domain is given by

$$N_\delta := \{(\underline{t}, \underline{x}, \underline{o}) \,|\, (1), (2), (3)\}.$$

(1) We have $\underline{t} \in \mathcal{U}_\delta(\bar{\underline{t}})$, and $t_{\hat{\mathrm{e}}} \geq 0$ for each gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}$ (in particular $t_{\hat{\mathrm{e}}} \in [0, \delta)$ for each additional gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$).
(2) We have $(\underline{x}, \underline{o}) \in \mathcal{U}_\delta(\check{\underline{x}}, \check{\underline{o}})$.
(3) We impose the constraint $\phi^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}, \underline{o}) = \mathrm{Id}$ for all gluing edges $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$.

Now it suffices to prove that Ξ is a diffeomorphism on N_δ for small enough $\delta > 0$. To do this, we transform the problem yet again by eliminating the dependent variables in condition (3) above. Applying Lemma 7.27, we see that for each gluing edge $\mathrm{e} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$, the derivative $D_{(t_{\hat{\mathrm{e}}}, x^-_{\hat{\mathrm{e}}}, x^+_{\hat{\mathrm{e}}})}\phi^{\hat{\mathrm{e}}}(\bar{t}_{\hat{\mathrm{e}}}, \check{\underline{x}}, \check{\underline{o}})$ is invertible. Then by the implicit function theorem, the variables $t_{\hat{\mathrm{e}}}$, $x^-_{\hat{\mathrm{e}}}$, and $x^+_{\hat{\mathrm{e}}}$ can be expressed as functions of other variables in $(\underline{x}_{\hat{\mathrm{v}}-}, \underline{o}_{\hat{\mathrm{v}}-})$ and $(\underline{x}_{\hat{\mathrm{v}}+}, \underline{o}_{\hat{\mathrm{v}}+})$. Thus we define a set $\acute{N}_\delta$ by eliminating the components $t_{\hat{\mathrm{e}}}$, $x^-_{\hat{\mathrm{e}}}$, and $x^+_{\hat{\mathrm{e}}}$ for each $\mathrm{e} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$ from the set N_δ. More precisely, we have

$$\acute{N}_\delta = \left\{\left((t_{\hat{\mathrm{e}}})_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}}, (x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}, \hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}}, (x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}, \hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}}, (\underline{o}_{\hat{\mathrm{v}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}}\right)\right\}, \tag{7.72}$$

By abuse of notation, we still write each element of $\acute{N}_\delta$ as $(\underline{t}, \underline{x}, \underline{o})$. We now define a map $\acute{\Xi}$ on $\acute{N}_\delta$

$$\acute{\Xi}(\underline{t}, \underline{x}, \underline{o}) = \left((\acute{\phi}^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}, \underline{o}))_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}}, (x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}, \hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}}, (\underline{o}_{\hat{\mathrm{v}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}}\right), \tag{7.73}$$

where $\acute{\phi}^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}, \underline{o})$ is defined from $\phi^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}, \underline{o})$ in (7.71) by expressing marked points $(x^-_{\hat{\mathrm{f}}}, x^+_{\hat{\mathrm{f}}})$ for $\hat{\mathrm{f}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$ in terms of other variables.

It follows from Lemma 7.28 that for an additional gluing edge $\hat{e} \in \hat{E}^{ag}$, the derivative $D_{(t_{\hat{e}}, x^-_{\hat{e}}, x^+_{\hat{e}})} \acute{\phi}^{\hat{e}}(\underline{0}, \underline{\check{x}}, \underline{\check{o}})$ is invertible. Moreover, the (η, a) coordinates given in Lemma 7.28 show that we have derivative $\partial_y \acute{\phi}(\underline{0}, \underline{\check{x}}, \underline{\check{o}}) = 0$ for all variables y in $(\underline{t}, \underline{x}, \underline{o})$ other than $t_{\hat{e}}, x^-_{\hat{e}}, x^+_{\hat{e}}$. This combined with the simple form of derivatives of other components implies that the derivative $D\acute{\Xi}(\underline{0}, \underline{\check{x}}, \underline{\check{o}})$ is invertible. Thus for $\delta > 0$ small enough, $\acute{N}$ is a manifold with boundary and $\acute{\Xi}$ is a diffeomorphism on $\acute{N}_\delta$. This shows that the original set M^{slc}_δ is a manifold with boundary and the original map Ω is a diffeomorphism on M^{slc}_δ.

(1) Similarly we prove (1), the submersion result of Ω on $M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$. Analogous to Lemma 7.29, the inverse of the map

$$\begin{aligned} \Psi : \mathcal{U}_\delta(\underline{\bar{x}}, \underline{\bar{o}}) \times \mathcal{U}_\delta(\bar{\xi}) &\to \mathcal{U}_\lambda(\bar{\xi}(\underline{\bar{x}}, \underline{\bar{o}})) \times \mathcal{U}_\lambda(\bar{\xi}) \\ (\underline{x}, \underline{o}, \xi) &\mapsto \big(\xi(\underline{x}, \underline{o}), \underline{\xi}\big) \end{aligned}$$

is of the form

$$\begin{aligned} \Gamma : \mathcal{U}_\delta(\bar{\xi}(\underline{\bar{x}}, \underline{\bar{o}})) \times \mathcal{U}_\delta(\bar{\xi}) &\to \mathcal{U}_\lambda(\underline{\bar{x}}, \underline{\bar{o}}) \times \mathcal{U}_\lambda(\bar{\xi}) \\ (\underline{x}, \underline{o}, \underline{\xi}) &\mapsto \big(\xi^{-1}(\underline{x}, \underline{o}), \xi\big) \,, \end{aligned}$$

As before, we change coordinates and study the map $\Omega \circ \Gamma$. Denote the following marked points by

$$(\underline{\check{x}}, \underline{\check{o}}) := \underline{\bar{\xi}}(\underline{\bar{x}}, \underline{\bar{o}}).$$

We now define a set of generalized family of strip coordinates near $\check{x}^\pm_{\hat{e}}$ for each gluing edge $\hat{e} \in \hat{E}^g$ by

$$g^\pm_{\hat{e}}(x^\pm_{\hat{e}}, \xi^{\hat{v}^\pm}, \cdot) := \xi^{\hat{v}^\pm} \circ h^\pm_{\hat{e}}\left(\left(\xi^{\hat{v}^\pm}\right)^{-1}(x^\pm_{\hat{e}}), \cdot \right). \tag{7.74}$$

The first component of Ω is transformed to a generalized shift map

$$\phi^{\hat{e}}(t_{\hat{e}}, \underline{x}, \underline{\xi}) = g^-_{\hat{e}}(x^-_{\hat{e}}, \xi^{\hat{v}^-}, \cdot) \circ L_{R(t_{\hat{e}})} \circ g^+_{\hat{e}}(x^+_{\hat{e}}, \xi^{\hat{v}^+}, \cdot)^{-1}. \tag{7.75}$$

As before, the composite map $\Xi := \Omega \circ \Gamma$ of the form

$$\Xi(\underline{t}, \underline{x}, \underline{o}, \underline{\xi}) = \Big((\phi^{\hat{e}}(t_{\hat{e}}, \underline{x}, \underline{\xi}))_{\hat{e} \in \hat{E}^{ag}}, (x_{\hat{v}, \hat{e}})_{\hat{v} \in \hat{V}^m, \hat{e} \notin \hat{E}^g}, (\underline{o}_{\hat{v}})_{\hat{v} \in \hat{V}^m} \Big) \,,$$

whose domain is given by

$$N_\delta := \big\{ (\underline{t}, \underline{x}, \underline{o}, \underline{\xi}) \,|\, (1), (2), (3) \big\} \,.$$

(1) We have $\underline{t} \in \mathcal{U}_\delta(\underline{\bar{t}})$, and $t_{\hat{e}} \geq 0$ for each gluing edge $\hat{e} \in \hat{E}^g$ (in particular $t_{\hat{e}} \in [0, \delta)$ for each additional gluing edge $\hat{e} \in \hat{E}^{ag}$).
(2) We have $(\underline{x}, \underline{o}) \in \mathcal{U}_\delta(\underline{\check{x}}, \underline{\check{o}})$ and $\xi \in \mathcal{U}_\delta(\underline{\bar{\xi}})$.
(3) We impose the constraint $\phi^{\hat{e}}(t_{\hat{e}}, \underline{x}, \underline{\xi}) = \mathrm{Id}$ for all gluing edges $\hat{e} \in \hat{E}^g_{\underline{\bar{r}}}$.

Similar to the proof of (2), it follows from Lemma 7.27 and Lemma 7.28 that the map Ξ is a a submersion on N_δ, and so it the original map Ω on M_δ.

(3) Lastly we prove (3). We have

$$\xi^{\hat{v}^-} \circ \phi^{\hat{e}}(t_{\hat{e}}, \underline{x}) \circ (\xi^{\hat{v}^+})^{-1} = \phi^{\hat{e}}(t_{\hat{e}}, \underline{\xi}(\underline{x}), \underline{\xi}), \tag{7.76}$$

where the latter is the generalized shift map given in (7.75). It follows from Lemma 7.28 that the (η, a) coordinates of $\xi^{\hat{v}^-} \circ \phi^{\hat{e}}(0, \underline{x}) \circ (\xi^{\hat{v}^+})^{-1}$ is given by $\left(\xi^{\hat{v}^-}(x^-_{\hat{e}}) \cdot \left(\xi^{\hat{v}^+}(x^+_{\hat{e}})\right)^{-1},\ \xi^{\hat{v}^+}(x^+_{\hat{e}})\right)$ □

Proof of Lemma 7.26. We first analyze the derivative of Ω in local coordinates. Denote the image of $(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}})$ under Ω in Lemma 7.24 by

$$((\underline{\bar{\eta}}, \underline{\bar{a}}), \underline{\bar{y}}, \underline{\bar{p}}) := \Omega(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}).$$

More precisely, Lemma 7.24 (3) implies that the first component is given by $(\underline{\bar{\eta}}, \underline{\bar{a}}) = (\bar{\eta}_{\hat{e}}, \bar{a}_{\hat{e}})_{\hat{e} \in \hat{E}^{ag}}$, with $\bar{\eta}_{\hat{e}} = \bar{\xi}^{\hat{v}^-}(\bar{x}^-_{\hat{e}}) \cdot \left(\bar{\xi}^{\hat{v}^+}(\bar{x}^+_{\hat{e}})\right)^{-1}$ and $\bar{a}_{\hat{e}} = \bar{\xi}^{\hat{v}^+}(\bar{x}^+_{\hat{e}})$. Also, the last two components are given by boundary marked points $\underline{\bar{y}} = (\bar{y}_{\hat{v},\hat{e}})_{\hat{v} \in \hat{V}^m, \hat{e} \notin \hat{E}^g}$ with $\bar{y}_{\hat{v},\hat{e}} = \bar{\xi}^{\hat{v}}(\bar{x}_{\hat{v},\hat{e}})$, and interior marked points $\underline{\bar{p}} = (\underline{\bar{p}}_{\hat{v}})_{\hat{v} \in \hat{V}^m}$ with $\underline{\bar{p}}_{\hat{v}} = \bar{\xi}^{\hat{v}}(\underline{\bar{o}}_{\hat{v}})$. Thus Lemma 7.24 (1) defines a map with smaller target space

$$\Omega : M_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) \to \mathcal{U}_\varepsilon((\underline{\bar{\eta}}, \underline{\bar{a}}), \underline{\bar{y}}, \underline{\bar{p}}).$$

We now set up a chart around the point $((\underline{\bar{\eta}}, \underline{\bar{a}}), \underline{\bar{y}}, \underline{\bar{p}})$. For each additional gluing edge $\hat{e} \in \hat{E}^{ag}$, recall that $\bar{a}_{\hat{e}} = \bar{\xi}^{\hat{v}^+}(\bar{x}^+_{\hat{e}})$ lies on the boundary ∂D. Choose a smooth chart $f_{\hat{e}} : B(\bar{a}_{\hat{e}}) \to B(0)$ with $f_{\hat{e}}(\bar{a}_{\hat{e}}) = 0$, where $B(\bar{a}_{\hat{e}})$ is a neighborhood of $\bar{a}_{\hat{e}}$ in the disk D, and $B(0)$ is a neighborhood of 0 in the upper-half plane $\{\mathrm{Im} z \geq 0\}$. For each point $((\underline{\eta}, \underline{a}), \underline{y}, \underline{p}) \in \mathcal{U}_\varepsilon((\underline{\bar{\eta}}, \underline{\bar{a}}), \underline{\bar{y}}, \underline{\bar{p}})$, we denote the real and imaginary parts of $a_{\hat{e}}$ in the coordinates of $f_{\hat{e}}$ by

$$a^1_{\hat{e}} := \mathrm{Re} f_{\hat{e}}(a_{\hat{e}}), \quad a^2_{\hat{e}} := \mathrm{Im} f_{\hat{e}}(a_{\hat{e}}). \tag{7.77}$$

Recall that the element $\xi^{\hat{v}^-} \circ \phi^{\hat{e}}(0, \underline{x}) \circ (\xi^{\hat{v}^+})^{-1}$ can be thought of as a generalized shift map $\phi^{\hat{e}}(t_{\hat{e}}, \underline{\xi}(\underline{x}), \underline{\xi})$ as in (7.76). We view $((\underline{\eta}, \underline{a}), \underline{y}, \underline{p}) := \Omega(\underline{t}, \underline{x}, \underline{o}, \underline{\xi})$ as a function of $(\underline{t}, \underline{x}, \underline{o}, \underline{\xi})$. The calculation in (7.67) implies that at a point $(\underline{\check{t}}, \underline{\check{x}}, \underline{\check{o}}, \underline{\check{\xi}})$ with $\check{t}_{\hat{e}} = 0$, the derivative of the imaginary part of $a_{\hat{e}}$ satisfies

$$\partial_{t_{\hat{e}}} a^2_{\hat{e}} > 0. \tag{7.78}$$

We shall use this result later in computing the derivative $\partial_{t_{\hat{e}}} t'_{\hat{e}'}$.

We now study the derivatives of the inverse of the map Ω defined on the slice

$$\Omega : M^{\mathrm{slc}}_\delta(\underline{\bar{t}}, \underline{\bar{x}}, \underline{\bar{o}}, \underline{\bar{\xi}}) \to \mathcal{U}_\varepsilon((\underline{\bar{\eta}}, \underline{\bar{a}}), \underline{\bar{y}}, \underline{\bar{p}}).$$

Lemma 7.24 (2) implies that it is a diffeomorphism onto its image. Thus viewing $(\underline{t}, \underline{x}, \underline{o}, \underline{\xi}) := \Omega^{-1}((\underline{\eta}, \underline{a}), \underline{y}, \underline{p})$ as a function of $((\underline{\eta}, \underline{a}), \underline{y}, \underline{p})$, at a point $((\underline{\check{\eta}}, \underline{\check{a}}), \underline{\check{y}}, \underline{\check{p}})$ with imaginary part $\check{a}^2_{\hat{e}} = 0$, we have

$$\partial_{\eta_{\hat{e}}} t_{\hat{e}} = 0, \quad \partial_{a^1_{\hat{e}}} t_{\hat{e}} = 0. \tag{7.79}$$

This is because for all points $((\underline{\eta}, \underline{a}), \underline{y}, \underline{p})$ near $((\underline{\check{\eta}}, \underline{\check{a}}), \underline{\check{y}}, \underline{\check{p}})$ with fixed $a^2_{\hat{e}} = 0$, the element $\xi^{\hat{v}^-} \circ \phi^{\hat{e}}(t_{\hat{e}}, \underline{x}) \circ (\xi^{\hat{v}^+})^{-1}$ lies on the boundary of $\overline{\mathrm{Aut}}(D)$ and thus we have $t_{\hat{e}} = 0$.

We now compute the derivative $\partial_{t_{\hat{e}}} t'_{\hat{e}'}$ in the statement of the lemma. By the chain rule, we have

$$\begin{aligned} 1 = \partial_{t'_{\hat{e}'}} t'_{\hat{e}'} &= \partial_{\eta_{\hat{e}}} t'_{\hat{e}'} \cdot \partial_{t'_{\hat{e}'}} \eta_{\hat{e}} + \partial_{a^1_{\hat{e}}} t'_{\hat{e}'} \cdot \partial_{t'_{\hat{e}'}} a^1_{\hat{e}} + \partial_{a^2_{\hat{e}}} t'_{\hat{e}'} \cdot \partial_{t'_{\hat{e}'}} a^2_{\hat{e}} \\ &= \partial_{a^2_{\hat{e}}} t'_{\hat{e}'} \cdot \partial_{t'_{\hat{e}'}} a^2_{\hat{e}}, \end{aligned}$$

where the last equality is due to $\partial_{\eta_{\hat{e}}} t'_{\hat{e}'} = 0$ and $\partial_{a^1_{\hat{e}}} t'_{\hat{e}'} = 0$ in (7.79). Due to $\partial_{t'_{\hat{e}'}} a^2_{\hat{e}} > 0$ in (7.78), we have $\partial_{a^2_{\hat{e}}} t'_{\hat{e}'} > 0$. Similarly we have

$$\begin{aligned} \partial_{t_{\hat{e}}} t'_{\hat{e}'} &= \partial_{\eta_{\hat{e}}} t'_{\hat{e}'} \cdot \partial_{t_{\hat{e}}} \eta_{\hat{e}} + \partial_{a^1_{\hat{e}}} t'_{\hat{e}'} \cdot \partial_{t_{\hat{e}}} a^1_{\hat{e}} + \partial_{a^2_{\hat{e}}} t'_{\hat{e}'} \cdot \partial_{t_{\hat{e}}} a^2_{\hat{e}} \\ &= \partial_{a^2_{\hat{e}}} t'_{\hat{e}'} \cdot \partial_{t_{\hat{e}}} a^2_{\hat{e}} > 0. \end{aligned}$$

This proves the lemma. □

7.2.2. *Proof of Theorem 7.21.*

We have trivial biholomorphic equivalence

$$(\hat{\zeta}, \underline{\hat{\psi}})(\#(\underline{0}, \hat{\mu})) = \#(\underline{0}, \hat{\mu}),$$

with $\hat{\zeta} : \hat{\mathrm{T}} \to \hat{\mathrm{T}}$ being the identity isomorphism and each biholomorphism being $\hat{\psi}_{\hat{\mathrm{v}}} = \mathrm{Id}$, and the bijection of edges $\hat{\chi} : \hat{\mathrm{E}} \to \hat{\mathrm{E}}$ in (7.19) is the identity map. We now find $\delta, \delta' > 0$ such that if there are $(\underline{r}, \mu) \in \mathcal{U}^{\mathrm{slc}}_{\delta}(\underline{0}, \hat{\mu})$ and $(\underline{r}', \mu') \in \mathcal{U}^{\mathrm{slc}}_{\delta'}(\underline{0}, \hat{\mu})$ with

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mu)) = \#(\underline{r}', \mu'), \tag{7.80}$$

then $\hat{\zeta}$ covers ζ, we have equality of gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} = \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}'}$, and lastly we have matching marked points

$$\psi_{\mathrm{v}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) = x'_{\hat{\mathrm{v}},\hat{\mathrm{e}}}, \quad \psi_{\mathrm{v}}(o_{\hat{\mathrm{v}},j}) = o'_{\hat{\mathrm{v}},j}, \tag{7.81}$$

for gluing vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$, non-gluing edges $\hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} = \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}'}$, and indices $1 \le j \le n(\underline{o}_{\hat{\mathrm{v}}})$.

The covering property follows directly from Theorem 7.12, and the equality $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}} = \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}'}$ from Lemma 7.17. We now find δ, δ' such that equation (7.81) holds. For each main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}$ and index j, we choose an open neighborhood $B(\hat{x}_{\hat{\mathrm{v}},j})$ of the interior marked point $\hat{x}_{\hat{\mathrm{v}},j}$, similar to open neighborhoods $B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$ of boundary marked points $\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ in Remark 5.9; moreover, we choose these open neighborhoods mutually disjoint on each disk $D_{\hat{\mathrm{v}}}$.

For main vertices $\hat{\mathrm{v}}$, we denote the glued vertices $\mathrm{v} := [\hat{\mathrm{v}}]_{\underline{r}}$, and $\mathrm{v}' := [\hat{\mathrm{v}}]_{\underline{r}'}$, and denote the biholomorphisms ψ_{v} in the standard models on $D_{\hat{\mathrm{v}}}$ and $D_{\hat{\mathrm{v}}}$ by

$$\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}} := \left(\Phi^{\hat{\mathrm{v}}}_{\mathrm{v}'}\right)^{-1} \circ \psi_{\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{\mathrm{v}}. \tag{7.82}$$

There exists a positive number Δ' such that for all marked points $(\underline{x}', \underline{o}') \in \mathcal{U}_{\Delta'}(\hat{\underline{x}}, \hat{\underline{o}})$ and all main vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}$, we have $x'_{\hat{\mathrm{v}},\hat{\mathrm{e}}} \in B(\hat{x}'_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$ and $o'_{\hat{\mathrm{v}},j} \in$

$B(\hat{x}'_{\hat{\mathrm{v}},j})$. By Theorem 7.12, there are $\delta, \delta' > 0$ such that if we have biholomorphic equivalence (7.80), then we have estimate

$$d_{\mathrm{Aut}(D)}(\mathrm{Id}, \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}) < \Delta'/2. \tag{7.83}$$

Without loss of generality, we assume

$$\delta < \Delta'/2, \quad \delta' < \Delta'$$

because the above estimate persists if we decrease δ and δ'. In particular, we have an estimate in the uniform norm of disk automorphisms

$$\|\mathrm{Id} - \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}\|_{C^0} < \Delta'/2. \tag{7.84}$$

For each non-gluing edge $\hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}$, we now show $\psi_{\mathrm{v}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) = x'_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$. We have

$$\begin{aligned} |\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}} - \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})| &\le |\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}} - x_{\hat{\mathrm{v}},\hat{\mathrm{e}}}| + |x_{\hat{\mathrm{v}},\hat{\mathrm{e}}} - \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})| \\ &< \Delta'/2 + \Delta'/2 < \Delta'. \end{aligned} \tag{7.85}$$

We observe that both boundary marked points $x'_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ and $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$ are glued marked points in $\#^{\hat{\mathrm{v}}}_{\underline{r}'}(\underline{x}')_{\mathrm{v}'}$, and both are Δ'-close to the marked point $\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ due to $\delta' < \Delta'$ and estimate (7.85). Then by the choice of Δ', both $x'_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ and $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$ lie in the open set $B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$. But $\hat{\mathrm{e}}$ is a non-gluing edge, so by Lemma 7.6 (2) there is only one marked point in $\#^{\hat{\mathrm{v}}}_{\underline{r}',\underline{x}'}(\underline{x}', \underline{o}')_{\mathrm{v}'}$ that lies in the open set $B(\hat{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}})$. Hence we have $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(x_{\hat{\mathrm{v}},\hat{\mathrm{e}}}) = x'_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ as in (7.81).

We now prove $\psi_{\mathrm{v}}(o_{\hat{\mathrm{v}},j}) = o'_{\hat{\mathrm{v}},j}$ in a similar fashion. As before, we have

$$\begin{aligned} |\hat{o}_{\hat{\mathrm{v}},j} - \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(o_{\hat{\mathrm{v}},j})| &\le |\hat{o}_{\hat{\mathrm{v}},j} - o_{\hat{\mathrm{v}},j}| + |o_{\hat{\mathrm{v}},j} - \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(o_{\hat{\mathrm{v}},j})| \\ &< \Delta'/2 + \Delta'/2 < \Delta'. \end{aligned} \tag{7.86}$$

Both interior marked points $o'_{\hat{\mathrm{v}},j}$ and $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(o_{\hat{\mathrm{v}},j})$ are glued marked points in $\#^{\hat{\mathrm{v}}}_{\underline{r}',\underline{x}'}(\underline{o}')_{\mathrm{v}'}$, and both are Δ'-close to the marked point $\hat{o}_{\hat{\mathrm{v}},j}$ due to $\delta' < \Delta'$ and estimate (7.86). By the choice of Δ', both $o'_{\hat{\mathrm{v}},j}$ and $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(o_{\hat{\mathrm{v}},j})$ lie in the open set $B(\hat{o}_{\hat{\mathrm{v}},j})$. But there is only one marked point in $\#^{\hat{\mathrm{v}}}_{\underline{r}',\underline{x}'}(\underline{x}', \underline{o}')_{\mathrm{v}'}$ that lies in the open set $B(\hat{o}_{\hat{\mathrm{v}},j})$. Hence we have $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}}(o_{\hat{\mathrm{v}},j}) = o'_{\hat{\mathrm{v}},j}$ as in (7.81). This finishes the proof of equation (7.81).

We shall further restrict $\delta, \delta' > 0$ such that for $(\underline{r}, \mu) \in \mathcal{U}^{\mathrm{slc}}_{\delta}(\underline{0}, \hat{\mu})$ and $(\underline{r}', \mu') \in \mathcal{U}^{\mathrm{slc}}_{\delta'}(\underline{0}, \hat{\mu})$ with equivalence $(\zeta, \underline{\psi})(\#(\underline{r}, \mu)) = \#(\underline{r}', \mu')$, we have the injectivity result $(\underline{r}, \mu) = (\underline{r}', \mu')$. Let $\underline{\hat{\xi}} = (\xi^{\hat{\mathrm{v}}})_{\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}}$ be the tuple of identity disk automorphisms, i.e., $\xi^{\hat{\mathrm{v}}} = \mathrm{Id}$, and let $\alpha : M^{\mathrm{slc}}_{\hat{\delta}}(\underline{0}, \underline{\hat{\ell}}, \underline{\hat{x}}, \underline{\hat{o}}, \underline{\hat{\xi}}) \to M^{\mathrm{slc}}_{\hat{\delta}}(\underline{0}, \underline{\hat{\ell}}, \underline{\hat{x}}, \underline{\hat{o}}, \underline{\hat{\xi}})$ be the map in Claim 7.23 restricted to the slice. Going through the construction of α in this case, we can show that α is the identity map. Indeed, for each set of gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}$, the map $\alpha_{\hat{\mathrm{E}}^{\mathrm{g}}}$ in (7.33) restricted to the slice has the form

$$\begin{aligned} \alpha_{\hat{\mathrm{E}}^{\mathrm{g}}} : M_{\hat{\delta}}(\underline{0}, \underline{\hat{\ell}}) \times M^{\mathrm{slc}}_{\hat{\delta}}(\underline{0}, \underline{\hat{x}}, \underline{\hat{o}}, \underline{\hat{\xi}}) &\to M_{\hat{\delta}}(\underline{0}, \underline{\hat{\ell}}) \times M^{\mathrm{slc}}_{\hat{\delta}}(\underline{0}, \underline{\hat{x}}, \underline{\hat{o}}, \underline{\hat{\xi}}) \\ ((\underline{s}, \underline{\ell}), (\underline{t}, \underline{x}, \underline{o}, \underline{\xi})) &\mapsto ((\underline{s}', \underline{\ell}'), (\underline{t}', \underline{x}', \underline{o}', \underline{\xi}')), \end{aligned}$$

where parameters $\underline{s} = (s_{\hat{e}})_{\hat{e}\in\hat{E}^{nd}\setminus\hat{E}^{g}}$ and $\underline{s}' = (s'_{\hat{e}})_{\hat{e}\in\hat{E}^{nd}\setminus\hat{E}^{g}}$ are non-positive, and parameters $\underline{t} = (t_{\hat{e}})_{\hat{e}\in\hat{E}^{nd}\setminus\hat{E}^{g}}$ and $\underline{t}' = (t'_{\hat{e}})_{\hat{e}\in\hat{E}^{nd}\setminus\hat{E}^{g}}$ are non-negative. Going through the construction of the non-gluing components $M_{\hat{\delta}}(\underline{0}, \underline{\hat{\ell}}) \to M_{\hat{\delta}}(\underline{0}, \underline{\hat{\ell}})$ with $(\underline{s}, \underline{\ell}) \mapsto (\underline{s}', \underline{\ell}')$, the form of equation (7.40) turns into $\ell_{\hat{e}} = \ell'_{\hat{e}}$ or $s_{\hat{e}} = s'_{\hat{e}}$. Therefore the non-gluing components of the map $\alpha_{\hat{E}^{g}}$ is the identity. As for the gluing components of the map $M^{slc}_{\hat{\delta}}(\underline{0}, \underline{\hat{x}}, \underline{\hat{o}}, \underline{\hat{\xi}}) \to M^{slc}_{\hat{\delta}}(\underline{0}, \underline{\hat{x}}, \underline{\hat{o}}, \underline{\hat{\xi}})$ constructed in (7.47), the map $\Omega^{-1} \circ \Omega$ is the identity map, where Ω is defined in Lemma 7.24.

We now choose the positive Δ' in the previous analysis so that for any disk automorphism ψ close to the identity map $d_{\mathrm{Aut}(D)}(\mathrm{Id}, \psi) < \Delta'/2$, we have $d_{\mathrm{Aut}(D)}(\mathrm{Id}, \psi^{-1}) < \hat{\delta}$, and we choose δ, δ' as before with the additional condition $\delta, \delta' < \hat{\delta}$. For such choices, we now show that for $(\underline{r}, \mu) \in \mathcal{U}^{slc}_{\delta}(\underline{0}, \hat{\mu})$ and $(\underline{r}', \mu') \in \mathcal{U}^{slc}_{\delta'}(\underline{0}, \hat{\mu})$ with equivalence

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mu)) = \#(\underline{r}', \mu'), \tag{7.87}$$

we have the injectivity result $(\underline{r}, \mu) = (\underline{r}', \mu')$. Firstly, denote the gluing edges by $\hat{E}^{g} := \hat{E}^{g}_{\underline{r}} = \hat{E}^{g}_{\underline{r}'}$, and write the gluing parameters as $\underline{r} = (\underline{s}, \underline{t})$ and $\underline{r}' = (\underline{s}', \underline{t}')$ where $\underline{s}$ and $\underline{s}'$ are non-positive, and $\underline{t}$ and $\underline{t}'$ are non-negative. Moreover, we define disk automorphisms $\underline{\xi}' = (\xi'^{\hat{v}})_{\hat{v}\in\hat{V}^{m}}$ by

$$\xi'^{\hat{v}} := (\psi^{\hat{v},\hat{v}})^{-1}. \tag{7.88}$$

We now verify that

(1) the element $(\underline{t}', \underline{x}', \underline{o}', \underline{\xi}')$ lies in the domain $M^{slc}_{\hat{\delta}}(\underline{0}, \underline{\hat{x}}, \underline{\hat{o}}, \underline{\hat{\xi}})$, and
(2) we have $\alpha_{\hat{E}^{g}}((\underline{s}, \underline{\ell}), (\underline{t}, \underline{x}, \underline{o}, \underline{\hat{\xi}})) = ((\underline{s}', \underline{\ell}'), (\underline{t}', \underline{x}', \underline{o}', \underline{\xi}'))$.

Indeed, property (1) is satisfied due to our choices of Δ' and δ'. We now show property (2). It follows from the equivalence (7.87) that by the definition of the non-gluing components of $\alpha_{\hat{E}^{g}}$ (see (7.38)), $(\underline{s}', \underline{\ell}')$ is the image of $(\underline{s}, \underline{\ell})$. It suffices to show

$$\Omega(\underline{t}, \underline{x}, \underline{o}, \underline{\hat{\xi}}) = \Omega(\underline{t}', \underline{x}', \underline{o}', \underline{\xi}'),$$

where Ω is the map in Lemma 7.24. By equation (7.81) and definition (7.88), we have

$$x_{\hat{v},\hat{e}} = (\psi^{\hat{v},\hat{v}})^{-1}(x'_{\hat{v},\hat{e}}) = \xi'^{\hat{v}}(x'_{\hat{v},\hat{e}}), \quad x_{\hat{v},j} = (\psi^{\hat{v},\hat{v}})^{-1}(x'_{\hat{v},j}) = \xi'^{\hat{v}}(x'_{\hat{v},j}).$$

This shows that the second and the third components of Ω match on both sides. It only suffices to prove $\phi^{\hat{e}}(t_{\hat{e}}, \underline{x}) = \xi'^{\hat{v}^-} \circ \phi^{\hat{e}}(t'_{\hat{e}}, \underline{x}') \circ (\xi'^{\hat{v}^+})^{-1}$ for each gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}$. We have

$$
\begin{aligned}
&\xi'^{\hat{\mathrm{v}}^-} \circ \phi^{\hat{\mathrm{e}}}(t'_{\hat{\mathrm{e}}}, \underline{x}') \circ (\xi'^{\hat{\mathrm{v}}^+})^{-1} \\
&= (\psi^{\hat{\mathrm{v}}^-, \hat{\mathrm{v}}^-})^{-1} \circ \phi^{\hat{\mathrm{e}}}(t'_{\hat{\mathrm{e}}}, \underline{x}') \circ \psi^{\hat{\mathrm{v}}^+, \hat{\mathrm{v}}^+} \\
&= \left(\Phi_{\mathrm{v}}^{\hat{\mathrm{v}}^-}\right)^{-1} \circ \psi_{\mathrm{v}}^{-1} \circ \Phi_{\mathrm{v}'}^{\hat{\mathrm{v}}^-} \circ \phi^{\hat{\mathrm{e}}}(t'_{\hat{\mathrm{e}}}, \underline{x}') \circ \left(\Phi_{\mathrm{v}'}^{\hat{\mathrm{v}}^+}\right)^{-1} \circ \psi_{\mathrm{v}} \circ \Phi_{\mathrm{v}}^{\hat{\mathrm{v}}^+} \\
&= \left(\Phi_{\mathrm{v}}^{\hat{\mathrm{v}}^-}\right)^{-1} \circ \Phi_{\mathrm{v}}^{\hat{\mathrm{v}}^+} \\
&= \phi^{\hat{\mathrm{e}}}(t_{\hat{\mathrm{e}}}, \underline{x}).
\end{aligned}
$$

The first equality is by definition (7.88), and the second equality is by definition (7.82). The last two equalities follow from Lemma 7.4 (2). This finishes proving property (2). Thus we have $\alpha(\underline{r}, \underline{\ell}, \underline{x}, \underline{o}, \hat{\underline{\xi}}) = (\underline{r}', \underline{\ell}', \underline{x}', \underline{o}', \underline{\xi}')$, but on the other hand, α is the identity map. Hence in particular, we have $(\underline{r}, \underline{\ell}, \underline{x}, \underline{o}) = (\underline{r}', \underline{\ell}', \underline{x}', \underline{o}')$. Choosing $\hat{\varepsilon}_{\mathrm{inj}}(\hat{\mu})$ to be the minimum of δ and δ', we have the injectivity result.

8. Proof for the Topology and the Atlas of the Quotient Space of Disk Trees

8.1. **Sc-smoothness of Splicing.**

In this section, we prove the sc-smoothness result in the splicing arising from gluing (Section 6.2), and we shall do so by adapting the sc-smoothness analysis in [7] to our setting.

Recall from Section 6.2 that F^{nd} is the space of strip nodal maps, and recall the smooth cut-off function $\beta : \mathbb{R} \to [0,1]$ with properties

- $\beta(s) = 1$ for $s \leq -1$ and $\beta(s) = 0$ for $s \geq 1$,
- $\beta'(s) < 0$ for $s \in (-1, 1)$,
- $\beta(s) + \beta(-s) = 1$ for all s.

We denote the shifted cut-off function by

$$\beta_{s_0}(s) := \beta(s - s_0).$$

For a strip nodal map $(\xi^+, \xi^-) \in F^{\mathrm{nd}}$, the plus gluing in (6.3) is given by

$$\oplus_r(\xi^+, \xi^-) = \beta_{R/2}\, \xi^+ + (1 - \beta_{R/2})\, \xi^-(\cdot - R),$$

and its "complementary map" the minus gluing $\ominus_r$ in (6.4) is given by

$$
\begin{aligned}
\ominus_r(\xi^+, \xi^-) = &- (1 - \beta_{R/2})\big(\xi^+ - \tfrac{1}{2}([\xi^+]_{R/2} + [\xi^-]_{-R/2})\big) \\
&+ \beta_{R/2}\big(\xi^-(\cdot - R) - \tfrac{1}{2}([\xi^+]_{R/2} + [\xi^-]_{-R/2})\big),
\end{aligned}
$$

where $[\xi^+]_{R/2}$ and $[\xi^-]_{-R/2}$ are the real parts of the integrals

$$[\xi^+]_{R/2} := \mathrm{Re} \int_0^\pi \xi^+(R/2, t)dt, \quad [\xi^-]_{-R/2} := \mathrm{Re} \int_0^\pi \xi^-(-R/2, t)dt.$$

We now prove Lemma 6.8, that the total gluing $\boxdot_r = (\oplus_r, \ominus_r)$ is a linear sc-isomorphism. This result is analogous to Theorem 1.27 of [7].

Proof of Lemma 6.8. For each (u,v) in the target space $G_r = H^{3,\delta_0}([0,R]\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n)\times H^{3,\delta_0}_{\text{op-lim}}(\mathbb{R}\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n)$, we now solve the linear equation $\boxdot_r(\xi^+,\xi^-)=(u,v)$ for $(\xi^+,\xi^-)\in F^{\text{nd}}$. We define a non-vanishing smooth function $\gamma:\mathbb{R}\to\mathbb{R}$ by

$$\gamma = \beta^2 + (1-\beta)^2, \tag{8.1}$$

and denote the shifted function by

$$\gamma_{s_0}(s) := \beta(s-s_0).$$

Analogous to the proof Theorem 1.27 of [7], solving the linear equation

$$\boxdot_r(\xi^+,\xi^-)=(u,v)$$

amounts to inverting a matrix. The solution to the above equation is given by

$$\begin{bmatrix}\xi^+\\ \xi^-(\cdot-R)\end{bmatrix} = \frac{1}{\gamma_{R/2}}\begin{bmatrix}\beta_{R/2} & -(1-\beta_{R/2})\\ 1-\beta_{R/2} & \beta_{R/2}\end{bmatrix}\begin{bmatrix}u\\ (2\beta_{R/2}-1)[u]_{R/2}+v\end{bmatrix}. \tag{8.2}$$

Note that the non-vanishing smooth function γ arises from taking the determinant of the matrix $\begin{bmatrix}\beta_{R/2} & (1-\beta_{R/2})\\ -(1-\beta_{R/2}) & \beta_{R/2}\end{bmatrix}$ in the total gluing. $\square$

For the rest of this section, we choose R to be the exponential gluing profile given by $\varphi(r)=e^{1/r}-1$.

We now derive the formula for the splicing $(r,\xi^+,\xi^-)\mapsto(\zeta^+,\zeta^-)$ in (6.5) and prove its sc-smoothness as stated in Theorem 6.9. Recall from equation (6.5) that for $r\in(0,\varepsilon)$, the splicing $\pi_r(\xi^+,\xi^-)=(\zeta^+,\zeta^-)$ is defined by

$$\begin{aligned}\oplus_r(\zeta^+,\zeta^-) &= \oplus_r(\xi^+,\xi^-),\\ \ominus_r(\zeta^+,\zeta^-) &= 0.\end{aligned}$$

Through a computation using the solution (8.2), we have

$$\begin{aligned}\zeta^+ &= \left(1-\frac{\beta_{R/2}}{\gamma_{R/2}}\right)\frac{1}{2}([\xi^+]_{R/2}+[\xi^-]_{-R/2}) \\ &\quad+\frac{\beta_{R/2}^2}{\gamma_{R/2}}\xi^+ + \frac{\beta_{R/2}(1-\beta_{R/2})}{\gamma_{R/2}}\xi^-(\cdot-R),\end{aligned} \tag{8.3}$$

$$\begin{aligned}\zeta^- &= \left(1-\frac{1-\beta_{-R/2}}{\gamma_{-R/2}}\right)\frac{1}{2}([\xi^+]_{R/2}+[\xi^-]_{-R/2}) \\ &\quad+\frac{\beta_{-R/2}(1-\beta_{-R/2})}{\gamma_{-R/2}}\xi^+(\cdot+R) + \frac{(1-\beta_{-R/2})^2}{\gamma_{-R/2}}\xi^-.\end{aligned} \tag{8.4}$$

It is easier to analyze the sc-smoothness of the splicing $\pi_r: F^{\text{nd}}\to F^{\text{nd}}$ by viewing each strip nodal map (ξ^+,ξ^-) as the pair $(\eta^+ + c, \eta^- + c)$. Here c is the common limit of ξ^+ and ξ^-, and we have $\eta^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$. In this view, the splicing $(r, \eta^+ + c, \eta^- + c) \mapsto (\zeta^+, \zeta^-)$ has the formula

$$
\begin{aligned}
\zeta^+ &= c + \left(1 - \frac{\beta_{R/2}}{\gamma_{R/2}}\right) \frac{1}{2}([\eta^+]_{R/2} + [\eta^-]_{-R/2}) \\
&\quad + \frac{\beta_{R/2}^2}{\gamma_{R/2}}\eta^+ + \frac{\beta_{R/2}(1-\beta_{R/2})}{\gamma_{R/2}}\eta^-(\cdot - R),
\end{aligned} \tag{8.5}
$$

$$
\begin{aligned}
\zeta^- &= c + \left(1 - \frac{1-\beta_{-R/2}}{\gamma_{-R/2}}\right) \frac{1}{2}([\eta^+]_{R/2} + [\eta^-]_{-R/2}) \\
&\quad + \frac{\beta_{-R/2}(1-\beta_{-R/2})}{\gamma_{-R/2}}\eta^+(\cdot + R) + \frac{(1-\beta_{-R/2})^2}{\gamma_{-R/2}}\eta^-.
\end{aligned} \tag{8.6}
$$

We now analyze the sc-smoothness of (8.5) and (8.6) term by term. The following result is adapted from Proposition 2.17 of [7] to our setting.

Proposition 8.1. *We have the following sc-smoothness results.*

(1) *The maps*

$$H^{3,\delta_0}_{\lim}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n) \to \mathbb{R}^n, \quad \xi^\pm \mapsto c$$

are sc-smooth, where c is the limit of $\xi^\pm$.

(2) *The maps*

$$
\begin{aligned}
[0,\varepsilon) \times H^{3,\delta_0}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n) &\to H^{3,\delta_0}(\mathbb{R}^+ \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n), \\
(r, \eta^\pm) &\mapsto \left(1 - \tfrac{\beta_{R/2}}{\gamma_{R/2}}\right)[\eta^\pm]_{\pm R/2}, \\
(0, \eta^\pm) &\mapsto 0, \\
[0,\varepsilon) \times H^{3,\delta_0}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n) &\to H^{3,\delta_0}(\mathbb{R}^- \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n), \\
(r, \eta^\pm) &\mapsto \left(1 - \tfrac{1-\beta_{-R/2}}{\gamma_{-R/2}}\right)[\eta^\pm]_{\pm R/2}, \\
(0, \eta^\pm) &\mapsto 0,
\end{aligned}
$$

are sc-smooth, with vanishing derivatives at $r = 0$.

(3) *The maps*

$$
\begin{aligned}
[0,\varepsilon) \times H^{3,\delta_0}(\mathbb{R}^+ \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n) &\to H^{3,\delta_0}(\mathbb{R}^+ \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n), \\
(r, \eta^+) &\mapsto \tfrac{\beta_{R/2}^2}{\gamma_{R/2}}\eta^+, \\
(0, \eta^+) &\mapsto \eta^+, \\
[0,\varepsilon) \times H^{3,\delta_0}(\mathbb{R}^- \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n) &\to H^{3,\delta_0}(\mathbb{R}^- \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n), \\
(r, \eta^-) &\mapsto \tfrac{(1-\beta_{-R/2})^2}{\gamma_{-R/2}}\eta^-, \\
(0, \eta^-) &\mapsto \eta^-,
\end{aligned}
$$

are sc-smooth. Moreover, their difference with the identity map Id *has vanishing derivatives at $r = 0$.*

(4) *The maps*

$$\begin{aligned}
[0,\varepsilon)\times H^{3,\delta_0}(\mathbb{R}^+\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n) &\to H^{3,\delta_0}(\mathbb{R}^-\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n),\\
(r,\eta^+) &\mapsto \tfrac{\beta_{-R/2}(1-\beta_{-R/2})}{\gamma_{-R/2}}\eta^+(\cdot+R),\\
(0,\eta^+) &\mapsto 0,\\
[0,\varepsilon)\times H^{3,\delta_0}(\mathbb{R}^-\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n) &\to H^{3,\delta_0}(\mathbb{R}^+\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n),\\
(r,\eta^-) &\mapsto \tfrac{\beta_{R/2}(1-\beta_{R/2})}{\gamma_{R/2}}\eta^-(\cdot-R),\\
(0,\eta^-) &\mapsto 0,
\end{aligned}$$

are sc-smooth, with vanishing derivatives at $r=0$*.*

We now use the above term by term analysis to prove the sc-smoothness of the splicing π_r.

Proof of Theorem 6.9. For $r\in(-\varepsilon,0]$, the splicing is defined to be $\pi_r=\mathrm{Id}$. The derivatives match with the term by term analysis in Proposition 8.1 for $r\geq 0$. $\square$

We now show the term by term sc-smoothness in Proposition 8.1 by quoting [7].

Proof of Proposition 8.1. (1) is given by Lemma 2.18 [7]: the map $\xi^\pm\mapsto c$ is an sc-operator and hence sc-smoothness. (2) follows from Lemma 2.19 and Lemma 2.20 of [7]. (3) follows from the sc-smoothness of maps Γ_1 and Γ_3 of Proposition 8.2, since $\frac{\beta^2}{\gamma}$ and $\frac{(1-\beta)^2}{\gamma}$ are both constant outside of a compact interval and they have the desired limits. Lastly, (4) follows from the sc-smoothness of maps Γ_2 and Γ_4 of Proposition 8.2, since $\frac{\beta(1-\beta)}{\gamma}$ is compactly supported. $\square$

The following more general sc-smoothness result is adapted from Proposition 2.8 of [7] to our setting.

Proposition 8.2. *The following maps*

$$\Gamma_i:[0,\varepsilon)\times H^{3,\delta_0}(\mathbb{R}\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n)\to H^{3,\delta_0}(\mathbb{R}\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n)$$

are sc-smooth. Moreover, for $i=2,4$ *we have vanishing derivatives* $T^k\Gamma_i=0$ *at* $r=0$*; for* $i=1,3$ *we have vanishing derivatives* $T^k(\Gamma_i-\mathrm{Id})=0$ *at* $r=0$*.*

(1) *Let* $f_1:\mathbb{R}\to\mathbb{R}$ *be a smooth function which is constant outside of a compact interval with* $f_1(\infty)=0$ *and* $f_1(-\infty)=1$*. Define*

$$\Gamma_1(r,\xi)=\begin{cases} f_1(\cdot-R/2)\,\xi, & r>0\\ \xi, & r=0.\end{cases}$$

(2) *Let* $f_2:\mathbb{R}\to\mathbb{R}$ *be a compactly supported smooth function. Define*

$$\Gamma_2(r,\xi)=\begin{cases} f_2(\cdot-R/2)\,\xi(\cdot-R), & r>0\\ 0, & r=0.\end{cases}$$

(3) *Let $f_3 : \mathbb{R} \to \mathbb{R}$ be a smooth function which is constant outside of a compact interval with $f_3(\infty) = 1$ and $f_3(-\infty) = 0$. Define*

$$\Gamma_3(r, \xi) = \begin{cases} f_3(\cdot + R/2)\,\xi, & r > 0 \\ \xi, & r = 0. \end{cases}$$

(4) *Let $f_4 : \mathbb{R} \to \mathbb{R}$ be a compactly supported smooth function. Define*

$$\Gamma_4(r, \xi) = \begin{cases} f_4(\cdot + R/2)\,\xi(\cdot + R), & r > 0 \\ 0, & r = 0. \end{cases}$$

8.2. **Closedness of the Biholomorphic Equivalence in $\mathfrak{X}$.**

In this section we prove the Hausdorff property of the topology on $\mathfrak{X}$ in Theorem 4.6. More precisely, we show that for any distinct pair $\kappa, \kappa' \in \mathfrak{X}$, there exist neighborhoods $\mathfrak{U}_\varepsilon(\kappa; \hat{\tau})$ and $\mathfrak{U}_{\varepsilon'}(\kappa'; \hat{\tau}')$ defined in (4.17) with empty intersection. We shall prove it by using the following result regarding the closedness of the biholomorphic equivalence (see Section 7.1 for the analogous result for the Deligne-Mumford space).

Firstly, by using Proposition 5.23 we fix a choice of everywhere transversal constraint Σ for $\hat{\tau}$ and $\hat{\tau}'$, and fix any positive $\varepsilon < \hat{\varepsilon}$ and $\varepsilon' < \hat{\varepsilon}'$ as given in the proposition. For convenience, we abbreviate

$$\mathcal{U}_\varepsilon(\hat{\underline{r}}, \hat{\tau}) := \mathcal{U}_\varepsilon(\hat{\underline{r}}) \times \mathcal{U}_\varepsilon(\hat{\tau})$$

and let

$$(\bar{\underline{r}}, \bar{\tau}) \in \mathcal{U}_\varepsilon(\underline{0}, \hat{\tau}), \quad (\bar{\underline{r}}', \bar{\tau}') \in \mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\tau}')$$

be arbitrarily given.

Proposition 8.3. *Assume there are sequences $(\underline{r}_n, \tau_n) \to (\bar{\underline{r}}, \bar{\tau})$, $(\underline{r}'_n, \tau'_n) \to (\bar{\underline{r}}', \bar{\tau}')$, and a sequence of biholomorphisms $(\zeta_n, \underline{\psi}_n)$ with*

$$(\zeta_n, \underline{\psi}_n)(\#(\underline{r}_n, \tau_n)) = \#(\underline{r}'_n, \tau'_n).$$

Then there exists a biholomorphism $(\bar{\zeta}, \bar{\underline{\psi}})$ with

$$(\bar{\zeta}, \bar{\underline{\psi}})(\#(\bar{\underline{r}}, \bar{\tau})) = \#(\bar{\underline{r}}', \bar{\tau}').$$

Proposition 8.3 is a weaker version of Theorem 8.4. We now show the Hausdorff property.

Proof of Hausdorff property. Suppose this topology is not Hausdorff. Then there are distinct $\kappa, \kappa' \in \mathfrak{X}$ such that for all $\varepsilon, \varepsilon' > 0$, their neighborhoods $\mathfrak{U}_\varepsilon(\kappa; \hat{\tau})$ and $\mathfrak{U}_{\varepsilon'}(\kappa'; \hat{\tau}')$ have non-empty intersection. In other words, there exist sequences $(\underline{r}_n, \tau_n) \to (\underline{0}, \hat{\tau})$ and $(\underline{r}'_n, \tau'_n) \to (\underline{0}, \hat{\tau}')$, and a sequence of biholomorphisms $(\zeta_n, \underline{\psi}_n)$ such that $(\zeta_n, \underline{\psi}_n)(\#(\underline{r}_n, \tau_n)) = \#(\underline{r}'_n, \tau'_n)$. Applying Proposition 8.3, we get a biholomorphism $(\hat{\zeta}, \hat{\underline{\psi}})$ with $(\hat{\zeta}, \hat{\underline{\psi}})(\hat{\tau}) = \hat{\tau}'$. This contradicts with the assumption that $\kappa \neq \kappa'$. $\square$

Recall that commutative diagram (7.4) specifies the covering property between two tree isomorphisms $\bar{\zeta}$ and ζ, and (7.5) defines a distance on standard model between two biholomorphisms $\bar{\psi}_{\bar{\mathrm{v}}}$ and ψ_{v}. We now state the convergence result which implies Proposition 8.3. (This is the corresponding result to Theorem 7.11 for disk trees.)

Theorem 8.4. *Assume there are sequences $(\underline{r}_n, \tau_n) \to (\underline{\bar{r}}, \bar{\tau})$, $(\underline{r}'_n, \tau'_n) \to (\underline{\bar{r}}', \bar{\tau}')$, and a sequence of biholomorphisms $(\zeta_n, \underline{\psi}_n)$ with*

$$(\zeta_n, \underline{\psi}_n)(\#(\underline{r}_n, \tau_n)) = \#(\underline{r}'_n, \tau'_n).$$

Then there exists a biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ with

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\tau})) = \#(\underline{\bar{r}}', \bar{\tau}')$$

and a subsequence of $(\zeta_n, \underline{\psi}_n)$ (also indexed by n) such that $\bar{\zeta}$ covers ζ_n in the sense that (7.4) commutes; moreover for every main vertex $\bar{\mathrm{v}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$, we denote $\bar{\mathrm{v}}' := \bar{\zeta}(\bar{\mathrm{v}})$ and $\mathrm{v}_n := \bar{p}_{\underline{r}_n}(\bar{\mathrm{v}})$, then for all gluing vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ and $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$, we have $d^{\hat{\mathrm{v}}, \hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n, \mathrm{v}_n}) \to 0$ as defined in (7.5).

We shall prove this theorem in Section 8.2.1. The proofs of this series of results mainly exploit their counterparts in the Deligne-Mumford space by using the everywhere stabilization maps induced by Σ in Proposition 5.23

$$\mathrm{st} : \mathcal{U}_{\varepsilon}(\hat{\tau}) \to \mathcal{U}_{\hat{\nu}}(\mathrm{st}(\hat{\tau})), \quad \mathrm{st}' : \mathcal{U}_{\varepsilon'}(\hat{\tau}') \to \mathcal{U}_{\hat{\nu}'}(\mathrm{st}'(\hat{\tau}')).$$

The following estimate is useful in proving the injectivity of charts Θ constructed in (6.12). (This is the corresponding result to Theorem 7.12 for disk trees.)

Theorem 8.5. *Assume there is a biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ with*

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\tau})) = \#(\underline{\bar{r}}', \bar{\tau}').$$

Then given $\lambda > 0$, there are $\delta, \delta' > 0$ such that if there are $(\underline{r}, \tau) \in \mathcal{U}_{\delta}(\underline{\bar{r}}, \bar{\tau})$ and $(\underline{r}', \tau') \in \mathcal{U}_{\delta'}(\underline{\bar{r}}', \bar{\tau}')$ and a biholomorphism $(\zeta, \underline{\psi})$ with

$$(\zeta, \underline{\psi})(\#(\underline{r}, \tau)) = \#(\underline{r}', \tau'),$$

then $\bar{\zeta}$ covers ζ in the sense that (7.4) commutes; moreover for every main vertex $\bar{\mathrm{v}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$, we denote $\bar{\mathrm{v}}' := \bar{\zeta}(\bar{\mathrm{v}})$ and $\mathrm{v} := \bar{p}_{\underline{r}}(\bar{\mathrm{v}})$, then for all gluing vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ and $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$, we have $d^{\hat{\mathrm{v}}, \hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{\mathrm{v}}) < \lambda$ as defined in (7.5).

Proof. Firstly, it follows from Proposition 5.23 that we have

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \mathrm{st}(\bar{\tau}))) = \#(\underline{\bar{r}}', \mathrm{st}'(\bar{\tau}')).$$

By Theorem 7.12, there are $\rho, \rho' > 0$ such that if there are $(\underline{r}, \mu) \in \mathcal{U}_{\rho}(\underline{\bar{r}}, \mathrm{st}(\bar{\tau}))$ and $(\underline{r}', \mu') \in \mathcal{U}_{\rho'}(\underline{\bar{r}}', \mathrm{st}'(\bar{\tau}'))$ and a biholomorphism $(\zeta, \underline{\psi})$ with

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mu)) = \#(\underline{r}', \mu'),$$

then $(\bar{\zeta}, \underline{\bar{\psi}})$ covers $(\zeta, \underline{\psi})$, and the desired distance estimate $d^{\hat{\mathrm{v}}, \hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{\mathrm{v}}) < \lambda$ holds. By the continuity of the everywhere stabilization maps (Lemma 5.22), there exist δ, δ' such that the images of the disk tree neighborhoods are contained in the Deligne-Mumford neighborhoods

$$\mathrm{st}(\mathcal{U}_\delta(\bar{\tau})) \subset \mathcal{U}_\rho(\mathrm{st}(\bar{\tau})), \quad \mathrm{st}'(\mathcal{U}_{\delta'}(\bar{\tau}')) \subset \mathcal{U}_{\rho'}(\mathrm{st}'(\bar{\tau}')).$$

Then for $(\underline{r}, \tau) \in \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\tau})$ and $(\underline{r}', \tau') \in \mathcal{U}_{\delta'}(\underline{\bar{r}}', \bar{\tau}')$ satisfying $(\zeta, \underline{\psi})(\#(\underline{r}, \tau)) = \#(\underline{r}', \tau')$, Proposition 5.23 implies $(\zeta, \underline{\psi})(\#(\underline{r}, \mathrm{st}(\tau))) = \#(\underline{r}', \mathrm{st}'(\tau'))$. By the choice of δ and δ', we conclude $\bar{\zeta}$ covers ζ and $d^{\hat{\mathrm{v}}, \hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{\mathrm{v}}) < \lambda$. $\square$

We prove Theorem 8.4 in the next section. We shall do so by studying the standard model of the glued map, corresponding to the standard model of the glued marked points in Definition 7.5. Let $(\underline{r}, \tau) \in \mathcal{U}_\varepsilon(\underline{0}, \hat{\tau})$ be arbitrarily given.

Definition 8.6. Let $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$ be a main glued vertex. For each gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$, we define

$$\#^{\hat{\mathrm{v}}}_{\underline{r}, \underline{x}}(\underline{u})_{\mathrm{v}} := \#_{\underline{r}, \underline{x}}(\underline{u})_{\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{\mathrm{v}}$$

to be the **standard model of the glued map** on $D_{\hat{\mathrm{v}}}$, where Φ is the standard pullback in Definition 7.5.

We now make a simple but useful observation of the standard model of the glued map.

Lemma 8.7. *Let $\mathrm{v} \in \#_{\underline{r}}(\hat{\mathrm{V}})$ be a main glued vertex, and fix a gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\mathrm{v}}$.*

(1) *Suppose the vertex $\hat{\mathrm{v}}$ is not the root of the gluing tree $\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}}$. Then for each point outside of shrunk strip neighborhoods (see (3.3)) of gluing edges $z \in D_{\hat{\mathrm{v}}} \setminus \bigsqcup_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}} N(x_{\hat{\mathrm{v}}, \hat{\mathrm{e}}}; -(R_{\hat{\mathrm{e}}}/2 - 1))$, we have*

$$\#^{\hat{\mathrm{v}}}_{\underline{r}, \underline{x}}(\underline{u})_{\mathrm{v}}(z) = u_{\hat{\mathrm{v}}}(z).$$

(2) *Suppose the vertex $\hat{\mathrm{v}}$ is the root of the gluing tree $\hat{\mathrm{T}}^{\mathrm{g}}_{\mathrm{v}}$, and let $\hat{\mathrm{f}}$ be the outgoing edge of $\hat{\mathrm{v}}$. Then for each point outside of the strip neighborhood of edge $\hat{\mathrm{f}}$ and shrunk strip neighborhoods of gluing edges $z \in D_{\hat{\mathrm{v}}} \setminus (\bigsqcup_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}}} N(x_{\hat{\mathrm{v}}, \hat{\mathrm{e}}}; -(R_{\hat{\mathrm{e}}}/2 - 1)) \sqcup N(x_{\hat{\mathrm{v}}, \hat{\mathrm{f}}}))$, we also have*

$$\#^{\hat{\mathrm{v}}}_{\underline{r}, \underline{x}}(\underline{u})_{\mathrm{v}}(z) = u_{\hat{\mathrm{v}}}(z).$$

Proof. This results follows directly from the gluing construction. For each gluing edge $\hat{\mathrm{e}}$, in the plus gluing (4.14) we have

$$\oplus_{r_{\hat{\mathrm{e}}}}(u^{h^+}_{\hat{\mathrm{e}}}, u^{h^-}_{\hat{\mathrm{e}}}) := \beta\,(\cdot - R_{\hat{\mathrm{e}}}/2)\, u^{h^+}_{\hat{\mathrm{e}}} + (1 - \beta\,(\cdot - R_{\hat{\mathrm{e}}}/2))\, u^{h^-}_{\hat{\mathrm{e}}}(\cdot - R_{\hat{\mathrm{e}}}).$$

By the property of the cut-off function β defined before Remark 4.2, the restriction of the plus gluing on $[0, R_{\hat{\mathrm{e}}}/2 - 1] \times [0, \pi]$ is $u^{h^+}_{\hat{\mathrm{e}}}$, and the restriction on $[R_{\hat{\mathrm{e}}}/2 + 1, R_{\hat{\mathrm{e}}}] \times [0, \pi]$ is $u^{h^-}_{\hat{\mathrm{e}}}(\cdot - R_{\hat{\mathrm{e}}})$. This proves the result. $\square$

8.2.1. *Proof of Theorem 8.4.*

By picking a subsequence, we assume without loss of generality that gluing parameters $\underline{r}_n$ and $\underline{r}'_n$ give rise to *fixed* gluing edges $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}_n} = \hat{\mathrm{E}}^{\mathrm{g}}$ and $\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{r}'_n} = \hat{\mathrm{E}}'^{\mathrm{g}}$ for all n. Thus we get fixed trees $\#_{\hat{\mathrm{E}}^{\mathrm{g}}}(\hat{\mathrm{T}})$ and $\#_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\hat{\mathrm{T}}')$; moreover the sequence of tree isomorphisms ζ_n is identically equal to a fixed ζ due to the uniqueness of the ordered tree isomorphism (Theorem 2.1). Consequently, we have fixed vertex quotient maps $\bar{p}_{\hat{\mathrm{E}}^{\mathrm{g}}} : \#_{\underline{\bar{r}}}(\hat{\mathrm{V}}) \to \#_{\hat{\mathrm{E}}^{\mathrm{g}}}(\hat{\mathrm{V}})$ and $\bar{p}'_{\hat{\mathrm{E}}'^{\mathrm{g}}} : \#_{\underline{\bar{r}}'}(\hat{\mathrm{V}}') \to \#_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\hat{\mathrm{V}}')$, and fixed additional gluing edges $\hat{\mathrm{E}}^{\mathrm{ag}}$ and $\hat{\mathrm{E}}'^{\mathrm{ag}}$ (see Definition 7.8).

By the convergence assumption and the continuity of everywhere stabilization maps (Lemma 5.22), we have convergence $(\underline{r}_n, \mathrm{st}(\tau_n)) \to (\underline{\bar{r}}, \mathrm{st}(\bar{\tau}))$ and $(\underline{r}'_n, \mathrm{st}'(\tau'_n)) \to (\underline{\bar{r}}', \mathrm{st}'(\bar{\tau}'))$. By the biholomorphic equivalence assumption and Proposition 5.23, we have

$$(\zeta, \underline{\psi}_n)(\#(\underline{r}_n, \mathrm{st}(\tau_n))) = \#(\underline{r}'_n, \mathrm{st}'(\tau'_n)).$$

Then by Theorem 7.11, there exists a biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ with

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \mathrm{st}(\bar{\tau}))) = \#(\underline{\bar{r}}', \mathrm{st}'(\bar{\tau}')), \tag{8.7}$$

and a subsequence of $(\zeta, \underline{\psi}_n)$ (also indexed by n) such that $\bar{\zeta}$ covers ζ and we have

$$d^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar{\psi}_{\bar{\mathrm{v}}}, \psi_{n,\mathrm{v}}) \to 0. \tag{8.8}$$

It suffices to show that this biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ also yields equivalence

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\tau})) = \#(\underline{\bar{r}}', \bar{\tau}'). \tag{8.9}$$

To be more specific, we need to show that for corresponding main vertices $\bar{\mathrm{v}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$ and $\bar{\mathrm{v}}' = \bar{\zeta}(\bar{\mathrm{v}})$ we have equivalence of glued maps

$$\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}} \circ \bar{\psi}_{\bar{\mathrm{v}}}^{-1} = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{u}}')_{\bar{\mathrm{v}}'}, \tag{8.10}$$

and for corresponding edges $\bar{\mathrm{e}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{E}})$ and $\bar{\mathrm{e}}' = \bar{\zeta}(\bar{\mathrm{e}})$ we have equivalence of displaced Morse trajectories

$$\#_{\underline{\bar{r}}}(\underline{\bar{\gamma}})_{\bar{\mathrm{e}}} = \#_{\underline{\bar{r}}'}(\underline{\bar{\gamma}}')_{\bar{\mathrm{e}}'}. \tag{8.11}$$

Note that by the construction of everywhere stabilization (Definition 5.21), equation (8.7) directly implies the equivalence of glued marked points

$$\bar{\psi}_{\bar{\mathrm{v}}}(\#_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}},\bar{\mathrm{e}}}) = \#_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}',\bar{\mathrm{e}}'} \tag{8.12}$$

for corresponding vertices and edges.

We first show (8.10). Denote vertices $\mathrm{v} := \bar{p}_{\hat{\mathrm{E}}^{\mathrm{g}}}(\bar{\mathrm{v}})$ and $\mathrm{v}' := \zeta(\mathrm{v})$, and fix a pair of gluing vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ and $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}$. We denote the biholomorphism $\bar{\psi}_{\bar{\mathrm{v}}}$ and $\psi_{n.\mathrm{v}}$ in the standard model on $D_{\hat{\mathrm{v}}}$ and $D_{\hat{\mathrm{v}}'}$ by

$$\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} := \left(\bar{\Phi}^{\hat{\mathrm{v}}'}_{\bar{\mathrm{v}}'}\right)^{-1} \circ \bar{\psi}_{\bar{\mathrm{v}}} \circ \bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}, \quad \psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} := \left(\Phi^{\hat{\mathrm{v}}'}_{n,\mathrm{v}'}\right)^{-1} \circ \psi_{n,\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}},$$

where $\bar{\Phi}$ and Φ are standard pullbacks in Definition 7.5. To verify equation (8.10), it suffices to verify the equation in standard model

$$\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}} \circ \left(\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\right)^{-1} = \#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{u}}')_{\bar{\mathrm{v}}'} \tag{8.13}$$

On the other hand, the assumption $(\zeta, \underline{\psi}_n)(\#(\underline{r}_n, \tau_n))) = \#(\underline{r}'_n, \tau'_n)$ implies

$$\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}} \circ \left(\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\right)^{-1} = \#^{\hat{\mathrm{v}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{u}'_n)_{\mathrm{v}'}. \tag{8.14}$$

Moreover, it follows directly from (8.8) that we have $\left(\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\right)^{-1} \to \left(\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\right)^{-1}$ as disk automorphism, and thus it converges uniformly on the disk.

Let $(\#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}',\bar{\mathrm{e}}'})_{\hat{\mathrm{e}}'\in\hat{\mathrm{E}}'^{\mathrm{ag}}}$ be the set of glued marked points of additional gluing edges, where each edge $\hat{\mathrm{e}}'$ is identified with $\bar{\mathrm{e}}'$ by map (4.6). We now prove equation (8.13) by evaluating (8.14) at each point $z' \in D_{\hat{\mathrm{v}}'} \setminus (\#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}',\bar{\mathrm{e}}'})_{\hat{\mathrm{e}}'\in\hat{\mathrm{E}}'^{\mathrm{ag}}}$, away from the glued marked points of additional gluing edges

$$\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}} \circ \left(\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\right)^{-1}(z') = \#^{\hat{\mathrm{v}}'}_{\underline{r}'_n,\underline{x}'_n}(\underline{u}'_n)_{\mathrm{v}'}(z'). \tag{8.15}$$

It follows from Lemma 8.8 that the right-hand side of the above equation converges to $\#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{u}}')_{\bar{\mathrm{v}}'}(z')$. We now show that the left-hand side converges to $\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}} \circ \left(\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\right)^{-1}(z')$. For convenience, we denote

$$f_n := \#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}},\ g_n := \left(\psi_n^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\right)^{-1},\ \bar{f} := \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}},\ \bar{g} := \left(\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\right)^{-1}.$$

Then we have

$$d\big(f_n(g_n(z')), \bar{f}(\bar{g}(z'))\big) \le d\big(f_n(g_n(z')), \bar{f}(g_n(z'))\big) + d\big(\bar{f}(g_n(z')), \bar{f}(\bar{g}(z'))\big)$$

By the correspondence of glued marked points (8.12) in standard model, we have $\left(\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\right)^{-1}\left(\#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}'}\right) = \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}}}$. Thus by the choice of z', the sequence $g_n(z')$ converges to $\bar{g}(z')$, away from the glued marked points of additional gluing edges $(\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}},\bar{\mathrm{e}}})_{\hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{ag}}}$. Then by Lemma 8.8, the first term in the above inequality goes to 0. The second term goes to 0 by the continuity of $\bar{f}$ and the convergence $g_n \to \bar{g}$.

Thus the convergence on both sides of (8.15) shows equation (8.13) holds when evaluated at each point $z' \in D_{\hat{\mathrm{v}}'} \setminus (\#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}',\bar{\mathrm{e}}'})_{\hat{\mathrm{e}}'\in\hat{\mathrm{E}}'^{\mathrm{ag}}}$. Then equation (8.13) follows from continuity. This verifies (8.10).

We now prove equation (8.11). Via the edge identifying map (4.6), the pair of edges $\bar{\mathrm{e}} = (\bar{\mathrm{v}}, \bar{\mathrm{w}})$ and $\bar{\mathrm{e}}' = (\bar{\mathrm{v}}', \bar{\mathrm{w}}')$ are identified with $\hat{\mathrm{e}} = (\hat{\mathrm{v}}, \hat{\mathrm{w}}) \in \hat{\mathrm{E}} \setminus \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$ and $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}', \hat{\mathrm{w}}') \in \hat{\mathrm{E}}' \setminus \hat{\mathrm{E}}'^{\mathrm{g}}_{\underline{\bar{r}}'}$, respectively. Furthermore, via the vertex quotient map (Definition 7.8) we denote glued vertices $\mathrm{v} := \bar{p}_{\hat{\mathrm{E}}^{\mathrm{g}}}(\bar{\mathrm{v}})$, $\mathrm{v}' := \bar{p}_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\bar{\mathrm{v}}')$, $\mathrm{w} := \bar{p}_{\hat{\mathrm{E}}^{\mathrm{g}}}(\bar{\mathrm{w}})$, and $\mathrm{w}' := \bar{p}_{\hat{\mathrm{E}}'^{\mathrm{g}}}(\bar{\mathrm{w}}')$. Recall that we have decomposition (7.3) of gluing edges $\hat{\mathrm{E}}^{\mathrm{g}} = \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}} \sqcup \hat{\mathrm{E}}^{\mathrm{ag}}$, where $\hat{\mathrm{E}}^{\mathrm{ag}}$ is the additional gluing edges (Definition 7.8). By the assumption $\hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$, either we have $\hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}$ or $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$. The edge $\hat{\mathrm{e}}'$ has a similar dichotomy. Then it follows from Lemma 7.17 and the fact that $\bar{\zeta}$ covers ζ that the additional gluing edges are in one-to-one correspondence. Hence the edges $\hat{\mathrm{e}} = (\hat{\mathrm{v}}, \hat{\mathrm{w}})$ and $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}', \hat{\mathrm{w}}')$ satisfy either

(1) $\hat{\mathrm{e}} \notin \hat{\mathrm{E}}^{\mathrm{g}}$ and $\hat{\mathrm{e}}' \notin \hat{\mathrm{E}}'^{\mathrm{g}}$, i.e., the glued vertices satisfy $\mathrm{v} \neq \mathrm{w}$ and $\mathrm{v}' \neq \mathrm{w}'$, or

(2) $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$ and $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}'^{\mathrm{ag}}$, i.e., the glued vertices satisfy $\mathrm{v} = \mathrm{w}$ and $\mathrm{v}' = \mathrm{w}'$.

Suppose the pair $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$ is in case (1). We denote the edges $\mathrm{e} = (\mathrm{v}, \mathrm{w})$ and $\mathrm{e}' = (\mathrm{v}', \mathrm{w}')$. The convergence assumptions $(\underline{r}_n, \tau_n) \to (\underline{\bar{r}}, \bar{\tau})$ and $(\underline{r}'_n, \tau'_n) \to (\underline{\bar{r}}', \bar{\tau}')$ imply that the displaced Morse trajectories $\#_{\underline{r}_n}(\underline{\gamma}_n)_{\mathrm{e}}$ and $\#_{\underline{r}'_n}(\underline{\gamma}'_n)_{\mathrm{e}'}$ converge to $\#_{\underline{\bar{r}}}(\underline{\bar{\gamma}})_{\bar{\mathrm{e}}}$ and $\#_{\underline{\bar{r}}'}(\underline{\bar{\gamma}}')_{\bar{\mathrm{e}}'}$, respectively. It follows from the assumption $(\zeta, \underline{\psi}_n)(\#(\underline{r}_n, \tau_n)) = \#(\underline{r}'_n, \tau'_n)$ that we have $\#_{\underline{r}_n}(\underline{\gamma}_n)_{\mathrm{e}} = \#_{\underline{r}'_n}(\underline{\gamma}'_n)_{\mathrm{e}'}$. Therefore we have the desired equality of the displaced Morse trajectories $\#_{\underline{\bar{r}}}(\underline{\bar{\gamma}})_{\bar{\mathrm{e}}} = \#_{\underline{\bar{r}}'}(\underline{\bar{\gamma}}')_{\bar{\mathrm{e}}'}$.

It suffices to prove the result for $\hat{\mathrm{e}} = (\hat{\mathrm{v}}, \hat{\mathrm{w}})$ and $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}', \hat{\mathrm{w}}')$ in case (2) when $\mathrm{v} = \mathrm{w}$ and $\mathrm{v}' = \mathrm{w}'$. By the definition of additional gluing edges (Definition 7.8), we have $\bar{r}_{\hat{\mathrm{e}}} = 0$ and $\bar{r}'_{\hat{\mathrm{e}}'} = 0$. Hence both $\#_{\underline{\bar{r}}}(\underline{\bar{\gamma}})_{\bar{\mathrm{e}}}$ and $\#_{\underline{\bar{r}}'}(\underline{\bar{\gamma}}')_{\bar{\mathrm{e}}'}$ are constant trajectories with images $\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}}(\#_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}},\bar{\mathrm{e}}})$ and $\#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{u}}')_{\bar{\mathrm{v}}'}(\#_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}',\bar{\mathrm{e}}'})$, respectively. It follows from (8.10) and (8.12) that these two images match. Hence we have $\#_{\underline{\bar{r}}}(\underline{\bar{\gamma}})_{\bar{\mathrm{e}}} = \#_{\underline{\bar{r}}'}(\underline{\bar{\gamma}}')_{\bar{\mathrm{e}}'}$. This verifies equation (8.11), and we finish the proof of the theorem.

We now prove the supporting lemma used in the above proof. (The following convergence result is the corresponding result to Lemma 7.14 for disk trees.)

Lemma 8.8. *For every gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$, we have locally uniform convergence*

$$\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}} \to \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}}$$

on the disk away from the glued marked points of additional gluing edges $D_{\hat{\mathrm{v}}} \setminus (\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}},\bar{\mathrm{e}}})_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}}$.

Proof. For each gluing vertex $\hat{\mathrm{v}}$, we denote the set outside of shrunk strip neighborhoods of gluing edges by

$$C_{\hat{\mathrm{v}}} := D_{\hat{\mathrm{v}}} \setminus \bigsqcup_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}} N(\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}; -\bar{R}_{\hat{\mathrm{e}}}).$$

We first show that we have locally uniform convergence

$$\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}} \to \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}} \tag{8.16}$$

on $C_{\hat{\mathrm{v}}} \setminus \{\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}\}_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}}$, away from the marked points of additional gluing edges.

First assume that the vertex $\hat{\mathrm{v}}$ is not the root of the gluing tree $\hat{\mathrm{T}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$. We now show the uniform convergence around a given point z in the following two cases.

(1) Suppose the point z

- lies in the set $D_{\hat{\mathrm{v}}} \setminus \bigsqcup_{\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}} N(\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}; -(\bar{R}_{\hat{\mathrm{e}}}/2 - 1))$, and

- is not equal to $\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$ for any additional gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$.

Then by the assumption $(\underline{r}_n, \tau_n) \to (\underline{\bar{r}}, \bar{\tau})$, for n large enough there is a neighborhood $B(z)$ contained in $D_{\hat{\mathrm{v}}} \setminus \bigsqcup_{\hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}} N(x_{n,\hat{\mathrm{v}},\hat{\mathrm{e}}}; -(R_{n,\hat{\mathrm{e}}}/2-1))$. Moreover, for each additional gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{ag}}$, the gluing length $R_{n,\hat{\mathrm{e}}} \to \infty$, so the neighborhood $N(x_{n,\hat{\mathrm{v}},\hat{\mathrm{e}}}; -(R_{n,\hat{\mathrm{e}}}/2-1))$ shrinks to $\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}$. By the assumption on z, we conclude that for n large enough the neighborhood $B(z)$ does not intersect $N(x_{n,\hat{\mathrm{v}},\hat{\mathrm{e}}}; -(R_{n,\hat{\mathrm{e}}}/2-1))$ for additional gluing edges. Therefore, we have

$$B(z) \subset D_{\hat{\mathrm{v}}} \setminus \textstyle\bigsqcup_{\hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{g}}} N(x_{n,\hat{\mathrm{v}},\hat{\mathrm{e}}}; -(R_{n,\hat{\mathrm{e}}}/2-1)).$$

Then by Lemma 8.7 (1), for each $z' \in B(z)$, we have $\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}}(z') = u_{n,\hat{\mathrm{v}}}(z')$, which converges to $\bar{u}_{\hat{\mathrm{v}}}(z')$ uniformly on $B(z)$. Again by Lemma 8.7 (1), we have $\bar{u}_{\hat{\mathrm{v}}}(z') = \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}}(z')$. This establishes the locally uniform convergence (8.16) in this case.

(2) Suppose the point z lies in the glued strip $h^+_{\hat{\mathrm{e}}}\big(\bar{x}^+_{\hat{\mathrm{e}}}, (0, \bar{R}_{\hat{\mathrm{e}}}) \times [0,\pi]\big)$ for an incoming gluing edge $\hat{\mathrm{e}} \in \hat{\mathrm{E}}^{\mathrm{g}}_{\underline{\bar{r}}}$ of vertex $\hat{\mathrm{v}}$ (or $h^-_{\hat{\mathrm{e}}}\big(\bar{x}^-_{\hat{\mathrm{e}}}, (-\bar{R}_{\hat{\mathrm{e}}}, 0) \times [0,\pi]\big)$ for an outgoing gluing edge $\hat{\mathrm{e}}$). Let us focus on the case when $\hat{\mathrm{e}}$ is an incoming edge since the case for an outgoing edge is similar. For n large enough there is a neighborhood $B(z)$ contained in the glued strips $h^+_{\hat{\mathrm{e}}}\big(x^+_{n,\hat{\mathrm{e}}}, (0, R_{n,\hat{\mathrm{e}}}) \times [0,\pi]\big)$. Thus by the plus gluing construction (4.14), for all $z' \in B(z)$ we have

$$\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)(z') = \oplus_{r_{n,\hat{\mathrm{e}}}}(u^{h^+}_{n,\hat{\mathrm{e}}}, u^{h^-}_{n,\hat{\mathrm{e}}}) \circ h^+_{\hat{\mathrm{e}}}(x^+_{n,\hat{\mathrm{e}}}, \cdot)^{-1}(z').$$

Moreover, the value $\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})(z')$ is expressed similarly. Then by the assumption $(\underline{r}_n, \tau_n) \to (\underline{\bar{r}}, \bar{\tau})$, we have locally uniform convergence (8.16) in this case. Furthermore, case (1) and (2) together cover the locally uniform convergence on $C_{\hat{\mathrm{v}}} \setminus \{\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}\}_{\hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{ag}}}$.

Now assume that the vertex $\hat{\mathrm{v}}$ is the root of the gluing tree $\hat{\mathrm{T}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$, and let $\hat{\mathrm{f}}$ be the outgoing edge of $\hat{\mathrm{v}}$. Then the locally uniform convergence is established similarly by Lemma 8.7 (2) along with a special consideration on the strip neighborhood $N(\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{f}}})$. Suppose a neighborhood $B(z)$ is contained in $N(\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{f}}})$. Then the local expression of $\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}}$ is given by (4.15), and it converges to the local expression of $\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}}$ uniformly on $B(z)$.

This finishes proving the locally uniform convergence on $C_{\hat{\mathrm{v}}} \setminus \{\bar{x}_{\hat{\mathrm{v}},\hat{\mathrm{e}}}\}_{\hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{ag}}}$ for each gluing vertex $\hat{\mathrm{v}}$. We now prove the lemma. Firstly, we observe that the glued disk $\#_{\underline{\bar{r}},\underline{\bar{x}}}(D)_{\bar{\mathrm{v}}}$ is the union of $\bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}(C_{\hat{\mathrm{v}}})$ for all gluing vertices $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$. Thus for each point $z \in D_{\hat{\mathrm{v}}} \setminus (\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}},\bar{\mathrm{e}}})_{\hat{\mathrm{e}}\in\hat{\mathrm{E}}^{\mathrm{ag}}}$ away from the glued marked points of additional gluing edges, there exists a gluing vertex $\hat{\mathrm{w}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$ such that the image $\bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}(z)$ lies in $\bar{\Phi}^{\hat{\mathrm{w}}}_{\bar{\mathrm{v}}}(C_{\hat{\mathrm{w}}})$. Thus for n large enough we have

$$\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}}(z) = \#^{\hat{\mathrm{w}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}} \circ \left(\Phi^{\hat{\mathrm{w}}}_{n,\mathrm{v}}\right)^{-1} \circ \Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}}(z).$$

Note that convergence (7.16) from the proof of Lemma 7.14 is in the same context. Thus we have $\left(\Phi^{\hat{\mathrm{w}}}_{n,\mathrm{v}}\right)^{-1} \circ \Phi^{\hat{\mathrm{v}}}_{n,\mathrm{v}} \to \left(\bar{\Phi}^{\hat{\mathrm{w}}}_{\bar{\mathrm{v}}}\right)^{-1} \circ \bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}$ as disk automorphisms, and hence converges uniformly on the standard disk. By construction, the point $\left(\bar{\Phi}^{\hat{\mathrm{w}}}_{\bar{\mathrm{v}}}\right)^{-1} \circ \bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}(z)$ lies in $C_{\hat{\mathrm{w}}}$ and is away from marked points of additional gluing edges. Thus the standard model $\#^{\hat{\mathrm{w}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}}$ converges to $\#^{\hat{\mathrm{w}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}}$ uniformly around the point $\left(\bar{\Phi}^{\hat{\mathrm{w}}}_{\bar{\mathrm{v}}}\right)^{-1} \circ \bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}}(z)$. This shows $\#^{\hat{\mathrm{v}}}_{\underline{r}_n,\underline{x}_n}(\underline{u}_n)_{\mathrm{v}}$ converges to $\#^{\hat{\mathrm{w}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\mathrm{v}} \circ \left(\bar{\Phi}^{\hat{\mathrm{w}}}_{\bar{\mathrm{v}}}\right)^{-1} \circ \bar{\Phi}^{\hat{\mathrm{v}}}_{\bar{\mathrm{v}}} = \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}}$ uniformly around z. This finishes the proof of the lemma. □

8.2.2. *Continuity and Sc-smoothness of Glued Maps in Standard Model.*

As we have seen in Lemma 8.8, the locally uniform convergence of the glued map holds outside of certain marked points. Later on when we study the transition maps of M-polyfold charts, we need the continuity/sc-smoothness result of the glued map outside neighborhoods of certain marked points as well. To formalize this idea, we general the notion of H^{3,δ_0} maps defined in Section 3.3.

Let $\underline{x}$ and $\underline{\hat{y}}$ be boundary marked points on a disk such that the two sets are disjoint. Let $B(\hat{y}_j)$ be an arbitrary neighborhood in the disk D around each marked point y_j.

Similar to Definition 3.12, we define

$$\mathrm{Map}^{3,\delta_0}((D \setminus \textstyle\bigsqcup_j B(\hat{y}_j), \underline{x}), M; L)$$

to be the set of H^{3,δ_0} maps on $D \setminus \bigsqcup_j B(\hat{y}_j)$ with Lagrangian boundary condition on $(D \setminus \bigsqcup_j B(\hat{y}_j)) \cap \partial D$. Similar to Definition 3.16, we define

$$\mathrm{Sec}^{3,\delta_0}((D \setminus \textstyle\bigsqcup_j B(\hat{y}_j), \underline{x}), u^*TM; TL)$$

to be the set of H^{3,δ_0} sections on $D \setminus \bigsqcup_j B(\hat{y}_j)$ with Lagrangian boundary condition on $(D \setminus \bigsqcup_j B(\hat{y}_j)) \cap \partial D$. Analogous to the neighborhood in (3.7), we define a neighborhood

$$\mathcal{U}_\varepsilon(\underline{\hat{x}}, \hat{u}) := \left\{ \left(\underline{x}, \exp_{\hat{u}}(\xi) \circ \nu_{\underline{x}}^{-1}\right) \;\middle|\; \begin{array}{c} \xi \in \mathrm{Sec}^{3,\delta_0}((D \setminus B(\underline{\hat{y}}), \underline{\hat{x}}), \hat{u}^*TM; TL), \\ \underline{x} \in \mathcal{U}_\varepsilon(\underline{\hat{x}}),\ \|\xi\| < \varepsilon \end{array} \right\},$$

where $\nu_{\underline{x}} : D \setminus \bigsqcup_j B(\hat{y}_j) \to D \setminus \bigsqcup_j B(\hat{y}_j)$ is a family of marked points varying diffeomorphism similar to Lemma 3.11.

Recall that we have gluing parameters $\underline{\bar{r}}$ and a disk tree representative $\bar{\tau} = (\hat{\mathrm{T}}, \underline{\bar{\gamma}}, \underline{\bar{x}}, \underline{\bar{u}})$. Fix a gluing vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}$. Let $B(\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}},\bar{\mathrm{e}}})$ be an arbitrary neighborhood in the disk D, around each glued marked point $\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}},\bar{\mathrm{e}}}$ whose edge $\bar{\mathrm{e}}$ corresponds to an edge $\hat{\mathrm{e}}$ with $\bar{r}_{\hat{\mathrm{e}}} = 0$ via the map (4.6). We denote the set outside of these neighborhoods by

$$K := D \setminus \textstyle\bigsqcup_{\bar{r}_{\hat{\mathrm{e}}}=0} B(\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}},\bar{\mathrm{e}}}). \tag{8.17}$$

We have the continuity result of the glued marked points and map restricted to K.

Lemma 8.9. *The mapping to the restricted glued marked points and glued map in standard model*

$$\begin{aligned}\mathcal{U}_\delta(\underline{\bar{r}}, \bar{\tau}) &\to \mathcal{U}_\lambda\big(\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}}} \cap K,\ \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}}|_K\big)\\ (\underline{r}, \tau) &\mapsto \big(\#^{\hat{\mathrm{v}}}_{\underline{r}}(\underline{x})_{\mathrm{v}} \cap K,\ \#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{u})_{\mathrm{v}}|_K\big)\end{aligned}$$

is continuous.

Since the disk tree representative $\bar{\tau}$ in Lemma 8.8 is arbitrary, it implies the above result.

Recall from (6.12) that our chart of the quotient space of disk trees is given by

$$\Theta : \mathcal{K}^{\mathrm{slc}}_\varepsilon(\underline{0}, \hat{\tau}) \to \mathfrak{X}, \quad (\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}) \mapsto [\#](\underline{r}, \tau(\underline{\rho}, \underline{x}, \underline{\xi})),$$

where $\mathcal{K}^{\mathrm{slc}}_\varepsilon(\underline{0}, \hat{\tau})$ is the splicing core slice, and the disk tree representative $\tau(\underline{\rho}, \underline{x}, \underline{\xi})$ is of the form $(\hat{\mathrm{T}}, \underline{\gamma}(\underline{\rho}, \underline{\xi}), \underline{x}, \underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))$. For a neighborhood $B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}})$ in the splicing core slice, we have the sc-smoothness result of the glued marked points and map restricted to K.

Lemma 8.10. *The mapping to the restricted glued marked points and glued map in standard model*

$$\begin{aligned}B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) &\to \mathcal{U}_\lambda\big(\#^{\hat{\mathrm{v}}}_{\underline{\bar{r}}}(\underline{\bar{x}})_{\bar{\mathrm{v}}} \cap K,\ \#^{\hat{\mathrm{v}}}_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{u}(\underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}))_{\bar{\mathrm{v}}}|_K\big)\\ (\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}) &\mapsto \big(\#^{\hat{\mathrm{v}}}_{\underline{r}}(\underline{x})_{\mathrm{v}} \cap K,\ \#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))_{\mathrm{v}}|_K\big)\end{aligned}$$

is sc-smooth.

This result follows from a similar analysis in Lemma 8.8.

8.2.3. *Stabilization Revisited.*

Recall that Proposition 5.23 finds an *everywhere* transversal constraint (Definition 5.3) for disk tree representatives $\hat{\tau}$ and $\hat{\tau}'$ whose *everywhere* stabilization maps (Definition 5.21) behaves well with nearby equivalence. Later on when proving the injectivity of M-polyfold charts Θ, we need a similar result for a set of transversal constraints (Definition 5.1) and their stabilization maps (Definition 5.19). We now show with the help of the estimate in Theorem 8.5, that such stabilization maps indeed behave well with nearby equivalence.

Suppose the representatives are of the form

$$\bar{\tau} = (\hat{\mathrm{T}}, \underline{\bar{\gamma}}, \underline{\bar{x}}, \underline{\bar{u}}), \quad \bar{\tau}' = (\hat{\mathrm{T}}', \underline{\bar{\gamma}}', \underline{\bar{x}}', \underline{\bar{u}}').$$

Let $\underline{\bar{o}}$ and $\underline{\bar{o}}'$ be interior marked points at which the derivatives $D\underline{\bar{u}}$ and $D\underline{\bar{u}}'$ are injective, respectively, such that the marked points $\underline{\bar{o}}$ and $\underline{\bar{o}}'$ satisfy the stability condition (5.1).

Proposition 8.11. *Assume there is a biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ with*

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\tau})) = \#(\underline{\bar{r}}', \bar{\tau}'), \quad \underline{\bar{\psi}}(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{o}})) = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{o}}')$$

and let $\underline{\Sigma}' = (\Sigma'_{\hat{v}',j'})$ *be transversal constraints at* $\underline{\bar{o}}' = (\bar{o}'_{\hat{v}',j'})$ *for the disk tree representative* $\bar{\tau}'$. *Then* $\underline{\Sigma}'$ *are also transversal constraints at* $\underline{\bar{o}}$ *for* $\bar{\tau}$. *Furthermore,* $\underline{\Sigma}'$ *induce stabilization maps near* $\underline{\bar{o}}$ *and* $\underline{\bar{o}}'$

$$\mathrm{st} : \mathcal{U}_{\bar{\delta}}(\bar{\tau}) \to \mathcal{U}_{\bar{\nu}}(\mathrm{st}(\bar{\tau})), \quad \mathrm{st}' : \mathcal{U}_{\bar{\delta}'}(\bar{\tau}') \to \mathcal{U}_{\bar{\nu}'}(\mathrm{st}'(\bar{\tau}'))$$

such that if there are $(\underline{r}, \tau) \in \mathcal{U}_{\bar{\delta}}(\underline{\bar{r}}, \bar{\tau})$ *and* $(\underline{r}', \tau') \in \mathcal{U}_{\bar{\delta}'}(\underline{\bar{r}}', \bar{\tau}')$ *and a biholomorphism* $(\zeta, \underline{\psi})$ *with*

$$(\zeta, \underline{\psi})(\#(\underline{r}, \tau)) = \#(\underline{r}', \tau'),$$

then the same biholomorphism yields equivalence

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mathrm{st}(\tau))) = \#(\underline{r}', \mathrm{st}'(\tau')).$$

Proof. For each main vertex $\bar{\mathrm{v}} \in \#_{\underline{\bar{r}}}(\hat{\mathrm{V}})$ and corresponding vertex $\bar{\mathrm{v}}' := \bar{\zeta}(\bar{\mathrm{v}})$, the equivalence $\underline{\bar{\psi}}(\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{o}})) = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{o}}')$ provides an equation

$$\bar{\psi}_{\bar{\mathrm{v}}}(\bar{o}_{\hat{\mathrm{v}},j}) = \bar{o}'_{\hat{\mathrm{v}}',j'} \tag{8.18}$$

for corresponding gluing vertex/index pairs $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{g}}_{\bar{\mathrm{v}}}, j \leq n(\underline{\bar{o}}_{\hat{\mathrm{v}}})$ and $\hat{\mathrm{v}}' \in \hat{\mathrm{V}}'^{\mathrm{g}}_{\bar{\mathrm{v}}'}, j' \leq n(\underline{\bar{o}}'_{\hat{\mathrm{v}}'})$. The equivalence $(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\tau})) = \#(\underline{\bar{r}}', \bar{\tau}')$ implies equation $\#_{\underline{\bar{r}},\underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}} = \#_{\underline{\bar{r}}',\underline{\bar{x}}'}(\underline{\bar{u}}')_{\bar{\mathrm{v}}'} \circ \bar{\psi}_{\bar{\mathrm{v}}}$. Evaluating the derivatives on both sides of that equation at the point $\bar{o}_{\hat{\mathrm{v}},j}$, we conclude that the images of derivatives coincide $\mathrm{im}(D\bar{u}_{\hat{\mathrm{v}}}(\bar{o}_{\hat{\mathrm{v}},j})) = \mathrm{im}(D\bar{u}'_{\hat{\mathrm{v}}'}(\bar{o}'_{\hat{\mathrm{v}}',j'}))$. By assumption, each $\Sigma'_{\hat{\mathrm{v}}',j'}$ is a transversal constraint at $\bar{o}'_{\hat{\mathrm{v}}',j'}$ for $\bar{u}'_{\hat{\mathrm{v}}'}$. By the definition of transversal constraint (Definition 5.1), the above analysis shows that it is also a transversal constraint at $\bar{o}_{\hat{\mathrm{v}},j}$ for $\bar{u}_{\hat{\mathrm{v}}}$.

We now study the stabilization maps st and st$'$ induced by $\underline{\Sigma}'$. We denote the stabilizations by

$$\mathrm{st}(\tau) := (\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{o}), \quad \mathrm{st}'(\tau') := (\hat{\mathrm{T}}', \underline{\ell}', \underline{x}', \underline{o}').$$

We shall find $\bar{\delta}, \bar{\delta}' > 0$ such that if there are $(\underline{r}, \tau) \in \mathcal{U}_{\bar{\delta}}(\underline{\bar{r}}, \bar{\tau})$ and $(\underline{r}', \tau') \in \mathcal{U}_{\bar{\delta}'}(\underline{\bar{r}}', \bar{\tau}')$ and a biholomorphism $(\zeta, \underline{\psi})$ with

$$(\zeta, \underline{\psi})(\#(\underline{r}, \tau)) = \#(\underline{r}', \tau'), \tag{8.19}$$

then we have an equation of stabilizations

$$\psi_{\mathrm{v}}(o_{\hat{\mathrm{v}},j}) = o'_{\hat{\mathrm{v}}',j'} \tag{8.20}$$

for the same vertex/index pairs $(\hat{\mathrm{v}}, j)$ and $(\hat{\mathrm{v}}', j')$ as in (8.18). We shall do so by applying the estimate in Theorem 8.5.

For all vertices $\hat{\mathrm{v}}$ and $\hat{\mathrm{v}}'$ satisfying $\bar{\zeta}([\hat{\mathrm{v}}]_{\underline{\bar{r}}}) = [\hat{\mathrm{v}}']_{\underline{\bar{r}}'}$, we denote the glued vertices $\bar{\mathrm{v}} := [\hat{\mathrm{v}}]_{\underline{\bar{r}}}$, $\bar{\mathrm{v}}' := [\hat{\mathrm{v}}']_{\underline{\bar{r}}'}$, $\mathrm{v} := [\hat{\mathrm{v}}]_{\underline{r}}$, and $\mathrm{v}' := [\hat{\mathrm{v}}]_{\underline{r}'}$, and denote the biholomorphisms $\bar{\psi}_{\bar{\mathrm{v}}}$ and ψ_{v} in the standard models on $D_{\hat{\mathrm{v}}}$ and $D_{\hat{\mathrm{v}}'}$ by

$$\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} := \left(\bar{\Phi}^{\hat{\mathrm{v}}'}_{\bar{\mathrm{v}}'}\right)^{-1} \circ \bar{\psi}_{\bar{\mathrm{v}}} \circ \bar{\Phi}^{\mathrm{v}}_{\bar{\mathrm{v}}}, \quad \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} := \left(\Phi^{\hat{\mathrm{v}}'}_{\mathrm{v}'}\right)^{-1} \circ \psi_{\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{\mathrm{v}}.$$

Let $N > 0$ be a bound for the derivatives

$$|D\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}| < N.$$

By Proposition 13.13, there are $\delta', \bar{\nu}' > 0$ such that for each $(\underline{x}'_{\hat{\mathrm{v}}'}, u'_{\hat{\mathrm{v}}'}) \in \mathcal{U}_{\delta'}(\underline{\bar{x}}'_{\hat{\mathrm{v}}'}, \bar{u}'_{\hat{\mathrm{v}}'})$, there is *precisely one point* $o'(u'_{\hat{\mathrm{v}}'})_{j'}$ in the open neighborhood $B_{\bar{\nu}'}(\bar{o}'_{\hat{\mathrm{v}}',j'})$. Similarly, there are $\delta, \bar{\nu} > 0$ satisfying

$$N\bar{\nu} < \bar{\nu}'/2$$

such that for each $(\underline{x}_{\hat{\mathrm{v}}}, u_{\hat{\mathrm{v}}}) \in \mathcal{U}_{\delta}(\underline{\bar{x}}_{\hat{\mathrm{v}}}, \bar{u}_{\hat{\mathrm{v}}})$, there is *precisely one point* $o(u_{\hat{\mathrm{v}}})_j$ in the open neighborhood $B_{\bar{\nu}}(\bar{o}_{\hat{\mathrm{v}},j})$.

By Theorem 8.5, there are $\bar{\delta}, \bar{\delta}'$ such that for pairs $(\underline{r}, \tau) \in \mathcal{U}_{\bar{\delta}}(\underline{\bar{r}}, \bar{\tau})$ and $(\underline{r}', \tau') \in \mathcal{U}_{\bar{\delta}'}(\underline{\bar{r}}', \bar{\tau}')$ satisfying (8.19), we have

$$d_{\mathrm{Aut}(D)}(\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}, \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}) < \bar{\nu}'/2.$$

Without loss of generality, we assume

$$\bar{\delta} < \delta, \quad \bar{\delta}' < \delta'$$

because the above estimate persists if we decrease $\bar{\delta}$ and $\bar{\delta}'$. In particular, we have an estimate in uniform norm of disk automorphisms

$$\|\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'} - \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}\|_{C^0} < \bar{\nu}'/2. \tag{8.21}$$

We now prove equation (8.20). We have

$$\begin{aligned}
&|\bar{o}'_{\hat{\mathrm{v}}',j'} - \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(o_{\hat{\mathrm{v}},j})| \\
&= |\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar{o}_{\hat{\mathrm{v}},j}) - \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(o_{\hat{\mathrm{v}},j})| \\
&\le |\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(\bar{o}_{\hat{\mathrm{v}},j}) - \bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(o_{\hat{\mathrm{v}},j})| + |\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(o_{\hat{\mathrm{v}},j}) - \psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(o_{\hat{\mathrm{v}},j})| \\
&< N\bar{\nu} + \bar{\nu}'/2 < \bar{\nu}'.
\end{aligned} \tag{8.22}$$

The first equality follows from equation (8.18), and the second inequality follows from the mean value inequality on the disk automorphism $\bar{\psi}^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}$ and the uniform estimate (8.21). By the equivalence (8.19) and the definition of stabilization (Definition 5.19), both points $o'_{\hat{\mathrm{v}}',j'}$ and $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(o_{\hat{\mathrm{v}},j})$ are mapped by $u'_{\hat{\mathrm{v}}'}$ to the transversal constraint $\Sigma'_{\hat{\mathrm{v}}',j'}$. Moreover, both of them are $\bar{\nu}'$-close to the marked point $\bar{o}'_{\hat{\mathrm{v}}',j'}$ due to $\bar{\delta}' < \delta'$ and estimate (8.22). But there is only one such point in the neighborhood $B_{\bar{\nu}'}(\bar{o}'_{\hat{\mathrm{v}}',j'})$. Hence we have $\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}(o_{\hat{\mathrm{v}},j}) = o'_{\hat{\mathrm{v}}',j'}$. This finishes the proof of (8.20). In other words, we have equivalence of stabilizations

$$\underline{\psi}(\#_{\underline{r},\underline{x}}(\underline{o})) = \#_{\underline{r}',\underline{x}'}(\underline{o}'). \tag{8.23}$$

Lastly, it follows from the equivalence (8.19) and the construction of displaced length in (5.6) and (5.7) that we have equality of displaced renormalized lengths

$$\#_{\underline{r}}(\ell(\underline{\gamma}))_{\mathrm{e}} = \#_{\underline{r}'}(\ell(\underline{\gamma}'))_{\mathrm{e}'}. \tag{8.24}$$

By (8.23) and (8.24), we have the desired equivalence

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mathrm{st}(\tau))) = \#(\underline{r}', \mathrm{st}'(\tau')).$$

$\square$

8.3. **Openness of the Biholomorphic Equivalence in $\mathfrak{X}$.**

In this section, we show that the collection of neighborhoods $\mathfrak{U}_\varepsilon(\kappa;\hat{\tau};R)$ in (4.17) forms a basis for the quotient space of disk trees $\mathfrak{X}$. Combining it with the Hausdorff property in Section 8.2 we prove Theorem 4.6. In addition, we also prove Theorem 6.12: the collection of maps Θ in (6.12) forms an M-polyfold atlas for $\mathfrak{X}$.

We shall prove the theorems above by using the following results of the openness of the biholomorphic equivalence. More precisely, we show that if we have gluing $\#(\underline{\bar{r}},\bar{\tau})$ equivalent to gluing $\#(\underline{\bar{r}}',\bar{\tau}')$, then gluing near $\#(\underline{\bar{r}},\bar{\tau})$ is equivalent to gluing near $\#(\underline{\bar{r}}',\bar{\tau}')$ (see Section 7.2 for analogous results for the Deligne-Mumford space). For the following results, we fix representatives $\hat{\tau}=(\hat{\mathrm{T}},\underline{\hat{\gamma}},\underline{\hat{x}},\underline{\hat{u}})$ and $\hat{\tau}'=(\hat{\mathrm{T}}',\underline{\hat{\gamma}}',\underline{\hat{x}}',\underline{\hat{u}}')$ of disk trees κ and κ'. For convenience, we abbreviate

$$\mathcal{U}_\varepsilon(\underline{\hat{r}},\hat{\tau}):=\mathcal{U}_\varepsilon(\underline{\hat{r}})\times\mathcal{U}_\varepsilon(\hat{\tau}).$$

We shall use the following result to prove the collection of neighborhoods in (4.17) forms a basis for the quotient space of disk trees $\mathfrak{X}$; we also use it to show the continuity of each chart Θ in (6.12). As we shall see, this result follows from an analogous result of Theorem 8.16.

Theorem 8.12. *Let R be any gluing profile. Assume there is a pair $(\underline{\bar{r}},\bar{\tau})\in\mathcal{U}_\varepsilon(\underline{0},\hat{\tau})$ and $(\underline{\bar{r}}',\bar{\tau}')\in\mathcal{U}_{\varepsilon'}(\underline{0},\hat{\tau}')$ with*

$$[\#](\underline{\bar{r}},\bar{\tau})=[\#](\underline{\bar{r}}',\bar{\tau}').$$

Then given $\delta'>0$, there is $\delta>0$ with inclusion

$$[\#](\mathcal{U}_\delta(\underline{\bar{r}},\bar{\tau}))\subset\Theta'(\mathcal{K}^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}',\bar{\tau}')).$$

We shall use the following result to show the continuity of the inverse of each chart Θ. This result is a consequence of Theorem 8.16.

Theorem 8.13. *Let R be any gluing profile. Assume there is a pair $(\underline{\bar{r}},\bar{\tau})\in\mathcal{U}_\varepsilon(\underline{0},\hat{\tau})$ and $(\underline{\bar{r}}',\underline{\bar{\rho}}',\underline{\bar{x}}',\underline{\bar{\xi}}')\in\mathcal{K}^{\mathrm{slc}}_{\varepsilon'}(\underline{0},\hat{\tau}')$ with*

$$[\#](\underline{\bar{r}},\bar{\tau})=\Theta'(\underline{\bar{r}}',\underline{\bar{\rho}}',\underline{\bar{x}}',\underline{\bar{\xi}}').$$

Then given $\delta'>0$ and a neighborhood $B_{\delta'}(\underline{\bar{r}}',\underline{\bar{\rho}}',\underline{\bar{x}}',\underline{\bar{\xi}}')\subset\mathcal{K}^{\mathrm{slc}}_{\varepsilon'}(\underline{0},\hat{\tau}')$ in the splicing core slice, there is $\delta>0$ with inclusion

$$[\#](\mathcal{U}_\delta(\underline{\bar{r}},\bar{\tau}))\subset\Theta'(B_{\delta'}(\underline{\bar{r}}',\underline{\bar{\rho}}',\underline{\bar{x}}',\underline{\bar{\xi}}')).$$

Note the similarity between the above two theorems. In Theorem 8.13, we denote the disk tree representative $\bar{\tau}:=\tau'(\underline{\bar{\rho}}',\underline{\bar{x}}',\underline{\bar{\xi}}')$ given by (6.12). Then Theorem 8.13 shows the containment of a neighborhood of $(\underline{\bar{r}}',\bar{\tau}')$ in a splicing core slice based at $(\underline{0},\hat{\tau}')$. On the other hand, Theorem 8.12 shows the containment of a neighborhood of $(\underline{\bar{r}}',\bar{\tau}')$ in a splicing core slice based at $(\underline{\bar{r}}',\bar{\tau}')$ itself.

We shall use the following result in showing the sc-smoothness of the transition maps of charts. We will prove this result later on in this section.

Theorem 8.14. *Let R be given by the exponential gluing profile $\varphi(r) = e^{1/r} - e$. Assume there is a pair $(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) \in \mathcal{K}^{\mathrm{slc}}_{\varepsilon}(\underline{0}, \hat{\tau})$ and $(\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}') \in \mathcal{K}^{\mathrm{slc}}_{\varepsilon'}(\underline{0}, \hat{\tau}')$ with*

$$\Theta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) = \Theta'(\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}').$$

Then there are $\delta, \delta' > 0$ such that for neighborhoods $B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) \subset \mathcal{K}^{\mathrm{slc}}_{\varepsilon}(\underline{0}, \hat{\tau})$ and $B_{\delta'}(\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}') \subset \mathcal{K}^{\mathrm{slc}}_{\varepsilon'}(\underline{0}, \hat{\tau}')$ in splicing core slices, there exists an sc-smooth map

$$\Lambda : B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) \to B_{\delta'}(\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')$$

with $\Lambda(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) = (\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')$ such that for $(\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}) \in B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}})$ we have

$$\Theta(\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}) = \Theta'(\Lambda(\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi})).$$

We shall use the following result to show the injectivity of each chart Θ, and we shall prove it by using Proposition 8.11 to pass to the Deligne-Mumford space and then using the injectivity result in $\mathfrak{DM}$ (Theorem 7.21).

Theorem 8.15. *Let R be any gluing profile. There exists an injectivity radius $\hat{\varepsilon} = \hat{\varepsilon}_{\mathrm{inj}}(\hat{\tau}) > 0$ such that if $(\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}), (\underline{r}', \underline{\rho}', \underline{x}', \underline{\xi}') \in \mathcal{U}^{\mathrm{slc}}_{\hat{\varepsilon}}(\underline{0}, \hat{\tau})$ satisfy*

$$\Theta(\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}) = \Theta(\underline{r}', \underline{\rho}', \underline{x}', \underline{\xi}'),$$

then we have $(\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}) = (\underline{r}', \underline{\rho}', \underline{x}', \underline{\xi}')$.

We now show that the collection of neighborhoods $\mathfrak{U}_\varepsilon(\kappa; \hat{\tau}; R)$ in (4.17) forms a basis for quotient space of disk trees.

Proof of Theorem 4.6. Fix a gluing profile R. We now show that the collection $\{\mathfrak{U}_\varepsilon(\kappa; \hat{\tau}; R)\}$ forms a basis. For convenience, we drop the gluing profile R in our notation. Clearly, this collection covers the quotient space of disk trees $\mathfrak{X}$. Suppose there is an element κ in the intersection $\mathfrak{U}_{\varepsilon'}(\kappa'; \hat{\tau}') \cap \mathfrak{U}_{\varepsilon''}(\kappa''; \hat{\tau}'')$. Let $\bar{\tau}$ be a representative of κ. We claim that there exists $\delta > 0$ with

$$\mathfrak{U}_\delta(\kappa; \bar{\tau}) \subset \mathfrak{U}_{\varepsilon'}(\kappa'; \hat{\tau}'), \tag{8.25}$$

and analogously there is $\delta > 0$ with $\mathfrak{U}_\delta(\kappa; \bar{\tau}) \subset \mathfrak{U}_{\varepsilon''}(\kappa''; \hat{\tau}'')$. Thus the neighborhood $\mathfrak{U}_\delta(\kappa; \bar{\tau})$ lies in the intersection $\mathfrak{U}_{\varepsilon'}(\kappa'; \hat{\tau}') \cap \mathfrak{U}_{\varepsilon''}(\kappa''; \hat{\tau}'')$, proving the collection forms a basis. We now find $\delta > 0$ satisfying (8.25). Firstly, since the element κ lies in $\mathfrak{U}_{\varepsilon'}(\kappa'; \hat{\tau}')$, there is $(\underline{\bar{r}}', \bar{\tau}') \in \mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\tau}')$ with $[\#](\underline{0}, \bar{\tau}) = \kappa = [\#](\underline{\bar{r}}', \bar{\tau}')$. We choose $\delta' > 0$ with $\mathcal{U}_{\delta'}(\underline{\bar{r}}', \bar{\tau}') \subset \mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\tau}')$. Then by Theorem 8.12, there exists $\delta > 0$ with

$$[\#](\mathcal{U}_\delta(\underline{0}, \bar{\tau})) \subset \Theta'(\mathcal{K}^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}', \bar{\tau}')).$$

By the construction of the splicing core slice, we have

$$[\#](\mathcal{U}_\delta(\underline{0}, \bar{\tau})) \subset \Theta'(\mathcal{K}^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}', \bar{\tau}')) \subset [\#](\mathcal{U}_{\delta'}(\underline{\bar{r}}', \bar{\tau}')) \subset [\#](\mathcal{U}_{\varepsilon'}(\underline{0}, \hat{\tau}')),$$

where by definition the first set is the neighborhood $\mathfrak{U}_\delta(\kappa; \bar{\tau})$ and the last set the neighborhood $\mathfrak{U}_{\varepsilon'}(\kappa'; \hat{\tau}')$. This proves (8.25).

The Hausdorff property of this topology is proved after Proposition 8.3. Furthermore, the topology is independent of the choices of gluing profiles R because for two gluing profiles R and R', the transition $R'^{-1} \circ R$ is a homeomorphism. This finishes the proof of the theorem. $\square$

We now show that the collection of maps Θ in (6.12) forms an sc-smooth atlas for $\mathfrak{X}$.

Proof of Theorem 5.17. We show that for a given representative $\hat{\tau}$ of an element of $\mathfrak{X}$, there exists $\varepsilon > 0$ such that the map

$$\Theta : \mathcal{K}^{\mathrm{slc}}_{\varepsilon}(\underline{0}, \hat{\tau}) \to \mathfrak{X}, \quad (\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}) \mapsto [\#](\underline{r}, \tau(\underline{\rho}, \underline{x}, \underline{\xi})).$$

is a homeomorphism onto its image. First of all, it follows from Theorem 8.15 that for $\varepsilon < \hat{\varepsilon}_{\mathrm{inj}}(\hat{\tau})$, the map Θ is injective.

We now prove the continuity of Θ. Let $\mathfrak{V}$ be an arbitrary open set in $\mathfrak{X}$, we now show that $\Theta^{-1}(\mathfrak{V})$ is open. It suffices to show that for a given $(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) \in \Theta^{-1}(\mathfrak{V})$, there exists a neighborhood $B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) \subset \mathcal{K}^{\mathrm{slc}}_{\varepsilon}(\underline{0}, \hat{\tau})$ in the splicing core slice with

$$B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) \subset \Theta^{-1}(\mathfrak{V}). \tag{8.26}$$

Denote $\kappa := \Theta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}})$. Since $\mathfrak{V}$ is open, there is a basis element $\mathfrak{U}_\delta(\kappa; \bar{\tau}')$ with $\mathfrak{U}_\delta(\kappa; \bar{\tau}') \subset \mathfrak{V}$. Note that by definition we have $\Theta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) = \kappa = [\#](\underline{0}, \bar{\tau}')$. Applying Theorem 8.12, there exists $\delta > 0$ with

$$[\#](\mathcal{U}_\delta(\underline{\bar{r}}, \bar{\tau})) \subset \Theta'(\mathcal{K}^{\mathrm{slc}}_{\delta'}(\underline{0}, \bar{\tau}')).$$

Thus we have

$$\Theta(B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}})) \subset [\#](\mathcal{U}_\delta(\underline{\bar{r}}, \bar{\tau})) \subset \Theta'(\mathcal{K}^{\mathrm{slc}}_{\delta'}(\underline{0}, \bar{\tau}')) \subset [\#](\mathcal{U}_{\delta'}(\underline{0}, \bar{\tau}')),$$

where the last set is the neighborhood $\mathfrak{U}_{\delta'}(\kappa; \bar{\tau}')$ contained in $\mathfrak{V}$. This proves (8.26).

We now prove the continuity of Θ^{-1}. Let V be an arbitrary open subset of the domain $\mathcal{K}^{\mathrm{slc}}_{\varepsilon}(\underline{0}, \hat{\tau})$, we now show that the image $\Theta(V)$ is open. It suffices to prove that for a given $\kappa \in \Theta(V)$ with a representative $\bar{\tau}'$, there exists $\delta > 0$ with

$$\mathfrak{U}_\delta(\kappa; \bar{\tau}') \subset \Theta(V). \tag{8.27}$$

Firstly, since the element κ lies in $\Theta(V)$, there is $(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) \in V$ with $[\#](\underline{0}, \bar{\tau}') = \kappa = \Theta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}})$. We choose a neighborhood $B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) \subset \mathcal{K}^{\mathrm{slc}}_{\varepsilon}(\underline{0}, \hat{\tau})$ in the splicing core slice with $B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}}) \subset V$. Then by Theorem 8.13 (with the primed and the un-primed swapped), there exists $\delta' > 0$ with

$$\mathfrak{U}_{\delta'}(\kappa; \bar{\tau}') = [\#](\mathcal{U}_{\delta'}(\underline{0}, \bar{\tau}')) \subset \Theta(B_\delta(\underline{\bar{r}}, \underline{\bar{\rho}}, \underline{\bar{x}}, \underline{\bar{\xi}})).$$

The last set is contained in the image $\Theta(V)$. This proves (8.27).

Lastly, we prove the transition map between two charts is smooth. Suppose there are two charts $\Theta : \mathcal{K}^{\mathrm{slc}}_{\varepsilon}(\underline{0}, \hat{\tau}) \to \mathfrak{X}$ and $\Theta' : \mathcal{K}^{\mathrm{slc}}_{\varepsilon'}(\underline{0}, \hat{\tau}') \to \mathfrak{X}$ whose images have a non-empty overlap $\mathrm{im}(\Theta) \cap \mathrm{im}(\Theta') \neq \emptyset$. Then the smoothness of the transition map $\Theta'^{-1} \circ \Theta$ follows directly from the smoothness of Λ in Theorem 7.20. $\square$

We now prove the injectivity result of Theorem 8.15.

Proof of Theorem 8.15. We have trivial biholomorphic equivalence

$$(\hat{\zeta}, \underline{\hat{\psi}})(\#(\underline{0}, \hat{\tau})) = \#(\underline{0}, \hat{\tau}),$$

with $\hat{\zeta} : \hat{\mathrm{T}} \to \hat{\mathrm{T}}$ being the identity isomorphism and each biholomorphism being $\hat{\psi}_{\hat{\mathrm{v}}} = \mathrm{Id}$. Moreover, let $\underline{\hat{o}}$ be the interior marked points chosen in Remark 6.10 (5) at which the derivatives $D\underline{\hat{u}}$ are injective, let $\underline{\Sigma}$ be the transversal constraints at $\underline{\hat{o}}$ chosen in the same remark, and lastly, let st be the stabilization map induced by $\underline{\Sigma}$ near $\underline{\hat{o}}$ (see Definition 5.19). Then Proposition 8.11 shows that there exist $\hat{\delta}, \hat{\delta}' > 0$ such that if there are $(\underline{r}, \tau) \in \mathcal{U}_{\hat{\delta}}(\underline{0}, \hat{\tau})$ and $(\underline{r}', \tau') \in \mathcal{U}_{\hat{\delta}'}(\underline{0}, \hat{\tau}')$ and a biholomorphism $(\zeta, \underline{\psi})$ with

$$(\zeta, \underline{\psi})(\#(\underline{r}, \tau)) = \#(\underline{r}', \tau'),$$

then we have equivalence

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mathrm{st}(\tau))) = \#(\underline{r}', \mathrm{st}(\tau')). \tag{8.28}$$

Let $\nu = \hat{\varepsilon}_{\mathrm{inj}}(\mathrm{st}(\hat{\tau}))$ be the injectivity radius of the stabilization $\mathrm{st}(\hat{\tau})$ given by Theorem 7.21. Then by the continuity of the stabilization map in Lemma 5.20, there is $\varepsilon > 0$ with containment

$$\mathrm{st}(\mathcal{U}_{\varepsilon}(\hat{\tau})) \subset \mathcal{U}_{\nu}(\mathrm{st}(\hat{\tau})).$$

We choose the injectivity radius $\hat{\varepsilon} := \hat{\varepsilon}_{\mathrm{inj}}(\hat{\tau})$ to be the minimum of $\hat{\delta}, \hat{\delta}', \varepsilon$, and we now prove that this radius satisfies the injectivity property. Suppose we have $(\underline{r}, \underline{\rho}, \underline{x}, \underline{\xi}), (\underline{r}', \underline{\rho}', \underline{x}', \underline{\xi}') \in \mathcal{K}^{\mathrm{slc}}_{\hat{\varepsilon}}(\underline{0}, \hat{\tau})$ such that there is a biholomorphism $(\zeta, \underline{\psi})$ with

$$(\zeta, \underline{\psi})(\#(\underline{r}, \tau(\underline{\rho}, \underline{x}, \underline{\xi}))) = \#(\underline{r}', \tau(\underline{\rho}', \underline{x}', \underline{\xi}')). \tag{8.29}$$

Here the disk tree representative $\tau(\underline{\rho}, \underline{x}, \underline{\xi}) = (\hat{\mathrm{T}}, \underline{\gamma}(\underline{\rho}, \underline{\xi}), \underline{x}, \underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))$ is constructed in (6.12). Then it follows from our choice of $\hat{\varepsilon}$ and equation (8.28) that we have

$$(\zeta, \underline{\psi})(\#(\underline{r}, \mathrm{st}(\tau(\underline{\rho}, \underline{x}, \underline{\xi})))) = \#(\underline{r}', \mathrm{st}(\tau(\underline{\rho}', \underline{x}', \underline{\xi}'))).$$

By the requirement of sections $\underline{\xi}$ in Definition 6.11 (1) and the construction of maps $\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))$ in (6.12), the stabilizations $\mathrm{st}(\tau(\underline{\rho}, \underline{x}, \underline{\xi}))$ and $\mathrm{st}(\tau(\underline{\rho}', \underline{x}', \underline{\xi}'))$ lie in the neighborhood slice $\mathcal{U}^{\mathrm{slc}}_{\nu}(\mathrm{st}(\hat{\tau}))$. By the injectivity result of the Deligne-Mumford in Theorem 7.21, we conclude

$$(\underline{r}, \mathrm{st}(\tau(\underline{\rho}, \underline{x}, \underline{\xi}))) = (\underline{r}', \mathrm{st}(\tau(\underline{\rho}', \underline{x}', \underline{\xi}'))). \tag{8.30}$$

By the uniqueness of the biholomorphism (Proposition 5.6), $(\zeta, \underline{\psi})$ is the identity biholomorphism, and by the construction of the stabilization map

(Definition 5.19), we have an equality of boundary marked points $\underline{x} = \underline{x}'$. Thus equation (8.29) translates to

$$\#_{\underline{r}}(\underline{\gamma}(\underline{\rho}, \underline{\xi})) = \#_{\underline{r}}(\underline{\gamma}(\underline{\rho}', \underline{\xi}')), \quad \#_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi})) = \#_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho}', \underline{x}, \underline{\xi}')).$$

By the construction of displaced Morse trajectories (4.7) and (4.8) and the construction in (6.12), the first equation shows an equality of Morse trajectories $\underline{\rho} = \underline{\rho}'$. Finally by Remark 6.14, the equation of glued maps $\#_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi})) = \#_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}'))$ shows an equality of sections $\underline{\xi} = \underline{\xi}'$. This proves the injectivity result. $\square$

We now state the openness result which implies Theorem 8.12 and Theorem 8.13.

Theorem 8.16. *Let R be any gluing profile. Assume there is a pair $(\underline{\bar{r}}, \bar{\tau}) \in \mathcal{U}_\varepsilon(\underline{0}, \hat{\tau})$ and $(\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}') \in \mathcal{K}^{\mathrm{slc}}_{\varepsilon'}(\underline{0}, \hat{\tau}')$ with*

$$[\#](\underline{\bar{r}}, \bar{\tau}) = \Theta'(\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}').$$

Then there are $\delta, \delta' > 0$ such that for the neighborhood $B_{\delta'}(\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}') \subset \mathcal{K}^{\mathrm{slc}}_{\varepsilon'}(\underline{0}, \hat{\tau}')$ in the splicing core slice, there exists a continuous map

$$\Lambda : \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\tau}) \to B_{\delta'}(\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')$$

with $\Lambda(\underline{\bar{r}}, \bar{\tau}) = (\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')$, and for each $(\underline{r}, \tau) \in \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\mu})$ we have

$$[\#](\underline{r}, \tau) = \Theta'(\Lambda(\underline{r}, \tau)).$$

Proof of Theorem 8.12. Analogously to Theorem 8.16, there exists a continuous map to the splicing core slice

$$\Lambda : \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\tau}) \to \mathcal{K}^{\mathrm{slc}}_{\delta'}(\underline{\bar{r}}', \bar{\tau}'),$$

and the result follows. $\square$

Proof of Theorem 8.13. This result follows directly from the map Λ in Theorem 8.16 and its continuity. $\square$

We now prove the above openness result by using the corresponding result Theorem 7.22 in the Deligne-Mumford space.

Proof of Theorem 8.16. We shall prove this result by passing to the Deligne-Mumford space using stabilization maps, and then using the continuity of the biholomorphic equivalence provided by Theorem 7.22. By assumption, there is a biholomorphism $(\bar{\zeta}, \underline{\bar{\psi}})$ with

$$(\bar{\zeta}, \underline{\bar{\psi}})(\#(\underline{\bar{r}}, \bar{\tau})) = \#(\underline{\bar{r}}', \tau'(\underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')), \tag{8.31}$$

where the disk tree representatives are of the form

$$\bar{\tau} = (\hat{\mathrm{T}}, \underline{\bar{\gamma}}, \underline{\bar{x}}, \underline{\bar{u}}), \quad \tau'(\underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}') = (\hat{\mathrm{T}}', \underline{\gamma}'(\underline{\bar{\rho}}', \underline{\bar{\xi}}'), \underline{\bar{x}}', \underline{u}'(\underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')).$$

The latter is defined in (6.12). In order to set up the right stabilization maps, we first find transversal constraints $\underline{\Sigma}'$ and interior marked points $\underline{\bar{o}}$ and $\underline{\bar{o}}'$ with the following properties.

- The derivatives $D\underline{\bar{u}}$ and $D\underline{u}'(\underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')$ are injective at $\underline{\bar{o}}$ and $\underline{\bar{o}}'$, respectively.
- $\underline{\Sigma}'$ are transversal constrains at marked points $\underline{\bar{o}}$ for the disk tree representative $\bar{\tau}$, and are also transversal constraints at $\underline{\bar{o}}'$ for $\tau'(\underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')$.
- The marked points $\underline{\bar{o}}$ and $\underline{\bar{o}}'$ satisfy the stability condition (5.1).

We shall obtain the desired transversal constraints and interior marked points as follows. We start with the transversal constraints $\underline{\hat{\Sigma}}'$ and the interior marked points $\underline{\hat{o}}'$ chosen in Remark 6.10 (5) for the splicing core slice $\mathcal{K}^{\mathrm{slc}}_{\varepsilon'}(\underline{0}, \hat{\tau}')$. By the requirement of sections $\underline{\bar{\xi}}'$ in Definition 6.11 (1) and the construction of maps in (6.12), $\underline{u}'(\underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}'))$ map marked points $\underline{\hat{o}}'$ to the transversal constraints $\underline{\hat{\Sigma}}'$. Using the biholomorphisms between glued disks $\underline{\bar{\psi}} : \#_{\underline{\bar{r}}, \underline{\bar{x}}}(\underline{D}) \to \#_{\underline{\bar{r}}', \underline{\bar{x}}'}(\underline{D})$, we obtain interior marked points $\underline{\hat{o}}$ which satisfy the equation

$$\underline{\bar{\psi}} : \#_{\underline{\bar{r}}, \underline{\bar{x}}}(\underline{\hat{o}}) \to \#_{\underline{\bar{r}}', \underline{\bar{x}}'}(\underline{\hat{o}}').$$

By construction, each marked point $\hat{o}_{\hat{\mathrm{v}}}$ has a corresponding marked point $\hat{o}'_{\hat{\mathrm{v}}'}$ with $\bar{\psi}_{\bar{\mathrm{v}}}(\hat{o}_{\hat{\mathrm{v}}}) = \hat{o}'_{\hat{\mathrm{v}}'}$, where the vertices satisfy $[\hat{\mathrm{v}}]_{\underline{\bar{r}}} = \bar{\mathrm{v}}$, $[\hat{\mathrm{v}}']_{\underline{\bar{r}}'} = \bar{\mathrm{v}}'$, and $\bar{\zeta}(\bar{\mathrm{v}}) = \bar{\mathrm{v}}'$. Moreover, (8.31) implies equivalence

$$\#_{\underline{\bar{r}}, \underline{\bar{x}}}(\underline{\bar{u}})_{\bar{\mathrm{v}}} = \#_{\underline{\bar{r}}', \underline{\bar{x}}'}(\underline{u}'(\underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}'))_{\bar{\mathrm{v}}'} \circ \bar{\psi}_{\bar{\mathrm{v}}}.$$

Evaluating both sides of the above equation at the point $\hat{o}_{\hat{\mathrm{v}}}$, we find that the image $\bar{u}_{\hat{\mathrm{v}}}(\hat{o}_{\hat{\mathrm{v}}})$ lies in the transversal constraint $\hat{\Sigma}'_{\hat{\mathrm{v}}'}$, and the derivatives have the same image $\mathrm{im}(D\bar{u}_{\hat{\mathrm{v}}}(\hat{o}_{\hat{\mathrm{v}}})) = \mathrm{im}(Du'(\underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')_{\hat{\mathrm{v}}'}(\hat{o}'_{\hat{\mathrm{v}}'}))$. Thus $\underline{\hat{\Sigma}}'$ are also transversal constraints at $\underline{\hat{o}}$ for the disk tree representative $\bar{\tau}$. However, this approach has one issue: the marked points $\underline{\hat{o}}$ do not necessarily satisfy the stability condition (5.1). We rectify this situation by adding transversal constraints and interior marked points by using Lemma 5.1. We denote the augmented transversal constraints and interior marked points by $\underline{\Sigma}'$ and $\underline{\bar{o}}$, and use the biholomorphisms $\underline{\bar{\psi}}$ to push forward the marked points and obtain $\underline{\bar{o}}'$. Naturally, the marked points $\underline{\bar{o}}'$ contain the original marked points $\underline{\hat{o}}'$, and for each non-ghost vertex $\hat{\mathrm{v}}'$, we index them as

$$\bar{o}'_{\hat{\mathrm{v}}',1} := \hat{o}'_{\hat{\mathrm{v}}'}. \tag{8.32}$$

Thus we find the desired transversal constraints and interior marked points.

We now show that for some $\delta, \delta' > 0$, for each $(\underline{r}, \tau) \in \mathcal{U}_\delta(\underline{\bar{r}}, \bar{\tau})$ we can solve the following equation for $(\underline{r}', \underline{\rho}', \underline{x}', \underline{\xi}') \in B_{\delta'}(\underline{\bar{r}}', \underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')$ and a biholomorphism $(\zeta, \underline{\psi})$

$$(\zeta, \underline{\psi})(\#(\underline{r}, \tau)) = \#(\underline{r}', \tau'(\underline{\rho}', \underline{x}', \underline{\xi}')). \tag{8.33}$$

In order to solve this equation, we first solve the corresponding equation of their stabilizations (cf. Proposition 8.11). By construction, the stabilizations induced by the transversal constraints $\underline{\Sigma}'$ are of the form

$$\mathrm{st}(\bar{\tau}) := \bar{\mu} = (\hat{\mathrm{T}}, \underline{\bar{\ell}}, \underline{\bar{x}}, \underline{\bar{o}}), \quad \mathrm{st}'(\tau'(\underline{\bar{\rho}}', \underline{\bar{x}}', \underline{\bar{\xi}}')) := \bar{\mu}' = (\hat{\mathrm{T}}', \underline{\bar{\ell}}', \underline{\bar{x}}', \underline{\bar{o}}').$$

Note that for each tuple $(\underline{r},\tau) \in \mathcal{U}_\delta(\underline{\bar{r}},\bar{\tau})$, the stabilization $\mathrm{st}(\tau)$ lies in a neighborhood $\mathcal{U}_{\bar{\nu}}(\bar{\mu})$; for each tuple $(\underline{r}',\underline{\rho}',\underline{x}',\underline{\xi}') \in B_{\delta'}(\underline{\bar{r}}',\underline{\bar{\rho}}',\underline{\bar{x}}',\underline{\bar{\xi}}')$, the stabilization $\mathrm{st}'(\tau'(\underline{\rho}',\underline{x}',\underline{\xi}'))$ lies in a neighborhood slice $\mathcal{U}^{\mathrm{slc}}_{\bar{\nu}'}(\bar{\mu}')$ because for each non-ghost vertex $\hat{\mathrm{v}}'$ we have $o'_{\hat{\mathrm{v}}',1} \equiv \hat{o}_{\hat{\mathrm{v}}'}$ (see (8.32) and Definition 6.11 (1)).

By Theorem 7.22 and Proposition 7.18, there exist small $\nu,\nu > 0$, a large $k' > 0$, and a continuous map

$$\lambda : \mathcal{U}_\nu(\underline{\bar{r}},\bar{\mu}) \to \mathcal{U}^{\mathrm{slc}}_{\nu'}(\underline{\bar{r}}',\bar{\mu}'). \tag{8.34}$$

For each $(\underline{r},\mu) \in \mathcal{U}_\nu(\underline{\bar{r}},\bar{\mu})$, we denote $(\underline{r}',\mu') := \lambda(\underline{r},\mu)$. There is a biholomorphism $\left(\zeta_{\underline{r}},\underline{\psi}_{(\underline{r},\underline{x},\underline{o})}\right)$ with

$$\left(\zeta_{\underline{r}},\underline{\psi}_{(\underline{r},\underline{x},\underline{o})}\right)(\#(\underline{r},\mu)) = \#(\underline{r}',\mu'), \tag{8.35}$$

and the biholomorphism in standard model

$$\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}_{(\underline{r},\underline{x},\underline{o})} := (\Phi^{\hat{\mathrm{v}}'}_{\mathrm{v}'})^{-1} \circ \psi_{(\underline{r},\underline{x},\underline{o}),\mathrm{v}} \circ \Phi^{\hat{\mathrm{v}}}_{\mathrm{v}}$$

depends continuously on $(\underline{r},\underline{x},\underline{o})$. Furthermore, each additional gluing edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-,\hat{\mathrm{v}}^+)$ in $\hat{\mathrm{E}}^{\mathrm{ag}}_{\underline{r}}$ bijectively corresponds to an additional gluing edge $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}'^-,\hat{\mathrm{v}}'^+) = \bar{\chi}(\hat{\mathrm{e}})$ in $\hat{\mathrm{E}}'^{\mathrm{ag}}_{\underline{r}'}$, and we have inclusion of the glued strips of edges $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$

$$\psi_{(\underline{r},\underline{x},\underline{o}),\mathrm{v}}([0,R_{\hat{\mathrm{e}}}] \times [0,\pi]) \supset [k', R'_{\hat{\mathrm{e}}'} - k'] \times [0,\pi] \tag{8.36}$$

with $\mathrm{v} := [\hat{\mathrm{v}}^-]_{\underline{r}} = [\hat{\mathrm{v}}^+]_{\underline{r}}$. Each edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^-,\hat{\mathrm{v}}^+)$ with $\bar{r}_{\hat{\mathrm{e}}} = 0, r_{\hat{\mathrm{e}}} \le 0$ bijectively corresponds to an edge $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}'^-,\hat{\mathrm{v}}'^+) = \bar{\chi}(\hat{\mathrm{e}})$ with $\bar{r}'_{\hat{\mathrm{e}}'} = 0, r'_{\hat{\mathrm{e}}'} \le 0$, and we have inclusion of shrunk strip neighborhoods of edges $\hat{\mathrm{e}}$ and $\hat{\mathrm{e}}'$

$$\psi_{(\underline{r},\underline{x},\underline{o}),\mathrm{v}^\pm}(N(x^\pm_{\hat{\mathrm{e}}})) \supset N(x'^\pm_{\hat{\mathrm{e}}'}; -k') \tag{8.37}$$

with $\mathrm{v}^\pm := [\hat{\mathrm{v}}^\pm]_{\underline{r}}$. As we shall see, such inclusion will be used in solving the sections $\underline{\xi}'$ in equation (8.33).

For each disk tree representative $(\underline{r},(\hat{\mathrm{T}},\underline{\gamma},\underline{x},\underline{u}))$, let $(\underline{r},(\hat{\mathrm{T}},\underline{\ell},\underline{x},\underline{o}))$ be its stabilization and let $(\underline{r}',(\hat{\mathrm{T}}',\underline{\ell}',\underline{x}',\underline{o}'))$ be the tuple satisfying equation (8.35). In order to solve equation (8.33), we shall solve the following equation for Morse trajectories $\underline{\rho}'$ and sections $\underline{\xi}'$ in the neighborhood $B_{\delta'}(\underline{\bar{r}}',\underline{\bar{\rho}}',\underline{\bar{x}}',\underline{\bar{\xi}}')$

$$\left(\zeta_{\underline{r}},\underline{\psi}_{(\underline{r},\underline{x},\underline{o})}\right)(\#(\underline{r},(\hat{\mathrm{T}},\underline{\gamma},\underline{x},\underline{u}))) = \#\big(\underline{r}',(\hat{\mathrm{T}}',\underline{\gamma}'(\underline{\rho}',\underline{\xi}'),\underline{x}',\underline{u}'(\underline{\rho}',\underline{x}',\underline{\xi}'))\big). \tag{8.38}$$

For each glued edge $\mathrm{e} \in \#_{\underline{r}}(\hat{\mathrm{E}})$ and its corresponding edge $\mathrm{e}' := \zeta_{\underline{r}}(\mathrm{e})$, we solve the following equation for Morse trajectories $\underline{\rho}'$

$$\#_{\underline{r}}(\underline{\gamma})_{\mathrm{e}} = \#_{\underline{r}'}(\underline{\gamma}'(\underline{\rho}',\underline{\xi}'))_{\mathrm{e}'}. \tag{8.39}$$

By the construction of charts (6.12) and the displaced Morse trajectories in (4.7) and (4.8), the above equation involves $\underline{\rho}'$ only when the edge e'

corresponds to a non-nodal edge $\hat{e}' \in \hat{E}' \setminus \hat{E}'^{nd}$ via the map (4.6). Thus by construction, the solution to the above equation is given by

$$\underline{\rho}'_{\hat{e}'} = \#_{\underline{r}}(\underline{\gamma})_e.$$

Furthermore, the solution depends continuously on the input.

For main glued vertices $v \in \#_{\underline{r}}(\hat{V})$ and their corresponding vertices $v' := \zeta_{\underline{r}}(v)$, we solve the following equation for sections $\underline{\xi}'$

$$\#_{\underline{r},\underline{x}}(\underline{u})_v \circ \psi^{-1}_{(\underline{r},\underline{x},\underline{o}),v} = \#_{\underline{r}',\underline{x}'}(\underline{u}'(\underline{\rho}', \underline{x}', \underline{\xi}'))_{v'}. \tag{8.40}$$

Let $\hat{v} \in \hat{V}^m$ and $\hat{v}' \in \hat{V}'^m$ be main vertices satisfying $\bar{\zeta}([\hat{v}]_{\underline{\bar{r}}}) = [\hat{v}']_{\underline{\bar{r}}'}$. Then equation (8.40) in the standard model of $D_{\hat{v}}$ and $D_{\hat{v}'}$ is given by

$$\#^{\hat{v}}_{\underline{r},\underline{x}}(\underline{u})_v \circ \left(\psi^{\hat{v},\hat{v}'}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1} = \#^{\hat{v}'}_{\underline{r}',\underline{x}'}(\underline{u}'(\underline{\rho}', \underline{x}', \underline{\xi}'))_{v'}. \tag{8.41}$$

We shall use this equation in standard model to find a solution for sections $\underline{\xi}'$, and we will prove that the solution depends continuously on the input.

For each main vertex $\hat{v}' \in \hat{V}^m$, we first solve for the part of the section $\xi'_{\hat{v}'}$ on $D \setminus \bigsqcup N(\hat{x}'_{\hat{v}',\hat{e}'})$, outside of strip neighborhoods. Let $\hat{v}$ be a main vertex with $\bar{\zeta}([\hat{v}]_{\underline{\bar{r}}}) = [\hat{v}']_{\underline{\bar{r}}'}$. Then by construction (6.14) and equation (8.41), we have

$$\#^{\hat{v}}_{\underline{r},\underline{x}}(\underline{u})_v \circ \left(\psi^{\hat{v},\hat{v}'}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1}\big|_{D\setminus \bigsqcup N(x'_{\hat{v}',\hat{e}'})} = \exp_{\hat{u}'_{\hat{v}'}} \circ\, \xi'_{\hat{v}'} \circ \nu'^{-1}_{\underline{x}'_{\hat{v}'}}\big|_{D\setminus \bigsqcup N(x'_{\hat{v}',\hat{e}'})}, \tag{8.42}$$

where $\nu'_{\underline{x}'_{\hat{v}'}}$ is the chosen family of marked points varying diffeomorphisms. This solves the section $\xi'_{\hat{v}'}$ on $D \setminus \bigsqcup N(\hat{x}'_{\hat{v}',\hat{e}'})$. Note that the region $D \setminus \bigsqcup N(x'_{\hat{v}',\hat{e}'})$ avoids all glued marked points $\#^{\hat{v}'}_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{v}',\bar{f}'}$ whose edge $\bar{f}'$ corresponds to an edge $\hat{f}'$ with $\bar{r}'_{\hat{f}'} = 0$. Thus the solution depends continuously on the input, due to the continuity of the glued map $\#^{\hat{v}}_{\underline{r},\underline{x}}(\underline{u})_v$ (Lemma 8.9), the continuous dependence in the disk automorphism $\left(\psi^{\hat{v},\hat{v}'}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1}$ (Proposition 13.13), and the continuity of $\mathrm{Aut}(D)$ action (Proposition 6.4).

We now solve the sections $\underline{\xi}'$ on each strip neighborhood $N(\hat{x}'^{\pm}_{\hat{e}'})$ in their strip coordinates. Recall the notations of local expressions (6.15) and (6.9)

$$\hat{u}'^{h'^{\pm}}_{\hat{e}'} := \varphi'^{\pm}_{\hat{e}'} \circ \hat{u}'^{\pm}_{\hat{e}'} \circ h'^{\pm}_{\hat{e}'}(\hat{x}'^{\pm}_{\hat{e}'}, \cdot),$$

$$\xi'^{h'^{\pm}}_{\hat{e}'}(z) := D\varphi'^{\pm}_{\hat{e}'}(\hat{u}'^{\pm}_{\hat{e}'}(h'^{\pm}_{\hat{e}'}(\hat{x}'^{\pm}_{\hat{e}'}, z)))(\xi'^{\pm}_{\hat{e}'}(h'^{\pm}_{\hat{e}'}(\hat{x}'^{\pm}_{\hat{e}'}, z))),$$

where $\varphi'^{\pm}_{\hat{e}'}$ are the chosen smooth charts. We shall consider four cases classified by the edge type of $\hat{e}'$: (1) $\hat{e}' \in \hat{E}' \setminus \hat{E}'^{nd}$, (2) $\bar{r}'_{\hat{e}'} < 0$, (3) $\bar{r}'_{\hat{e}'} > 0$, and (4) $\bar{r}'_{\hat{e}'} = 0$.

(1) Suppose we have a non-nodal edge $\hat{e}' = (\hat{v}'^-, \hat{v}'^+) \in \hat{E}' \setminus \hat{E}'^{nd}$.

If $\hat{v}'^{\pm}$ is a main vertex, then choose any main vertex $\hat{v}^{\pm} \in \hat{V}^m$ satisfying $\bar{\zeta}([\hat{v}^{\pm}]_{\underline{\bar{r}}}) = [\hat{v}'^{\pm}]_{\underline{\bar{r}}'}$, and denote the glued vertex $v^{\pm} := [\hat{v}^{\pm}]_{\underline{r}}$. We study equation (8.41) on the strip neighborhood

$$N(x'^{\pm}_{\hat{\mathrm{e}}'}) \subset D_{\hat{\mathrm{v}}'^{\pm}}. \tag{8.43}$$

By construction (6.17) and the construction of glued maps (4.16), equation (8.41) induces an equation

$$\begin{aligned}\hat{u}'^{h'^{\pm}}_{\hat{\mathrm{e}}'} + \xi'^{h'^{\pm}}_{\hat{\mathrm{e}}'} + \alpha^{\pm} \cdot \varphi'^{\pm}_{\hat{\mathrm{e}}'}(\mathrm{ev}^{\pm}(\underline{\rho}'_{\hat{\mathrm{e}}}))\\ = \varphi'^{\pm}_{\hat{\mathrm{e}}'} \circ \#^{\hat{\mathrm{v}}^{\pm}}_{\underline{r},\underline{x}}(\underline{u})_{\mathrm{v}^{\pm}} \circ \left(\psi^{\hat{\mathrm{v}}^{\pm},\hat{\mathrm{v}}'^{\pm}}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1} \circ h'^{\pm}_{\hat{\mathrm{e}}'}(x'^{\pm}_{\hat{\mathrm{e}}'}, \cdot).\end{aligned} \tag{8.44}$$

We recall that α^- is the cut-off function in (6.13). This solves the local expression $\xi'^{h'^{\pm}}_{\hat{\mathrm{e}}'} \in H^{3,\delta_0}(\mathbb{R}^{\pm} \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$. Note that the region in (8.43) avoids all glued marked points $\#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}',\bar{\mathrm{f}}'}$ whose edge $\bar{\mathrm{f}}'$ corresponds to an edge $\hat{\mathrm{f}}'$ with $\bar{r}'_{\hat{\mathrm{f}}'} = 0$. Thus the solution depends continuously on the input, due to the continuity of the glued map $\#^{\hat{\mathrm{v}}^{\pm}}_{\underline{r},\underline{x}}(\underline{u})_{\mathrm{v}^{\pm}}$ (Lemma 8.9), the continuous dependence in the disk automorphism $\left(\psi^{\hat{\mathrm{v}}^{\pm},\hat{\mathrm{v}}'^{\pm}}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1}$ (Proposition 13.13), and the continuity of $\mathrm{Aut}(D)$ action (Proposition 6.4).

(2) Suppose we have an edge $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}'^-, \hat{\mathrm{v}}'^+)$ with gluing parameter $\bar{r}'_{\hat{\mathrm{e}}'} < 0$. Thus we have gluing parameter $r'_{\hat{\mathrm{e}}'} < 0$. As in (1), choose main vertices $\hat{\mathrm{v}}^{\pm} \in \hat{\mathrm{V}}^{\mathrm{m}}$ satisfying $\bar{\zeta}([\hat{\mathrm{v}}^{\pm}]_{\underline{\bar{r}}}) = [\hat{\mathrm{v}}'^{\pm}]_{\underline{\bar{r}}'}$, and denote the glued vertices $\mathrm{v}^{\pm} := [\hat{\mathrm{v}}^{\pm}]_{\underline{r}}$. We study equation (8.41) on the strip neighborhoods

$$N(x'^{\pm}_{\hat{\mathrm{e}}'}) \subset D_{\hat{\mathrm{v}}'^{\pm}}. \tag{8.45}$$

By construction (6.18) and the construction of glued maps (4.16), equation (8.41) induces an equation

$$\hat{u}'^{h'^{+}}_{\hat{\mathrm{e}}'} + \xi'^{h'^{+}}_{\hat{\mathrm{e}}'} = \varphi'_{\hat{\mathrm{e}}'} \circ \#^{\hat{\mathrm{v}}^{+}}_{\underline{r},\underline{x}}(\underline{u})_{\mathrm{v}^{+}} \circ \left(\psi^{\hat{\mathrm{v}}^{+},\hat{\mathrm{v}}'^{+}}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1} \circ h'^{+}_{\hat{\mathrm{e}}'}(x'^{+}_{\hat{\mathrm{e}}'}, \cdot), \tag{8.46}$$

and an equation

$$\begin{aligned}\hat{u}'^{h'^{-}}_{\hat{\mathrm{e}}'} + \xi'^{h'^{-}}_{\hat{\mathrm{e}}'} + \alpha^{-} \cdot \left[\varphi'_{\hat{\mathrm{e}}'}\left(\sigma^{\iota(r'_{\hat{\mathrm{e}}'})}(\varphi'^{-1}_{\hat{\mathrm{e}}'}(c(\xi'^{h'^{-}}_{\hat{\mathrm{e}}'})))\right) - c(\xi'^{h'^{-}}_{\hat{\mathrm{e}}'})\right]\\ = \varphi'_{\hat{\mathrm{e}}'} \circ \#^{\hat{\mathrm{v}}^{-}}_{\underline{r},\underline{x}}(\underline{u})_{\mathrm{v}^{-}} \circ \left(\psi^{\hat{\mathrm{v}}^{-},\hat{\mathrm{v}}'^{-}}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1} \circ h'^{-}_{\hat{\mathrm{e}}'}(x'^{-}_{\hat{\mathrm{e}}'}, \cdot).\end{aligned} \tag{8.47}$$

We recall that $\sigma : \mathbb{R} \times L \to L$ is the flow of the Morse-Smale pair (f, g) in (4.3), and $\iota : (-\infty, 0] \to (-\infty, 0]$ is a smooth increasing function chosen before Remark 4.2. Lastly, $c(\xi'^{h'^{-}}_{\hat{\mathrm{e}}'}) \in \mathbb{R}^n$ is the limit of the function. This solves the nodal strip map $(\xi'^{h'^{+}}_{\hat{\mathrm{e}}'}, \xi'^{h'^{-}}_{\hat{\mathrm{e}}'}) \in F^{\mathrm{nd}}$. The region in (8.45) avoids all glued marked points $\#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}',\bar{\mathrm{f}}'}$ whose edge $\bar{\mathrm{f}}'$ corresponds to an edge $\hat{\mathrm{f}}'$ with $\bar{r}'_{\hat{\mathrm{f}}'} = 0$. Thus the solution depends continuously on the input for the same reason as in (1).

(3) Suppose we have an edge $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}'^-, \hat{\mathrm{v}}'^+)$ with gluing parameter $\bar{r}'_{\hat{\mathrm{e}}'} > 0$. Thus we have gluing parameter $r'_{\hat{\mathrm{e}}'} > 0$. Choose a main vertex $\hat{\mathrm{v}} \in \hat{\mathrm{V}}^{\mathrm{m}}$ satisfying $\bar{\zeta}([\hat{\mathrm{v}}]_{\underline{\bar{r}}}) = [\hat{\mathrm{v}}'^{\pm}]_{\underline{\bar{r}}'}$, and denote the glued vertex $\mathrm{v} := [\hat{\mathrm{v}}]_{\underline{r}}$. We study equation (8.41) on the strip

$$h'^{+}_{\hat{\mathrm{e}}'}(x'^{+}_{\hat{\mathrm{e}}'}, [0, R'_{\hat{\mathrm{e}}'}] \times [0,\pi]) \subset D_{\hat{\mathrm{v}}'+}. \tag{8.48}$$

By construction (6.18) and the construction of glued maps (4.16), equation (8.41) induces an equation

(8.49)
$$\begin{aligned}\oplus_{r'_{\hat{\mathrm{e}}'}}\big(\hat{u}'^{h'+}_{\hat{\mathrm{e}}'} + \xi'^{h'+}_{\hat{\mathrm{e}}'},\, \hat{u}'^{h'-}_{\hat{\mathrm{e}}'} + \xi'^{h'-}_{\hat{\mathrm{e}}'}\big) &= \varphi'_{\hat{\mathrm{e}}'} \circ \#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{u})_{\mathrm{v}} \circ \Big(\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'+}_{(\underline{r},\underline{x},\underline{o})}\Big)^{-1} \circ h'^{+}_{\hat{\mathrm{e}}'}(x'^{+}_{\hat{\mathrm{e}}'},\cdot),\\ \ominus_{r'_{\hat{\mathrm{e}}'}}(\xi'^{h'+}_{\hat{\mathrm{e}}'},\, \xi'^{h'-}_{\hat{\mathrm{e}}'}) &= 0.\end{aligned}$$

We recall that the constraint on the minus gluing is due to the splicing construction (6.5). This solves the nodal strip map such that $(r'_{\hat{\mathrm{e}}'}, \xi'^{h'+}_{\hat{\mathrm{e}}'}, \xi'^{h'-}_{\hat{\mathrm{e}}'})$ lies in the splicing core. The region in (8.48) avoids all glued marked points $\#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}',\bar{\mathrm{f}}'}$ whose edge $\bar{\mathrm{f}}'$ corresponds to an edge $\hat{\mathrm{f}}'$ with $\bar{r}'_{\hat{\mathrm{f}}'} = 0$. Thus the solution depends continuously on the input for the same reason as in (1).

(4) Suppose we have an edge $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}'^{-}, \hat{\mathrm{v}}'^{+})$ with gluing parameter $\bar{r}'_{\hat{\mathrm{e}}'} = 0$.

The corresponding edge $\hat{\mathrm{e}} = (\hat{\mathrm{v}}^{-}, \hat{\mathrm{v}}^{+})$ with $\bar{\chi}(\hat{\mathrm{e}}) = \hat{\mathrm{e}}'$ also has gluing parameter $\bar{r}_{\hat{\mathrm{e}}} = 0$. Furthermore, we have $r_{\hat{\mathrm{e}}} > 0$ if and only if $r'_{\hat{\mathrm{e}}'} > 0$, in other words, additional gluing edges are in one-to-one correspondence. We now discuss the cases according to the signs of gluing parameters.

i) Suppose we have gluing parameters $r_{\hat{\mathrm{e}}} \le 0$ and $r'_{\hat{\mathrm{e}}'} \le 0$.

We study equation (8.41) on the shrunk strip neighborhood

$$N(x'^{\pm}_{\hat{\mathrm{e}}'}; -k') \subset D_{\hat{\mathrm{v}}'\pm}.$$

By the inclusion in (8.37) and the construction of glued maps (4.16), equation (8.41) induces an equation on the strip $[k', \infty) \times [0, \pi]$

$$\hat{u}'^{h'+}_{\hat{\mathrm{e}}'} + \xi'^{h'+}_{\hat{\mathrm{e}}'} = \varphi'_{\hat{\mathrm{e}}'} \circ \varphi^{-1}_{\hat{\mathrm{e}}} \circ u^{h+}_{\hat{\mathrm{e}}} \circ h^{+}_{\hat{\mathrm{e}}}(x^{+}_{\hat{\mathrm{e}}},\cdot)^{-1} \circ \Big(\psi^{\hat{\mathrm{v}}^{+},\hat{\mathrm{v}}'+}_{(\underline{r},\underline{x},\underline{o})}\Big)^{-1} \circ h'^{+}_{\hat{\mathrm{e}}'}(x'^{+}_{\hat{\mathrm{e}}'},\cdot), \tag{8.50}$$

and an equation on the strip $(-\infty, -k'] \times [0, \pi]$

$$\begin{aligned}&\hat{u}'^{h'-}_{\hat{\mathrm{e}}'} + \xi'^{h'-}_{\hat{\mathrm{e}}'} + \alpha^{-} \cdot \big[\varphi'_{\hat{\mathrm{e}}'}\big(\sigma^{\iota(r'_{\hat{\mathrm{e}}'})}(\varphi'^{-1}_{\hat{\mathrm{e}}'}(c(\xi'^{h'-}_{\hat{\mathrm{e}}'})))\big) - c(\xi'^{h'-}_{\hat{\mathrm{e}}'})\big]\\ &= \varphi'_{\hat{\mathrm{e}}'} \circ \varphi^{-1}_{\hat{\mathrm{e}}} \circ \big\{u^{h-}_{\hat{\mathrm{e}}} + \alpha^{-} \cdot \big[\varphi_{\hat{\mathrm{e}}}\big(\sigma^{\iota(r_{\hat{\mathrm{e}}})}(u^{-}_{\hat{\mathrm{e}}}(x^{-}_{\hat{\mathrm{e}}}))\big) - \varphi_{\hat{\mathrm{e}}}(u^{-}_{\hat{\mathrm{e}}}(x^{-}_{\hat{\mathrm{e}}}))\big]\big\}\\ &\quad \circ h^{-}_{\hat{\mathrm{e}}}(x^{-}_{\hat{\mathrm{e}}},\cdot)^{-1} \circ \Big(\psi^{\hat{\mathrm{v}}^{-},\hat{\mathrm{v}}'-}_{(\underline{r},\underline{x},\underline{o})}\Big)^{-1} \circ h'^{-}_{\hat{\mathrm{e}}'}(x'^{-}_{\hat{\mathrm{e}}'},\cdot).\end{aligned} \tag{8.51}$$

This solves the restricted sections $\xi'^{h'+}_{\hat{\mathrm{e}}'}|_{[k',\infty)\times[0,\pi]}$ and $\xi'^{h'-}_{\hat{\mathrm{e}}'}|_{(-\infty,-k']\times[0,\pi]}$.

ii) Suppose we have gluing parameters $r_{\hat{\mathrm{e}}} > 0$ and $r'_{\hat{\mathrm{e}}'} > 0$.

We study equation (8.41) on the strip

$$h'^{+}_{\hat{\mathrm{e}}'}(x'^{+}_{\hat{\mathrm{e}}'}, [k', R'_{\hat{\mathrm{e}}'} - k'] \times [0,\pi]) \subset D_{\hat{\mathrm{v}}'+}.$$

By the inclusion in (8.36) and the construction of glued maps (4.16), equation (8.41) induces an equation on the glued strip $[k', R'_{\hat{e}'} - k'] \times [0, \pi]$

(8.52)

$$\begin{aligned} \oplus_{r'_{\hat{e}'}} (\hat{u}'^{h'^+}_{\hat{e}'} + \xi'^{h'^+}_{\hat{e}'}, \hat{u}'^{h'^-}_{\hat{e}'} + \xi'^{h'^-}_{\hat{e}'}) &= \varphi'_{\hat{e}'} \circ \varphi_{\hat{e}}^{-1} \circ \oplus_{r_{\hat{e}}} (u^{h^+}_{\hat{e}}, u^{h^-}_{\hat{e}}) \\ &\qquad \circ h^+_{\hat{e}}(x^+_{\hat{e}}, \cdot)^{-1} \circ \left(\psi^{\hat{v}^+, \hat{v}'^+}_{(\underline{r}, \underline{x}, \underline{o})}\right)^{-1} \circ h'^+_{\hat{e}'}(x'^+_{\hat{e}'}, \cdot), \\ \ominus_{r'_{\hat{e}'}} (\xi'^{h'^+}_{\hat{e}'}, \xi'^{h'^-}_{\hat{e}'}) &= 0. \end{aligned}$$

This solves the restricted sections $\xi'^{h'^+}_{\hat{e}'}|_{[k', \infty) \times [0, \pi]}$ and $\xi'^{h'^-}_{\hat{e}'}|_{(-\infty, -k'] \times [0, \pi]}$. In both case i) and ii), we can solve the restriction on the finite strips $[0, k'] \times [0, \pi]$ and $[-k', 0] \times [0, \pi]$ in a similar way as in equation (8.42). The continuity of the solution uses a similar analysis as in the sc-smoothness of our splicing (see Section 8.1) and the continuity of parametrization on the strip (Proposition 13.16). We refer the readers to Section 5.1 of [8] for an analogous proof in the context of cylinder gluing. This finishes proving the theorem.

□

We now prove the sc-smoothness result of the transition maps of charts.

Proof of Theorem 8.14. We proceed in exactly the same way as the proof of Theorem 8.16. Analogous to solving equation (8.38) and proving the continuous dependence in the solution, we shall solve the following equation for Morse trajectories $\underline{\rho}'$ and sections $\underline{\xi}'$

(8.53)
$$\begin{aligned} &\left(\zeta_{\underline{r}}, \underline{\psi}_{(\underline{r}, \underline{x}, \underline{o})}\right) \left(\#\big(\underline{r}, (\hat{\mathrm{T}}, \underline{\gamma}(\underline{\rho}, \underline{\xi}), \underline{x}, \underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))\big)\right) \\ &= \#\big(\underline{r}', (\hat{\mathrm{T}}', \underline{\gamma}'(\underline{\rho}', \underline{\xi}'), \underline{x}', \underline{u}'(\underline{\rho}', \underline{x}', \underline{\xi}'))\big), \end{aligned}$$

and prove the sc-smooth dependence in the solution.

Analogous to equation (8.39), we have

(8.54)
$$\#_{\underline{r}}(\underline{\gamma}(\underline{\rho}, \underline{\xi}))_{\mathrm{e}} = \#_{\underline{r}'}(\underline{\gamma}'(\underline{\rho}', \underline{\xi}'))_{\mathrm{e}'}.$$

The solution is given by $\underline{\rho}'_{\hat{e}'} = \#_{\underline{r}}(\underline{\gamma}(\underline{\rho}, \underline{\xi}))_{\mathrm{e}}$ for each non-nodal edge $\hat{\mathrm{e}}' \in \hat{\mathrm{E}}' \setminus \hat{\mathrm{E}}'^{\mathrm{nd}}$, and it depends sc-smoothly on $\underline{\rho}$ and $\underline{\xi}$ due to the construction of charts (6.12), the displaced Morse trajectories (4.7) and (4.8), and the sc-smoothness of the limit in Proposition 8.1 (1).

We recall that $(\hat{\mathrm{T}}, \underline{\ell}, \underline{x}, \underline{o})$ is the stabilization of the disk tree representative $(\hat{\mathrm{T}}, \underline{\gamma}(\underline{\rho}, \underline{\xi}), \underline{x}, \underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))$. Analogous to equation (8.41), we use the equation in standard model

(8.55)
$$\#^{\hat{\mathrm{v}}}_{\underline{r}, \underline{x}}(\underline{u}(\underline{\rho}, \underline{x}, \underline{\xi}))_{\mathrm{v}} \circ \left(\psi^{\hat{\mathrm{v}}, \hat{\mathrm{v}}'}_{(\underline{r}, \underline{x}, \underline{o})}\right)^{-1} = \#^{\hat{\mathrm{v}}'}_{\underline{r}', \underline{x}'}(\underline{u}'(\underline{\rho}', \underline{x}', \underline{\xi}'))_{\mathrm{v}'}.$$

to find a solution for sections $\underline{\xi}'$ and prove that the solution depends sc-smoothly on the input. Similar to equation (8.42), we have

(8.56)
$$\#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho},\underline{x},\underline{\xi}))_{\mathrm{v}} \circ \left(\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1}\Big|_{D\setminus\bigsqcup N(x'_{\hat{\mathrm{v}}',\hat{\mathrm{e}}'})} = \exp_{\hat{u}'_{\hat{\mathrm{v}}'}} \circ \xi'_{\hat{\mathrm{v}}'} \circ \nu'^{-1}_{\underline{x}'_{\hat{\mathrm{v}}'}}\Big|_{D\setminus\bigsqcup N(x'_{\hat{\mathrm{v}}',\hat{\mathrm{e}}'})}.$$

This solves the section $\xi'_{\hat{\mathrm{v}}'}$ on $D \setminus \bigsqcup N(\hat{x}'_{\hat{\mathrm{v}}',\hat{\mathrm{e}}'})$. The region $D \setminus \bigsqcup N(x'_{\hat{\mathrm{v}}',\hat{\mathrm{e}}'})$ avoids all glued marked points $\#^{\hat{\mathrm{v}}'}_{\underline{\bar{r}}'}(\underline{\bar{x}}')_{\bar{\mathrm{v}}',\bar{\mathrm{f}}'}$ whose edge $\bar{\mathrm{f}}'$ corresponds to an edge $\hat{\mathrm{f}}'$ with $\bar{r}'_{\hat{\mathrm{f}}'} = 0$. Thus the solution depends sc-smoothly on the input, due to the sc-smoothness of the glued map $\#^{\hat{\mathrm{v}}}_{\underline{r},\underline{x}}(\underline{u}(\underline{\rho},\underline{x},\underline{\xi}))_{\mathrm{v}}$ (Lemma 8.10), the sc-smooth dependence in the disk automorphism $\left(\psi^{\hat{\mathrm{v}},\hat{\mathrm{v}}'}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1}$ (Theorem 7.22, Proposition 13.15, and Proposition 13.13), and the sc-smoothness of $\mathrm{Aut}(D)$ action (Proposition 6.4). Similarly, we can prove the sc-smooth dependence of the solution sections $\underline{\xi}'$ restricted to strip neighborhoods $N(\hat{x}'^{\pm}_{\hat{\mathrm{e}}'})$ in cases (1) - (3) of the proof of Theorem 8.16. We now consider case (4).

(4) Suppose we have an edge $\hat{\mathrm{e}}' = (\hat{\mathrm{v}}'^{-}, \hat{\mathrm{v}}'^{+})$ with gluing parameter $\bar{r}'_{\hat{\mathrm{e}}'} = 0$.
i) Suppose we have gluing parameters $r_{\hat{\mathrm{e}}} \le 0$ and $r'_{\hat{\mathrm{e}}'} \le 0$.
We study equation (8.55) on the strip

$$N(x'^{\pm}_{\hat{\mathrm{e}}'}; -k') \subset D_{\hat{\mathrm{v}}'^{\pm}}.$$

where $k' > 0$ is chosen so that the inclusion in (8.37) holds. Analogous to equation (8.50) and (8.51) in the proof of Theorem 8.16, we have an equation on the strip $[k', \infty) \times [0, \pi]$

$$\begin{aligned} \hat{u}'^{h'^+}_{\hat{\mathrm{e}}'} + \xi'^{h'^+}_{\hat{\mathrm{e}}'} = \varphi'_{\hat{\mathrm{e}}'} \circ \varphi^{-1}_{\hat{\mathrm{e}}} \circ \big(\hat{u}^{h^+}_{\hat{\mathrm{e}}} + \xi^{h^+}_{\hat{\mathrm{e}}}\big) \\ \circ\, h^{+}_{\hat{\mathrm{e}}}(x^{+}_{\hat{\mathrm{e}}}, \cdot)^{-1} \circ \left(\psi^{\hat{\mathrm{v}}^+,\hat{\mathrm{v}}'^+}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1} \circ h'^{+}_{\hat{\mathrm{e}}'}(x'^{+}_{\hat{\mathrm{e}}'}, \cdot), \end{aligned} \tag{8.57}$$

and an equation on the strip $(-\infty, -k'] \times [0, \pi]$

$$\begin{aligned} &\hat{u}'^{h'^-}_{\hat{\mathrm{e}}'} + \xi'^{h'^-}_{\hat{\mathrm{e}}'} + \alpha^- \cdot \left[\varphi'_{\hat{\mathrm{e}}'}\big(\sigma^{\iota(r'_{\hat{\mathrm{e}}'})}(\varphi'^{-1}_{\hat{\mathrm{e}}'}(c(\xi'^{h'^-}_{\hat{\mathrm{e}}'})))\big) - c(\xi'^{h'^-}_{\hat{\mathrm{e}}'})\right] \\ &= \varphi'_{\hat{\mathrm{e}}'} \circ \varphi^{-1}_{\hat{\mathrm{e}}} \circ \left\{\hat{u}^{h^-}_{\hat{\mathrm{e}}} + \xi^{h^-}_{\hat{\mathrm{e}}} + \alpha^- \cdot \left[\varphi_{\hat{\mathrm{e}}}\big(\sigma^{\iota(r_{\hat{\mathrm{e}}})}(\varphi^{-1}_{\hat{\mathrm{e}}}(c(\xi^{h^-}_{\hat{\mathrm{e}}})))\big) - c(\xi^{h^-}_{\hat{\mathrm{e}}})\right]\right\} \\ &\quad \circ h^{-}_{\hat{\mathrm{e}}}(x^{-}_{\hat{\mathrm{e}}}, \cdot)^{-1} \circ \left(\psi^{\hat{\mathrm{v}}^-,\hat{\mathrm{v}}'^-}_{(\underline{r},\underline{x},\underline{o})}\right)^{-1} \circ h'^{-}_{\hat{\mathrm{e}}'}(x'^{-}_{\hat{\mathrm{e}}'}, \cdot). \end{aligned} \tag{8.58}$$

This solves the restricted sections $\xi'^{h'^+}_{\hat{\mathrm{e}}'}|_{[k',\infty)\times[0,\pi]}$ and $\xi'^{h'^-}_{\hat{\mathrm{e}}'}|_{(-\infty,-k']\times[0,\pi]}$.
ii) Suppose we have gluing parameters $r_{\hat{\mathrm{e}}} > 0$ and $r'_{\hat{\mathrm{e}}'} > 0$.
We study equation (8.55) on the strip

$$h'^{+}_{\hat{\mathrm{e}}'}(x'^{+}_{\hat{\mathrm{e}}'}, [k', R'_{\hat{\mathrm{e}}'} - k'] \times [0, \pi]) \subset D_{\hat{\mathrm{v}}'^{+}},$$

where $k' > 0$ is chosen so that the inclusion in (8.36) holds. Analogous to equation (8.52), we have an equation on the glued strip $[k', R'_{\hat{e}'} - k'] \times [0, \pi]$

(8.59)

$$\begin{aligned}\oplus_{r'_{\hat{e}'}} \big(\hat{u}'^{h'^+}_{\hat{e}'} + \xi'^{h'^+}_{\hat{e}'},\, \hat{u}'^{h'^-}_{\hat{e}'} + \xi'^{h'^-}_{\hat{e}'}\big) &= \varphi'_{\hat{e}'} \circ \varphi^{-1}_{\hat{e}} \circ \oplus_{r_{\hat{e}}} \big(\hat{u}^{h^+}_{\hat{e}} + \xi^{h^+}_{\hat{e}},\, \hat{u}^{h^-}_{\hat{e}} + \xi^{h^-}_{\hat{e}}\big) \\ &\qquad \circ h^+_{\hat{e}}(x^+_{\hat{e}}, \cdot)^{-1} \circ \Big(\psi^{\hat{v}^+, \hat{v}'^+}_{(\underline{r}, \underline{x}, \underline{o})}\Big)^{-1} \circ h'^+_{\hat{e}'}(x'^+_{\hat{e}'}, \cdot), \\ \ominus_{r'_{\hat{e}'}} (\xi'^{h'^+}_{\hat{e}'},\, \xi'^{h'^-}_{\hat{e}'}) &= 0.\end{aligned}$$

This solves the restricted sections $\xi'^{h'^+}_{\hat{e}'}|_{[k', \infty) \times [0,\pi]}$ and $\xi'^{h'^-}_{\hat{e}'}|_{(-\infty, -k'] \times [0,\pi]}$.

By using a similar analysis as in the sc-smoothness of the splicing in Section 8.1, the sc-smoothness of parametrization on the strip (Proposition 13.16), and the choice of the smooth function ι (specified before Remark 4.2) with derivatives $\iota^{(k)}(0) = 0$, we can show that the map

$$\big(r_{\hat{e}}, \xi^{h^+}_{\hat{e}},\, \xi^{h^-}_{\hat{e}}\big) \mapsto \big(r'_{\hat{e}'}, \xi'^{h'^+}_{\hat{e}'}|_{[k', \infty) \times [0,\pi]},\, \xi'^{h'^-}_{\hat{e}'}|_{(-\infty, -k'] \times [0,\pi]}\big)$$

is sc-smooth and has vanishing $r_{\hat{e}}$ partial derivatives at $r_{\hat{e}} = 0$. We refer the readers to Section 5.1 of [8] for an analogous proof in the context of cylinder gluing. This finishes proving the theorem. □

9. $\bar{\partial}_J$ the Cauchy-Riemann Section

Ultimately, we shall produce an A_∞ algebra from counting the zero set of the perturbed Cauchy-Riemann section. In order to apply the polyfold perturbation result, here we give the bundle of complex anti-linear sections an M-polyfold strong bundle structure and show that the Cauchy-Riemann section is sc-Fredholm. We refer the readers to [7] and [8] for a detailed introduction to strong bundle and sc-Fredholm section.

9.1. $\mathfrak{Y}$ the Bundle of Complex Anti-linear Sections.

In this section, we study the bundle of complex anti-linear sections over the quotient space of disk trees and give it an M-polyfold strong bundle structure.

We define $\pi : \mathfrak{Y} \to \mathfrak{X}$ the **bundle of complex anti-linear sections** to be

$$\mathfrak{Y} := \{(\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}, \underline{\lambda}) \,|\, [\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}] \in \mathfrak{X}\}/_{\sim_{\mathrm{bihol}}}, \tag{9.1}$$

where the fiber consists of tuples $\underline{\lambda} = (\lambda_{\mathrm{v}})_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}}$ such that for each main vertex $\mathrm{v} \in \mathrm{V}^m$ and point $z \in D$,

$$\lambda_{\mathrm{v}}(z) : (T_z D, i) \to (T_{u_{\mathrm{v}}(z)} M, J(u_{\mathrm{v}}(z)))$$

is a complex anti-linear map. Moreover, we require each λ_{v} to have H^{2,δ_0} regularity (see Definition 9.1).

From now on, we shall always identify λ_{v} with the map applying to the first basis element

$$\lambda_{\mathrm{v}}\, \partial_s$$

in the (s,t) coordinates. Note that the anti-linearity determines the value on the other basis $\lambda_{\mathrm{v}}\,\partial_t = -J(u_{\mathrm{v}})\lambda_{\mathrm{v}}\,\partial_s$.

We now make the H^{2,δ_0} regularity precise. Fix $m \geq 2$ and weight $\delta \in (0,1)$, and fix boundary marked points $\underline{x} = (x_0, \ldots, x_k) \in \underline{MP}(\partial D)$ and disk map $u \in \mathrm{Map}^{m,\delta}((D,\underline{x}), M; L)$ (see Definition 3.12). Let $u^*TM \to D$ be the pullback bundle.

Definition 9.1. Choose C^∞ charts $\underline{\varphi}$ and strip coordinates $\underline{h}$ as in Definition 3.12. We define $\Lambda^{m,\delta}((D,\underline{x}), u^*TM)$ the Banach space of $\boldsymbol{H^{m,\delta}}$ **complex anti-linear sections of $\boldsymbol{u^*TM}$ with marked points $\underline{\boldsymbol{x}}$** to be the set of $\lambda : D \to u^*TM$ with the following properties.

- The restriction $\lambda|_{D\setminus\bigsqcup_i N(x_i;-1)}$ belongs to $H^m(D \setminus \bigsqcup_i N(x_i;-1))$, where each $N(x_i;-1)$ is the shrunk strip neighborhood.
- On the strip neighborhood $N(x_i)$, the local expression in strip coordinates
$$\lambda_i^{h^\pm}(z) := D\varphi_i(u(h_i^\pm(z))(\lambda(h_i^\pm(z))$$
on $\mathbb{R}^\pm\times[0,\pi]$ belongs to the weighted Sobolev space $\lambda_i^{h^\pm} \in H^{m,\delta}(\mathbb{R}^\pm\times[0,\pi],\mathbb{C}^n)$.

Choose a Riemannian metric g on M, we define a norm
$$\|\lambda\|^2 := \big\|\lambda|_{D\setminus\bigsqcup_i N(x_i;-1)}\big\|^2_{H^m} + \sum_{i=0}^k \big\|\lambda_i^{h^\pm}\big\|^2_{H^{m,\delta}}.$$

Remark 9.2. This space differs from the space of $H^{m,\delta}$ sections (Definition 3.16) in two aspects. Here we require $m \geq 2$ instead of $m \geq 3$. Furthermore, the local expression $\lambda_i^{h^\pm}$ lies in the space $H^{m,\delta}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n)$ instead of $H^{m,\delta}_{\lim}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$. In other words, $\lambda_i^{h^\pm}$ does not need to satisfy the boundary condition on $\mathbb{R}^n$, but its limit at $\pm\infty$ has to be 0.

We now define the **Cauchy-Riemann section** by

$$\bar\partial_J(u_{\mathrm{v}}) = \frac{1}{2}(\partial_s u_{\mathrm{v}} + J(u_{\mathrm{v}})\,\partial_t u_{\mathrm{v}}). \tag{9.2}$$

We also call the section it induces on the bundle $\bar\partial_J$. Later on we shall show that $\bar\partial_J : \mathfrak{X} \to \mathfrak{Y}$ is an sc-smooth sc-Fredholm section (Definition 1.4 of [7]). But first we establish an M-polyfold strong bundle structure (Definition 1.33 of [7]) on the bundle $\pi : \mathfrak{Y} \to \mathfrak{X}$ by constructing an sc-smooth splicing using the hat gluing (see Section 2.4 of [8]).

Similar to the space of strip nodal maps in Definition 6.7, we define
$$\widehat{F}^{\mathrm{nd}} := H^{2,\delta_0}(\mathbb{R}^+ \times [0,\pi], \mathbb{C}^n) \oplus H^{2,\delta_0}(\mathbb{R}^- \times [0,\pi], \mathbb{C}^n).$$

We shall define a splicing $(r, \lambda^+, \lambda^-) \mapsto \widehat{\pi}_r(\lambda^+, \lambda^-)$ for $r \in (-\varepsilon, \varepsilon)$ and $(\lambda^+, \lambda^-) \in \widehat{F}^{\mathrm{nd}}$. Recall the smooth cut-off function $\beta : \mathbb{R} \to [0,1]$ defined before Remark 4.2 with properties

- $\beta(s) = 1$ for $s \leq -1$ and $\beta(s) = 0$ for $s \geq 1$,
- $\beta'(s) < 0$ for $s \in (-1,1)$,

- $\beta(s) + \beta(-s) = 1$ for all s.

We denote the shifted cut-off function by

$$\beta_{s_0}(s) := \beta(s - s_0).$$

For $r \in (0, \varepsilon)$, we abbreviate $R(r)$ as R. We define the **hat plus gluing** by

$$\widehat{\oplus}_r(\lambda^+, \lambda^-) = \beta_{R/2}\, \lambda^+ + (1 - \beta_{R/2})\, \lambda^-(\cdot - R), \tag{9.3}$$

and the **hat minus gluing** by

$$\widehat{\ominus}_r(\lambda^+, \lambda^-) = -(1 - \beta_{R/2})\, \lambda^+ + \beta_{R/2}\, \lambda^-(\cdot - R). \tag{9.4}$$

Thus the **hat total gluing** $\widehat{\boxdot}_r = (\widehat{\oplus}_r, \widehat{\ominus}_r)$ maps from $\widehat{F}^{\mathrm{nd}}$ to the direct sum

$$\widehat{G}_r := H^2([0, R] \times [0, \pi], \mathbb{C}^n) \oplus H^{2,\delta_0}(\mathbb{R} \times [0, \pi], \mathbb{C}^n).$$

Similar to Theorem 2.42 (1) of [8], the following result shows that the hat total gluing is a linear sc-isomorphism.

Lemma 9.3. *Let R be any gluing profile. For $r \in (0, \varepsilon)$, the hat total gluing*

$$\widehat{\boxdot}_r : \widehat{F}^{\mathrm{nd}} \to \widehat{G}_r$$

is a linear sc-isomorphism.

This is the corresponding result to the total gluing in Theorem 6.8: one simply need to solve a linear equation to show that $\widehat{\boxdot}_r$ is an sc-isomorphism.

For $r \in (0, \varepsilon)$, we define $\widehat{\pi}_r : \widehat{F}^{\mathrm{nd}} \to \widehat{F}^{\mathrm{nd}}$ to be the linear projection onto $\ker(\widehat{\ominus}_r)$ along $\ker(\widehat{\oplus}_r)$. More precisely, we define

$$\widehat{\pi}_r(\lambda^+, \lambda^-) = (\zeta^+, \zeta^-), \tag{9.5}$$

with (ζ^+, ζ^-) satisfying

$$\widehat{\boxdot}_r(\zeta^+, \zeta^-) = (\widehat{\oplus}_r(\lambda^+, \lambda^-), 0).$$

Similar to Theorem 2.42 (2) of [8], the following result shows that the family of projections $\widehat{\pi}_r$ extends smoothly to $r \in (-\varepsilon, 0]$ as the identity map.

Theorem 9.4. *Let the gluing profile R be given by the exponential gluing profile $\varphi(r) = e^{\frac{1}{r}} - e$. Let $\widehat{\pi}_r : \widehat{F}^{\mathrm{nd}} \to \widehat{F}^{\mathrm{nd}}$ be the linear projection defined in (9.5) for $r \in (0, \varepsilon)$, and $\widehat{\pi}_r = \mathrm{Id}$ for $r \in (-\varepsilon, 0]$, then the map $(r, \lambda^+, \lambda^-) \mapsto \widehat{\pi}_r(\lambda^+, \lambda^-)$ is an sc-smooth splicing.*

Proof. Similar to formula (8.5) and (8.6), the splicing (9.5) is given by

$$\zeta^+ = \frac{\beta_{R/2}^2}{\gamma_{R/2}}\lambda^+ + \frac{\beta_{R/2}(1 - \beta_{R/2})}{\gamma_{R/2}}\lambda^-(\cdot - R),$$

$$\zeta^- = \frac{\beta_{-R/2}(1 - \beta_{-R/2})}{\gamma_{-R/2}}\lambda^+(\cdot + R) + \frac{(1 - \beta_{-R/2})^2}{\gamma_{-R/2}}\lambda^-.$$

The above maps are sc-smooth by corresponding sc-smoothness results of Proposition 8.1 (3) and (4) for H^{2,δ_0} maps. $\square$

For the bundle of complex anti-linear sections $\mathfrak{Y}$, we can construct a splicing analogous to (6.10) and an atlas analogous to (6.12), and give the bundle an M-polyfold strong bundle structure (see Definition 1.33 of [7], also see Section 3.4 of [8] for the polyfold strong bundle structure in the construction of Gromov-Witten invariants).

Proposition 9.5. *The bundle of complex anti-linear sections $\pi : \mathfrak{Y} \to \mathfrak{X}$ is an M-polyfold strong bundle.*

9.2. **Sc-smoothness of $\bar{\partial}_J$ the Cauchy-Riemann Section.**

We now show the sc-smoothness property of the $\bar{\partial}_J$ section. We refer the readers to Proposition 4.7 of [8] for the sc-smoothness of the Cauchy-Riemann section in the construction of Gromov-Witten invariants. Here we modify the proof to incorporate the map displacement defined in (4.15).

Theorem 9.6. *The Cauchy-Riemann section $\bar{\partial}_J : \mathfrak{X} \to \mathfrak{Y}$ is sc-smooth.*

Proof. Away from a nodal strip map, $\bar{\partial}_J$ section is classically smooth on each level, and hence sc-smooth. Here we prove the sc-smoothness near a nodal strip map $(\hat{u}^+, \hat{u}^-)$ which belongs to

$$\hat{u}^\pm \in H^{3+m,\delta_m}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n)$$

for all $m \geq 0$.

We first recall that the maps near the nodal strip map $(\hat{u}^+, \hat{u}^-)$ in the image of the chart (6.12) are of the following form. For gluing parameter $r > 0$ and strip nodal map $(\xi^+, \xi^-) \in F^{\mathrm{nd}}$, we have the plus gluing

$$\oplus_r(\hat{u}^+ + \xi^+, \hat{u}^- + \xi^-) = \beta_{R/2}\,(\hat{u}^+ + \xi^+) + (1 - \beta_{R/2})\,(\hat{u}^- + \xi^-).$$

For gluing parameter $r \leq 0$ and strip nodal map $(\xi^+, \xi^-) \in F^{\mathrm{nd}}$, we have the displaced pair, which we denote

$$\begin{aligned} &\oplus_r^+(\hat{u}^+ + \xi^+, \hat{u}^- + \xi^-) := \hat{u}^+ + \xi^+, \\ &\oplus_r^-(\hat{u}^+ + \xi^+, \hat{u}^- + \xi^-) := \hat{u}^- + \xi^- + \alpha^- \cdot [\sigma^{\iota(r)}(c(\xi^-)) - c(\xi^-)]. \end{aligned} \tag{9.6}$$

We recall that $\sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the Morse flow (4.3) in local coordinates, $\iota : (-\infty, 0] \to (-\infty, 0]$ is a smooth increasing function chosen before Remark 4.2, and $c(\xi^-)$ is the limit of ξ^- at $-\infty$.

We now derive a formula for the $\bar{\partial}_J$ section in the base coordinates (r, ξ^+, ξ^-) and fiber coordinates (λ^+, λ^-) given by the splicing in Theorem 9.4. We use a parallel transport to construct a smooth map Γ on a neighborhood of the diagonal of the target manifold M, such that for each point p and q closeby, Γ is a complex linear isomorphism

$$\Gamma(p, q) : (T_qM, J(q)) \to (T_pM, J(p)).$$

We refer the reader to Section 3.4 of [8] for the details of this construction. Then $\bar{\partial}_J$ in base coordinates $(r, \underline{\xi})$ is given by

$$f(r, \underline{\xi}) = \underline{\lambda} \tag{9.7}$$

as follows, where we abbreviate $\underline{\xi} := (\xi^+, \xi^-)$ and $\underline{\lambda} := (\lambda^+, \lambda^-)$.

For $r > 0$, the complex anti-linear sections $\underline{\lambda}$ are defined by equation

(9.8)
$$\Gamma(\oplus_r(\underline{\hat{u}}+\underline{\xi}), \oplus_r(\underline{\hat{u}}))\,\widehat{\oplus}_r(\underline{\lambda}) = \frac{1}{2}[\partial_s(\oplus_r(\underline{\hat{u}}+\underline{\xi})) + J(\oplus_r(\underline{\hat{u}}+\underline{\xi}))\,\partial_t(\oplus_r(\underline{\hat{u}}+\underline{\xi}))],$$
$$\widehat{\ominus}_r(\underline{\lambda}) = 0.$$

For $r \leq 0$, the complex anti-linear sections $\underline{\lambda}$ are defined by equation

(9.9)
$$\Gamma(\oplus_r^+(\underline{\hat{u}}+\underline{\xi}), \oplus_r^+(\underline{\hat{u}}))\,\lambda^+ = \frac{1}{2}[\partial_s(\oplus_r^+(\underline{\hat{u}}+\underline{\xi})) + J(\oplus_r^+(\underline{\hat{u}}+\underline{\xi}))\,\partial_t(\oplus_r^+(\underline{\hat{u}}+\underline{\xi}))],$$
$$\Gamma(\oplus_r^-(\underline{\hat{u}}+\underline{\xi}), \oplus_r^-(\underline{\hat{u}}))\,\lambda^- = \frac{1}{2}[\partial_s(\oplus_r^-(\underline{\hat{u}}+\underline{\xi})) + J(\oplus_r^-(\underline{\hat{u}}+\underline{\xi}))\,\partial_t(\oplus_r^-(\underline{\hat{u}}+\underline{\xi}))].$$

For convenience, we introduce the following notations. Let $A, B : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C}^n)$ be families of $\mathbb{R}$-linear maps derived from Γ and J

$$A(p,q) = \frac{1}{2}\Gamma(p,q)^{-1}, \quad B(p,q) = \frac{1}{2}\Gamma(p,q)^{-1}J(p).$$

Then $\bar{\partial}_J$ in base coordinates $f(r,\underline{\xi})$ can be re-written as follows.

For $r > 0$, the map $f(r,\underline{\xi})$ is given by

(9.10)
$$\widehat{\boxdot}_r^{-1}\begin{bmatrix} A(\oplus_r(\underline{\hat{u}}+\underline{\xi}), \oplus_r(\underline{\hat{u}}))\,\partial_s(\oplus_r(\underline{\hat{u}}+\underline{\xi})) + B(\oplus_r(\underline{\hat{u}}+\underline{\xi}), \oplus_r(\underline{\hat{u}}))\,\partial_t(\oplus_r(\underline{\hat{u}}+\underline{\xi})) \\ 0 \end{bmatrix}.$$

For $r \leq 0$, the map $f(r,\underline{\xi})$ is given by

(9.11)
$$\begin{bmatrix} A(\oplus_r^+(\underline{\hat{u}}+\underline{\xi}), \oplus_r^+(\underline{\hat{u}}))\,\partial_s(\oplus_r^+(\underline{\hat{u}}+\underline{\xi})) + B(\oplus_r^+(\underline{\hat{u}}+\underline{\xi}), \oplus_r^+(\underline{\hat{u}}))\,\partial_t(\oplus_r^+(\underline{\hat{u}}+\underline{\xi})) \\ A(\oplus_r^-(\underline{\hat{u}}+\underline{\xi}), \oplus_r^-(\underline{\hat{u}}))\,\partial_s(\oplus_r^-(\underline{\hat{u}}+\underline{\xi})) + B(\oplus_r^-(\underline{\hat{u}}+\underline{\xi}), \oplus_r^-(\underline{\hat{u}}))\,\partial_t(\oplus_r^-(\underline{\hat{u}}+\underline{\xi})) \end{bmatrix}.$$

We now decompose the map $f(r,\underline{\xi})$ as follows.

$$f(r,\underline{\xi}) = K(r,\underline{\xi})\,D_s^r(\underline{\hat{u}}+\underline{\xi}) + L(r,\underline{\xi})\,D_t^r(\underline{\hat{u}}+\underline{\xi}). \tag{9.12}$$

We define

(9.13)
$$D_s^r(\underline{\xi}) := \begin{cases} \widehat{\boxdot}_r^{-1}\begin{bmatrix} \partial_s(\oplus_r(\underline{\xi})) \\ 0 \end{bmatrix}, & r > 0, \\ \begin{bmatrix} \partial_s(\oplus_r^+(\underline{\xi})) \\ \partial_s(\oplus_r^-(\underline{\xi})) \end{bmatrix}, & r \leq 0. \end{cases} \quad D_t^r(\underline{\xi}) := \begin{cases} \widehat{\boxdot}_r^{-1}\begin{bmatrix} \partial_t(\oplus_r(\underline{\xi})) \\ 0 \end{bmatrix}, & r > 0, \\ \begin{bmatrix} \partial_t(\oplus_r^+(\underline{\xi})) \\ \partial_t(\oplus_r^-(\underline{\xi})) \end{bmatrix}, & r \leq 0. \end{cases}$$

Moreover, define an operator $K(r,\underline{\xi}) \in \mathcal{L}_{\mathbb{R}}(\mathbb{C}^{2n}, \mathbb{C}^{2n})$ by

(9.14)
$$K(r,\underline{\xi}) := \begin{cases} \widehat{\boxdot}_r^{-1}\begin{bmatrix} A(\oplus_r(\underline{\hat{u}}+\underline{\xi}), \oplus_r(\underline{\hat{u}})) & 0 \\ 0 & 0 \end{bmatrix}\widehat{\boxdot}_r, & r > 0, \\ \begin{bmatrix} A(\oplus_r^+(\underline{\hat{u}}+\underline{\xi}), \oplus_r^+(\underline{\hat{u}})) & 0 \\ 0 & A(\oplus_r^-(\underline{\hat{u}}+\underline{\xi}), \oplus_r^-(\underline{\hat{u}})) \end{bmatrix}, & r \leq 0. \end{cases}$$

Similarly, define an operator $L(r,\underline{\xi}) \in \mathcal{L}_{\mathbb{R}}(\mathbb{C}^{2n},\mathbb{C}^{2n})$ by replacing the A's by B in $K(r,\underline{\xi})$.

Writing $f(r,\underline{\xi})$ this way helps isolate the difficulty of the problem: $K(r,\underline{\xi})$ and $L(r,\underline{\xi})$ capture the subtlety of post-composing J, and $D_s^r(\underline{\hat{u}}+\underline{\xi})$ and $D_t^r(\underline{\hat{u}}+\underline{\xi})$ capture the subtlety of taking s and t derivatives.

We now prove that the map

$$(-\varepsilon,\varepsilon)\times F^{\mathrm{nd}} \to \widehat{F}^{\mathrm{nd}}, \quad (r,\underline{\xi}) \mapsto D_s^r(\underline{\xi})$$

is sc-smooth. For $r>0$, we have

$$\begin{aligned} & D_s^r(\underline{\xi}) \\ = & \widehat{\boxdot}_r^{-1}(\partial_s \widehat{\oplus}_r(\underline{\xi}),0) \\ = & \widehat{\boxdot}_r^{-1}(\beta'_{R/2}[\xi^+ - \xi^-(\cdot - R)],0) + \widehat{\boxdot}_r^{-1}(\widehat{\oplus}_r(\partial_s\underline{\xi}),0). \end{aligned}$$

The second equality follows from the product rule. By using a similar result to Proposition 8.2, one can show that the map defined by the first term

$$(r,\underline{\xi}) \mapsto \widehat{\boxdot}_r^{-1}(\beta'_{R/2}[\xi^+ - \xi^-(\cdot - R)],0)$$

extends to $r=0$ sc-smoothly by $(0,\underline{\xi}) \mapsto 0$. (For a more detailed proof, we refer the readers to (93) of [8]). Now by the definition of the hat splicing (9.5), the second term is $\widehat{\boxdot}_r^{-1}(\widehat{\oplus}_r(\partial_s\underline{\xi}),0) = \widehat{\pi}_r(\partial_s\underline{\xi})$. By Theorem 9.4, the map $(r,\underline{\xi}) \mapsto \widehat{\pi}_r(\partial_s\underline{\xi}) - \partial_s\underline{\xi}$ has vanishing derivatives at $r=0$. Thus the same is true for the map $(r,\underline{\xi}) \mapsto D_s^r(\underline{\xi}) - \partial_s\underline{\xi}$ when taking derivatives on $[0,\varepsilon)$. On the other hand, for $r \in (-\varepsilon, 0]$, by (9.6) we have

$$D_s^r(\underline{\xi}) - \partial_s\underline{\xi} = (0, \partial_s\alpha^- \cdot [\sigma^{\iota(r)}(c(\xi^-)) - c(\xi^-)]),$$

which has vanishing derivatives at $r=0$ due to $\iota^{(k)}(0)=0$ by choice. This finishes proving the sc-smoothness of $D_s^r(\underline{\xi})$.

By a similar analysis, we can show the sc-smoothness of $D_t^r(\underline{\xi})$. The sc-smoothness of $K(r,\underline{\xi})$ applying to $D_s^r(\underline{\xi})$ and $L(r,\underline{\xi})$ applying to $D_t^r(\underline{\xi})$ follows from Proposition 4.12 of [8] and an analogous analysis as above when $r \le 0$. $\square$

Similar to Theorem 4.17 of [8], the Cauchy-Riemann section is regularizing (Definition 1.34 of [7]).

Proposition 9.7. *The section $\bar{\partial}_J : \mathfrak{X} \to \mathfrak{Y}$ is regularizing.*

The regularizing property follows from the elliptic regularity for the J-holomorphic equation with totally real boundary condition, along with the estimates of the constant Cauchy-Riemann operator of the following proposition. This is an analogous result of Proposition 4.15 of [8].

Proposition 9.8. *Let J_0 be a constant almost-complex structure. The constant Cauchy-Riemann operator*

$$\bar{\partial}_{J_0} : H^{3,\delta_0}_{\text{op-lim}}(\mathbb{R}\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n) \to H^{2,\delta_0}(\mathbb{R}\times[0,\pi],\mathbb{C}^n;\mathbb{R}^n)$$

is an sc-isomorphism.

9.3. **Filled Section of $\bar{\partial}_J$ and the Polyfold Fredholm Property.**

In this section, we construct a filled section (Definition 1.35 of [7]) of the $\bar{\partial}_J$ section, and by using the analysis established in Section 4.5 of [8] we show that the $\bar{\partial}_J$ section is sc-Fredholm (Definition 1.40 of [7]). Roughly speaking, an sc-Fredholm section must be sc-smooth, regularizing (Definition 1.34 [7]), and has a sc-smooth filled section with certain contraction property.

Analogous to Section 4.5 of [8], the $\bar{\partial}_J$ section has a naturally defined filled section $\bar{f}(r, \underline{\xi})$. The filled section $\bar{f}(r, \underline{\xi}) = \underline{\lambda}$ is given as follows. For $r > 0$, the complex anti-linear sections $\underline{\lambda}$ are defined by equation

(9.15)

$$\begin{aligned}\Gamma(\oplus_r(\underline{\hat{u}}+\underline{\xi}), \oplus_r(\underline{\hat{u}}))\,\widehat{\oplus}_r(\underline{\lambda}) &= \frac{1}{2}[\partial_s(\oplus_r(\underline{\hat{u}}+\underline{\xi})) + J(\oplus_r(\underline{\hat{u}}+\underline{\xi}))\,\partial_t(\oplus_r(\underline{\hat{u}}+\underline{\xi}))],\\ \widehat{\ominus}_r(\underline{\lambda}) &= \bar{\partial}_{J(0)}(\ominus_r(\underline{\xi})).\end{aligned}$$

For $r \leq 0$, the complex anti-linear sections $\underline{\lambda}$ are defined by equation (9.9)

We now show the sc-smoothness of the filled section. This is the analogous result of Proposition 4.21 of [8].

Proposition 9.9. *The filled section of the $\bar{\partial}_J$ section*

$$(-\varepsilon, \varepsilon) \times F^{\mathrm{nd}} \to \widehat{F}^{\mathrm{nd}}, \quad (r, \underline{\xi}) \mapsto \bar{f}(r, \underline{\xi})$$

is sc-smooth.

Proof. Note that the filled section can be written as $\bar{f}(r, \underline{\xi}) = f(r, \underline{\xi}) + g(r, \underline{\xi})$, where $f(r, \underline{\xi})$ is given by (9.7), and $g(r, \underline{\xi})$ is defined as follows.

$$g(r, \underline{\xi}) := \begin{cases} \widehat{\boxdot}_r^{-1}(0, \bar{\partial}_{J(0)}(\ominus_r(\underline{\xi}))), & r > 0, \\ (0, 0), & r \leq 0. \end{cases}$$

This section does not involve map displacement as in (9.6). Its sc-smoothness follows from an analogous result of Proposition 4.11 of [8]. ☐

Combining the sc-smooth result (Theorem 9.6, Proposition 9.9), the regularizing property (9.7), and the sc^0 contraction germ associated to the filled section (Proposition 4.26 of [8]), we have the following sc-Fredholm result.

Theorem 9.10. *The Cauchy Riemann section $\bar{\partial}_J : \mathfrak{X} \to \mathfrak{Y}$ is an sc-Fredholm section.*

10. Transversality with Boundary and Corners in M-polyfolds

For readers who are not familiar with the general polyfold theory, we summarize several essential transversality notions for M-polyfolds with boundary and corners. Later in the paper, we will point out which transversality conditions we can achieve in the M-polyfold of disk trees. For a complete exposition, see [9].

We define the degeneracy index by passing to the local model (see (6.1)).

Definition 10.1. For an M-polyfold $\mathfrak{X}$ and a point $x \in \mathfrak{X}$, take an sc-smooth chart $(\mathfrak{U}, \phi, (O, C, E))$ around x. Define

$$d(x) := d_C(\phi^{-1}(x)).$$

We call $d : \mathfrak{X} \to \mathbb{N}_0$ the **degeneracy index**.

The notion of degeneracy index is well-defined, in that it is independent of the choice of sc-smooth chart around x. We can characterize the boundary by the degeneracy index (Definition 2.25 of [9]).

Definition 10.2. We define the **boundary of** $\mathfrak{X}$ to be the subset $\partial\mathfrak{X} := \{x \in \mathfrak{X} \,|\, d(x) \geq 1\}$.

At this point we recapitulate the concept of a *tame* M-polyfold, which has a "controlled" boundary and corners structure. (See Definition 2.37 of [9].)

Let C be a partial quadrant in an sc-Banach space E, and (O, C, E) an sc-smooth retract. Without loss of generality we can assume $C = [0, \infty)^k \times W$ and $E = \mathbb{R}^k \oplus W$. Let $x = (a_1, \ldots, a_k, w)$ be a point in O. Define the subspace E_x by

$$E_x := \{(b_1, \ldots, b_k, w) \in \mathbb{R}^k \oplus W \,|\, b_i = 0 \text{ if } a_i = 0\}.$$

Definition 10.3. An sc-smooth retraction $r : U \to U$ with image $O = r(U)$ is **tame** if

(1) $d_C(r(x)) = d_C(x)$ for $x \in U$, and
(2) for every smooth point $x \in O$ there exists an sc-subspace $A \subset E$, such that $E = T_xO \oplus A$ with $A \subset E_x$.

Moreover, a **tame M-polyfold** is an M-polyfold with an sc-smooth atlas whose charts are modeled on tame sc-smooth retracts.

Example 10.4. Let $E = \mathbb{R}^2$, then $E_{(0,0)} = \{(0, 0)\}$, $E_{(1,0)} = \mathbb{R} \times \{0\}$, $E_{(0,1)} = \{0\} \times \mathbb{R}$, and $E_{(1,1)} = \mathbb{R}^2$.

Remark 10.5. The quotient space of disk trees is tame, since the boundary and corners structure arises from Morse trajectory space (Section 2.2), but the splicing (6.10) is independent of the Morse trajectories.

Now define the *local reduced tangent space* $T_x^R O$ by

$$T_x^R O := T_xO \cap E_x.$$

In general for a point $x \in \mathfrak{X}$, we take an sc-smooth chart $(\mathfrak{U}, \phi, (O, C, E))$ around x, and define the *reduced tangent space* by passing to the local model,

$$T_x^R \mathfrak{X} := T\phi(x) \left(T^R_{\phi^{-1}(x)} O \right).$$

Remark 10.6. A tame M-polyfold has the following property regarding its reduced tangent space. (See Proposition 2.36 of [9].) For a smooth point x in a tame M-polyfold $\mathfrak{X}$, the codimension of the reduced tangent space $T_x^R\mathfrak{X}$ in $T_x\mathfrak{X}$ is the same as the degeneracy index $d(x)$.

Later in this paper, we shall show that the M-polyfold of disk trees is tame.

Now let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a tame M-polyfold strong bundle and $f : \mathfrak{X} \to \mathfrak{Y}$ an sc-Fredholm section (see Section 2.5 and 3.1 of [9]), we study the solution set $f^{-1}(0)$ and its relation with the boundary and corners structure.

Definition 10.7. Let x be a solution to $f(x) = 0$. We define the **linearized section** $Df(x) : T_x\mathfrak{X} \to \mathfrak{Y}_x$ as the the composition $df(x) : T_x\mathfrak{X} \to T_{(x,0)}\mathfrak{Y}$ and the projection $T_{(x,0)}\mathfrak{Y} \simeq T_x\mathfrak{X} \times \mathfrak{Y}_x \to \mathfrak{Y}_x$.

We now outline a transversality condition called *general position.* (See Definition 5.17 of [9].)

Definition 10.8. Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a tame M-polyfold strong bundle and f an sc-Fredholm section. We say f is in **general position** at a solution x to $f(x) = 0$ if the linearized section $Df(x) : T_x\mathfrak{X} \to \mathfrak{Y}_x$ is surjective and $\ker(Df(x))$ has an sc-complement contained in the reduced tangent space $T_x^R\mathfrak{X}$.

We have the following result on the solution set of a Fredholm section in general position. (See Theorem 5.18 of [9].)

Theorem 10.9. *Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a tame M-polyfold strong bundle, and f an sc-Fredholm section in general position at the solution set $\mathfrak{M} = f^{-1}(0)$. Then $\mathfrak{M}$ is a sub-M-polyfold of $\mathfrak{X}$ and the induced M-polyfold structure on $\mathfrak{M}$ is equivalent to that of a smooth manifold with boundary and corners. Moreover, for $x \in \mathfrak{M}$ we have $d_{\mathfrak{M}}(x) = d_{\mathfrak{X}}(x)$, where $d_{\mathfrak{M}}$ and $d_{\mathfrak{X}}$ are the respective degeneracy index of $\mathfrak{M}$ and $\mathfrak{X}$.*

Remark 10.10. Suppose x is a solution to $f(x) = 0$ with $Df(x)$ surjective. If x has degeneracy index $d(x)$ greater than Fredholm index $\mathrm{ind}_f(x)$, then by Remark 10.6 f is *not* in general position at x. We shall see such transversality failure come up later in Section 12.2.

There is a weaker transversality condition called *good position.* We first illustrate this concept locally. (See Definition 3.21 of [9].)

Let $C \subset E$ be a partial quadrant in an sc-Banach space, and $N \subset E$ a finite dimensional sc-subspace. We say N is in *good position to C* if

- $N \cap C$ has a non-empty interior in N, and
- there is an sc-complement P with $E = N \oplus P$ such that there exists $\varepsilon > 0$ so for $(n, p) \in N \oplus P$ with $|p| \leq \varepsilon|n|$, we have $n \in C$ if and only if $n + p \in C$.

Now let (O, C, E) be an sc-smooth retract. Without loss of generality assume $C = [0, \infty)^k \times W$ and $E = \mathbb{R}^k \oplus W$. Let $x = (a_1, \ldots, a_k, w)$ be a point in O. Define the subset C_x by

$$C_x := \{(b_1, \ldots, b_k, w) \in \mathbb{R}^k \oplus W \mid b_i \geq 0 \text{ if } a_i = 0\}.$$

Example 10.11. Let $E = \mathbb{R}^2$ and $C = [0, \infty)^2$, then $C_{(0,0)} = [0, \infty)^2$, $C_{(1,0)} = \mathbb{R} \times [0, \infty)$, $C_{(0,1)} = [0, \infty) \times \mathbb{R}$, and $C_{(1,1)} = \mathbb{R}^2$.

Now define the *local partial cone* C_xO by

$$C_xO := T_xO \cap C_x.$$

In general for a point $x \in \mathfrak{X}$, we take an sc-smooth chart $(\mathfrak{U}, \phi, (O, C, E))$ around x, and define the *partial cone* by passing to the local model,

$$C_x\mathfrak{X} := T\phi(x)\left(C_{\phi(x)}O\right).$$

We now summarize the notion of good position. (See Definition 3.53 of [9].)

Definition 10.12. Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a tame M-polyfold strong bundle, and $f : \mathfrak{X} \to \mathfrak{Y}$ a sc-Fredholm section. Then we say f is in **good position** at a solution x to $f(x) = 0$ if the linearized section $Df(x) : T_x\mathfrak{X} \to \mathfrak{Y}_x$ is surjective, and $\ker(Df(x))$ is in good position to the partial cone $C_x\mathfrak{X}$.

Remark 10.13. If a Fredholm section f is in general position at a solution x, then it is in good position at x. (See Lemma 5.26 of [9].) The converse is not true: suppose (O, C, E) is a retract such that $E = \mathbb{R}^2, C = [0, \infty)^2$ and $x = (0, 0)$ with tangent space $T_xO = \operatorname{span}\{(1, 1)\}$. In this case, T_xO has no complement contained in $T_x^RO = \{(0, 0)\}$, but it is in good position with $C_xO = [0, \infty)^2$.

Lastly we cite the following result on the solution set of a Fredholm section in good position. (See Theorem 3.57 of [9].)

Theorem 10.14. *Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a tame M-polyfold strong bundle, and f an sc-Fredholm section in good position at the solution set $\mathfrak{M} = f^{-1}(0)$. Then $\mathfrak{M}$ is a sub-M-polyfold of $\mathfrak{X}$ and the induced M-polyfold structure on $\mathfrak{M}$ is equivalent to that of a smooth manifold with boundary and corners.*

In the analysis of M-polyfold of disk trees in Section 12.2, we shall see a multitude of transversality phenomenon: the perturbed $\bar{\partial}_J$ section can achieve general position in most cases, but sometimes only good position, and occasionally even good position fails.

11. A_∞ Algebra and Concatenation Tree

In Morse theory, the moduli space of trajectories satisfying gradient flow equation yields a differential graded algebra, and consequently the Morse homology. Analogously, the moduli space of disk trees satisfying Cauchy-Riemann equation yields an A_∞ algebra. We describe this process as follows.

11.1. Curved A_∞ Algebra.

Let Λ be a ring with $\mathbb{Z}_2$ coefficients and C a Λ-module. For each $k \geq 0$ let $m^k : \bigotimes^k C \to C$ be a Λ-module homomorphism called the **k-th multiplication**. Define the total complex to be $\widetilde{C} := \oplus_{k=0}^{\infty} \bigotimes^k C$. For a pure tensor $P = p_k \otimes \cdots \otimes p_1$ we denote its length by

$$|P| = k.$$

We extend the multiplication to $\widetilde{m} : \widetilde{C} \to \widetilde{C}$ as follows. Given a pure tensor P, define

$$\widetilde{m}(P) = \sum_{P = P_2 \otimes P' \otimes P_1} P_2 \otimes m^{|P'|}(P') \otimes P_1, \quad (|P'|, |P_i| \text{ potentially } 0) \tag{11.1}$$

and then extend Λ-linearly to combinations of pure tensors. We call $(\widetilde{C}, \widetilde{m})$ a **curved A_∞ algebra** if it satisfies the **A_∞ algebra equation**

$$\widetilde{m} \circ \widetilde{m} = 0. \tag{11.2}$$

Note that the system of infinite equation starts with $m_1(m_0) = 0$. From now on, we shall refer to a curved A_∞ algebra simply as an A_∞ algebra.

We now discuss the concept of A_∞ homomorphism. Let $(\widetilde{C}, \widetilde{m})$ and $(\widetilde{D}, \widetilde{n})$ be A_∞ algebras, and for each $k \geq 0$ let $\varphi^k : \bigotimes^k C \to D$ be a Λ-module homomorphism. We extend it to $\widetilde{\varphi} : \widetilde{C} \to \widetilde{D}$ as follows. Given a pure tensor P, define

$$\widetilde{\varphi}(P) := \sum_{P = P_l \otimes \cdots \otimes P_1} \varphi^{|P_l|}(P_l) \otimes \cdots \otimes \varphi^{|P_1|}(P_1), \quad (|P_i| \text{ potentially } 0) \tag{11.3}$$

and then extend Λ-linearly to combinations of pure tensors. We call $\widetilde{\varphi} : (\widetilde{C}, \widetilde{m}) \to (\widetilde{D}, \widetilde{n})$ an A_∞ homomorphism if it satisfies the **A_∞ homomorphism equation**

$$\widetilde{\varphi} \circ \widetilde{m} = \widetilde{n} \circ \widetilde{\varphi}. \tag{11.4}$$

Let $\pi^1 : \widetilde{C} \to C$ be the projection to the 1-tensor component. The following lemma shows that in order to check $\widetilde{m} \circ \widetilde{m} = 0$ it suffices to check $\pi^1 \circ \widetilde{m} \circ \widetilde{m} = 0$. In practice, we will always verify the A_∞ equation this way.

Lemma 11.1. *Let C be a Λ-module and let $m^k : \bigotimes^k C \to C$ be the k-th multiplication for each $k \geq 0$. Then $\pi^1 \circ \widetilde{m} \circ \widetilde{m} = 0$ implies $\widetilde{m} \circ \widetilde{m} = 0$.*

There is a similar result on checking $\widetilde{\varphi}$ is an A_∞ homomorphism.

Lemma 11.2. *Let $(\widetilde{C}, \widetilde{m})$ and $(\widetilde{D}, \widetilde{n})$ be A_∞ algebras, and let $\varphi^k : \bigotimes^k C \to D$ be a Λ-module homomorphism for $k \geq 0$. Then $\pi^1 \circ (\widetilde{\varphi} \circ \widetilde{m} - \widetilde{n} \circ \widetilde{\varphi}) = 0$ implies $\widetilde{\varphi} \circ \widetilde{m} - \widetilde{n} \circ \widetilde{\varphi} = 0$.*

The above lemmas are proven on [13] page 11, while assuming $m^0 = 0$ and $n^0 = 0$. The general results are similar.

11.2. Type and Disk Tree A_∞ Algebra.

We introduce the notion of a type, which is crucial to constructing and understanding the disk tree A_∞ algebra.

We remind the readers that (M, ω) is a symplectic manifold, and L is a compact Lagrangian submanifold of M. In addition, we choose a compatible almost complex structure J on M, and a Morse-Smale pair (α, g) on the Lagrangian L. Let $\mathrm{crit}(L)$ be the set of Morse critical points.

Definition 11.3. A **type** is a tuple $Z = [P, q; \mu]$, where $P = p_k \otimes \cdots \otimes p_1$ and each p_i and q are Morse critical points of the Morse-Smale pair (α, g). $\mu \in \mathbb{Z}$ is the Maslov index of a type. For convenience, we denote

$$|Z|^{\text{in}} := |P| = k, \quad \mu(Z) := \mu$$

to be the number of incoming critical points and the Maslov index of type Z, respectively. We denote the set of all types by $\mathcal{Z}$.

As we have seen in the construction of $\mathfrak{X}$ the M-polyfold of disk trees, each type $Z = [p_k \otimes \cdots \otimes p_1, q; \mu]$ corresponds to a sub-M-polyfold $\mathfrak{X}(Z)$. It consists of disk trees with incoming critical points $p_k, \ldots, p_1$ and outgoing critical point q and Maslov index μ. Each $z = (\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}) \in \mathfrak{X}(Z)$ is a disk tree, which represents a relative homology class in $\overline{H}_2(M, L)$. We denote its pairing with the symplectic form by

$$\omega(z) = \sum_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}} \langle [u_{\mathrm{v}}]_{H_2}, [\omega]_{H^2} \rangle .$$

Thus, $\mathfrak{X}$ can be decomposed according to types, and each $\mathfrak{X}(Z)$ can be decomposed according to the pairing with the symplectic form (see (2.5)),

$$\mathfrak{X} = \bigsqcup_{Z \in \mathcal{Z}} \mathfrak{X}(Z), \text{ and } \mathfrak{X}(Z) = \bigsqcup_{\nu \geq 0} \mathfrak{X}(Z, \nu).$$

Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be the bundle of complex anti-linear sections (9.1). Let $\bar{\partial}_J : \mathfrak{X} \to \mathfrak{Y}$ be the Cauchy-Riemann section (9.2). Note that the Fredholm index $\mathrm{ind}_{\bar{\partial}_J}(z)$ is the same for all $z \in \mathfrak{X}(Z)$, and we refer to it as the **Fredholm index of type Z** $\mathrm{ind}_{\bar{\partial}_J}(Z)$. One can carry out a standard index computation by using Riemann-Roch theorem (Appendix C of [15]),

$$\mathrm{ind}_{\bar{\partial}_J}(Z) = \sum_{i=1}^{k} |p_i| - |q| + \mu - (k-1)(n-1) - 1, \tag{11.5}$$

where $|p_i|$ and $|q|$ are Morse indices.

Suppose $f : \mathfrak{X} \to \mathfrak{Y}$ is a section obtained by adding an sc^+ perturbation (Definition 2.27 of [8]) to $\bar{\partial}_J$ and it achieves general position (see Definition 10.8), we denote its **moduli space** by

$$\mathfrak{M}_f := f^{-1}(0) \text{ and } \mathfrak{M}_f(Z) := f^{-1}(0) \cap \mathfrak{X}(Z). \tag{11.6}$$

By Theorem 10.9, the solution set $\mathfrak{M}_f(Z)$ is a manifold of dimension $\mathrm{ind}_f(Z)$. Since the perturbation is sc^+, we have $\mathrm{ind}_f(Z) = \mathrm{ind}_{\bar{\partial}_J}(Z)$. From now on, if we omit the section in the Fredholm index, we assume it is the Fredholm index of $\bar{\partial}_J$.

Those types with index 0 play an important role in the A_∞ algebra. Thus we denote

$$[P, q] := [P, q; \mu], \tag{11.7}$$

where μ is the Maslov index such that $\mathrm{ind}([P, q; \mu]) = 0$.

We construct the **disk tree A_∞ algebra over $\mathbb{Z}_2$** as follows, using the **Novikov ring**

$$\Lambda = \left\{ \sum c_i e^{\nu_i} \,|\, c_i \in \mathbb{Z}_2, \nu_i \in [0,\infty) \right\}.$$

Here e is the quantum variable. In the sum, there are finitely many non-zero c_i with $\nu_i \leq N$ for any N. Let C be the complex generated by Morse critical points

$$C = \sum_{p \in \mathrm{crit}(L)} \Lambda \langle p \rangle.$$

The total complex $\widetilde{C} = \bigoplus_{k=0}^{\infty} \bigotimes^k C$ has an obvious bilinear form

$$\left\langle \sum \lambda_P P, \sum \lambda'_P P \right\rangle := \sum \lambda_P \lambda'_P \in \Lambda, \tag{11.8}$$

where we sum over pure tensors $P = p_k \otimes \cdots \otimes p_1$ with $p_i \in \mathrm{crit}(L)$. For each $k \geq 0$, define the k-th multiplication $m^k : \bigotimes^k C \to C$ as follows. For $R = r_k \otimes \cdots \otimes r_1$, define

$$\langle m^k(R), s \rangle := \sum_{z \in \mathfrak{M}_f([R,s])} e^{\omega(z)}. \tag{11.9}$$

Naturally, we have $m^k(R) = \sum_{s \in \mathrm{crit}(L)} \langle m^k(R), s \rangle s$. Note that $\mathfrak{M}_f([R,s])$ is a 0-dimensional manifold and thus countable.

This definition has two problems a priori: for a fixed $\nu \geq 0$, the set $\mathfrak{M}_f \cap \mathfrak{X}([R,s],\nu)$ is not necessarily finite, and the A_∞ equation $\widetilde{m} \circ \widetilde{m} = 0$ does not necessarily hold. We need certain compactness and compatibility result (see Section 11.5) to address these issues. In particular, $\widetilde{m} \circ \widetilde{m} = 0$ does hold if the perturbed $\bar{\partial}_J$ section f is compatible with concatenation, which is a crucial notion we discuss next.

11.3. **Concatenation.**

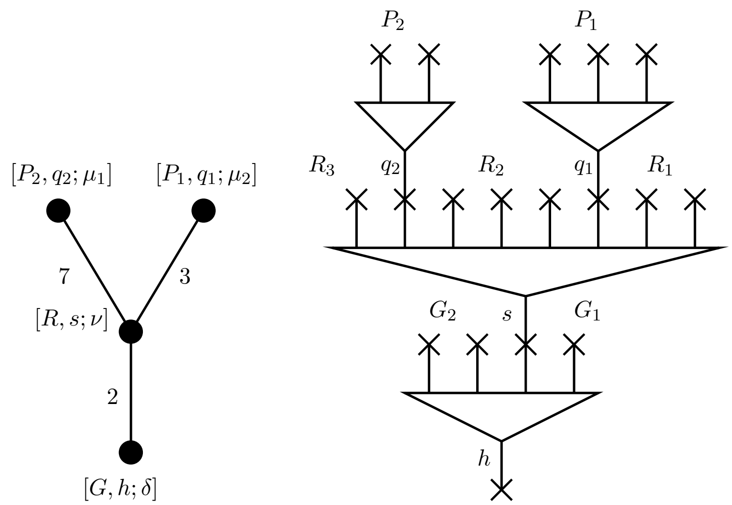
We now introduce the notion of concatenation tree, which specifies a way of concatenating types.

Definition 11.4. A **concatenation tree** is a tuple $\mathcal{T} = (\mathrm{T}, \underline{Z}, \underline{\iota})$. Here $\mathrm{T} = (\mathrm{V}, \mathrm{E})$ is the underlying ordered tree. $\underline{Z} \in \mathcal{Z}^{\mathrm{V}}$ is a tuple of types, or equivalently, $\underline{Z} : \mathrm{V} \to \mathcal{Z}$ assigns a type Z^{v} to each vertex v. Moreover $\underline{\iota} \in \mathbb{N}^{\mathrm{E}}$ is a tuple of natural numbers, or equivalently, $\underline{\iota} : \mathrm{E} \to \mathbb{N}$ specifies for each edge (v,w) that the outgoing critical point of type Z^{v} concatenates with the $\iota^{(\mathrm{v},\mathrm{w})}$-th incoming critical point of type Z^{w}.

Naturally we impose the following two conditions.

- *Injectivity condition*: for each $\mathrm{v} \in \mathrm{V}$, the restriction $\iota|_{\mathrm{E}^{\mathrm{in}}(\mathrm{v})}$ is injective.
- *Coincidence condition*: for $\mathrm{e} = (\mathrm{v},\mathrm{w})$, $Z^{\mathrm{v}} = [p_k \otimes \cdots \otimes p_1, q; \mu]$, and $Z^{\mathrm{w}} = [r_l \otimes \cdots \otimes r_1, s; \nu]$, we have $q = r_{\iota^{\mathrm{e}}}$.

We now construct the concatenated type $\circ_{\mathcal{T}}(\underline{Z})$.



(A) Concatenation tree $\mathcal{T}$ (B) Type representation of $\mathcal{T}$

FIGURE 3. (A) is an example of a concatenation tree, where each vertex is labeled a type Z^{v} and each edge a number ι^{e}. (B) is the type representation of this tree, where each cross represents a critical point and each triangle a disk tree. The concatenation type $\circ_{\mathcal{T}}(\underline{Z})$ is $[G_2 \otimes R_3 \otimes P_2 \otimes R_2 \otimes P_1 \otimes R_1 \otimes G_1, h; \mu_2 + \mu_1 + \nu + \delta]$.

Definition 11.5. Given a concatenation tree $\mathcal{T} = (\mathrm{T}, \underline{Z}, \underline{\iota})$, we define the **concatenated type** as follows.

In the case when $\mathcal{T}$ consists of a single edge (v, w), denote

$$[P, q; \mu] := Z^{\mathrm{v}}, \quad [R, s; \nu] := Z^{\mathrm{w}}.$$

The string $\bigotimes_{i \neq \iota^{(\mathrm{v},\mathrm{w})}} r_i$ is the product of the un-concatenated critical points in R, and it can be written as $R_2 \otimes R_1$. Here each R_j is the (potentially empty) product of critical points consecutive in R. We define the concatenated type by

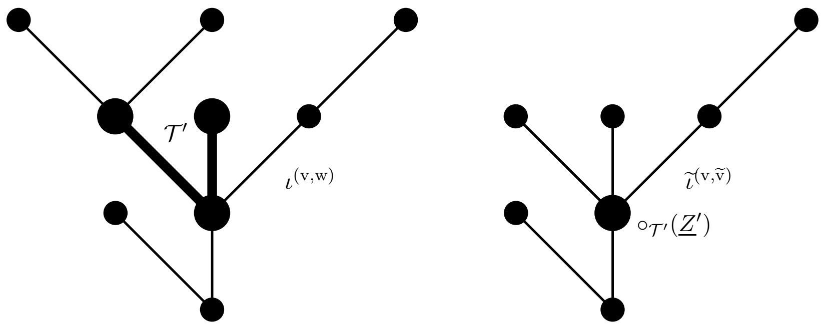
$$\circ_{\mathcal{T}}(\underline{Z}) = [R_2 \otimes P \otimes R_1, s; \mu + \nu]\,.$$

For a general concatenation tree, we define the concatenated type $\circ_{\mathcal{T}}(\underline{Z})$ inductively.

Remark 11.6. One can show by induction that the Fredholm index of the concatenated type is

$$\operatorname{ind}(\circ_{\mathcal{T}}(\underline{Z})) = \sum_{\mathrm{v} \in \mathrm{V}} \operatorname{ind}(Z^{\mathrm{v}}) + |\mathrm{E}|. \tag{11.10}$$

Definition 11.7. We say $\mathcal{T}'$ is a **sub-tree** of $\mathcal{T}$ if the underlying ordered tree T' is a sub-tree of T and $\underline{Z}', \underline{\iota}'$ are restrictions of $\underline{Z}, \underline{\iota}$. Equivalently, we call $\mathcal{T}$ a **super-tree** of $\mathcal{T}'$. This defines a partial order and we denote it by $\mathcal{T}' \leq \mathcal{T}$.



(A) Super-tree $\mathcal{T}$

(B) Concatenated-tree $\widetilde{\mathcal{T}}$

FIGURE 4. Associativity of concatenation: $\circ_{\mathcal{T}}(\underline{Z}) = \circ_{\widetilde{\mathcal{T}}}(\circ_{\mathcal{T}'}(\underline{Z}))$.

Remark 11.8. We now discuss the associativity of concatenation. First note that given a concatenation tree $\mathcal{T}$, the set

$$I_{\mathcal{T}} = \bigsqcup_{\mathrm{v}\in \mathrm{V}} \{\mathrm{v}\} \times \left(\{|Z^{\mathrm{v}}|^{\mathrm{in}}, \ldots, 1\} \setminus \underline{\iota}(\mathrm{E}^{\mathrm{in}}(\mathrm{v}))\right)$$

indexes all un-concatenated incoming critical points in $\mathcal{T}$. It is easy to see that there is a bijection $\alpha_{\mathcal{T}} : I_{\mathcal{T}} \to \{| \circ_{\mathcal{T}} (\underline{Z})|^{\mathrm{in}}, \ldots, 1\}$.

Given $\mathcal{T}'$ and a super-tree $\mathcal{T}$, we can construct the **concatenated-tree** $\widetilde{\mathcal{T}} = (\widetilde{\mathrm{T}}, \widetilde{\underline{Z}}, \widetilde{\underline{\iota}})$ from $\mathcal{T}$ by collapsing $\mathcal{T}'$ into a single vertex $\widetilde{\mathrm{v}}$. More precisely, define $\widetilde{\mathrm{V}} := (\mathrm{V} \setminus \mathrm{V}') \cup \{\widetilde{\mathrm{v}}\}$, and define a tuple $\circ_{\mathcal{T}'}(\underline{Z}) \in \mathcal{Z}^{\widetilde{\mathrm{V}}}$ by

$$\circ_{\mathcal{T}'}(\underline{Z})^{\mathrm{v}} := \begin{cases} Z^{\mathrm{v}}, & \mathrm{v} \in \mathrm{V} \setminus \mathrm{V}', \\ \circ_{\mathcal{T}'}(\underline{Z}'), & \mathrm{v} = \widetilde{\mathrm{v}}. \end{cases} \tag{11.11}$$

Let the tuple $\widetilde{\underline{Z}}$ be $\circ_{\mathcal{T}'}(\underline{Z})$. Lastly, an edge of the form $(\mathrm{v}, \widetilde{\mathrm{v}}) \in \widetilde{\mathrm{E}}$ corresponds to an edge $(\mathrm{v}, \mathrm{w}) \in \mathrm{E}$ with $\mathrm{w} \in \mathrm{V}'$. We define $\widetilde{\iota}^{(\mathrm{v},\widetilde{\mathrm{v}})}$ according to the bijection $\alpha_{\mathcal{T}'}$ as follows (see Figure 4).

$$\widetilde{\iota}^{(\mathrm{v},\widetilde{\mathrm{v}})} = \alpha_{\mathcal{T}'}\left(\mathrm{w}, \iota^{(\mathrm{v},\mathrm{w})}\right).$$

Conversely, given $\mathcal{T}'$ and $\widetilde{\mathcal{T}}$ with $\widetilde{\underline{Z}}^{\widetilde{\mathrm{v}}} = \circ_{\mathcal{T}'}(\underline{Z}')$, we can construct the super-tree $\mathcal{T}$ by using the inverse of $\alpha_{\mathcal{T}'}$.

One can easily show that

$$\circ_{\mathcal{T}}(\underline{Z}) = \circ_{\widetilde{\mathcal{T}}}\left(\circ_{\mathcal{T}'}(\underline{Z})\right). \tag{11.12}$$

This property is the **associativity of concatenation**: it does not matter how we concatenate the sub-tree first and then according to the concatenated-tree, it always ends up the same type.

11.4. **Concatenation and the Boundary and Corners Structure.**

Similar to the concatenated type in Definition 11.5, we define the concatenated disk tree. This operation directly relates to the boundary and corners structure of the M-polyfold of disk trees $\mathfrak{X}$.

Definition 11.9. Given a concatenation tree $\mathcal{T} = (\mathrm{T}, \underline{Z}, \underline{\iota})$ and a tuple of disk trees $\underline{z} \in \prod_{\mathrm{v}\in\mathrm{V}} \mathfrak{X}(Z^{\mathrm{v}})$, define the **concatenated disk tree** $\circ_{\mathcal{T}}(\underline{z}) \in \mathfrak{X}(\circ_{\mathcal{T}}(\underline{Z}))$ as follows.

In the case when $\mathcal{T}$ consists of a single edge (v, w), denote

$$[\mathrm{T}^{\mathrm{v}}, \underline{\gamma}^{\mathrm{v}}, \underline{x}^{\mathrm{v}}, \underline{u}^{\mathrm{v}}] := z^{\mathrm{v}}, \quad [\mathrm{T}^{\mathrm{w}}, \underline{\gamma}^{\mathrm{w}}, \underline{x}^{\mathrm{w}}, \underline{u}^{\mathrm{w}}] := z^{\mathrm{w}}, \quad i := \iota^{(\mathrm{v},\mathrm{w})}.$$

Moreover, let rt^{v} be the root of T^{v}, and $\mathrm{lf}_i^{\mathrm{w}}$ the $\iota^{(\mathrm{v},\mathrm{w})}$-th leaf of T^{w}. By construction, the root rt^{v} has one incoming edge $(\alpha, \mathrm{rt}^{\mathrm{v}})$ and no outgoing edge, and the leaf $\mathrm{lf}_i^{\mathrm{w}}$ has no incoming edge and one outgoing edge $(\mathrm{lf}_i^{\mathrm{w}}, \beta)$.

We construct the concatenated disk tree $\circ_{\mathcal{T}}(\underline{z}) = [\widetilde{\mathrm{T}}, \underline{\widetilde{\gamma}}, \underline{\widetilde{x}}, \underline{\widetilde{u}}]$ as follows. The tree $\widetilde{\mathrm{T}}$ is obtained by deleting the edges $(\alpha, \mathrm{rt}^{\mathrm{v}})$ from T^{v} and $(\mathrm{lf}_i^{\mathrm{w}}, \beta)$ from T^{w}, and connecting them by the edge (α, β), i.e.,

$$\widetilde{\mathrm{V}} := (\mathrm{V}^{\mathrm{v}} \sqcup \mathrm{V}^{\mathrm{w}}) \setminus \{\mathrm{rt}^{\mathrm{v}}, \mathrm{lf}_i^{\mathrm{w}}\},$$

$$\widetilde{\mathrm{E}} := (\mathrm{E}^{\mathrm{v}} \sqcup \mathrm{E}^{\mathrm{w}} \sqcup \{(\alpha, \beta)\}) \setminus \{(\alpha, \mathrm{rt}^{\mathrm{v}}), (\mathrm{lf}_i^{\mathrm{w}}, \beta)\}.$$

The generalized Morse trajectories at edge (α, β) are gotten by concatenating the trajectories at $(\alpha, \mathrm{rt}^{\mathrm{v}})$ with the ones at $(\mathrm{lf}_i^{\mathrm{w}}, \beta)$, i.e.,

$$\underline{\widetilde{\gamma}}_{\widetilde{\mathrm{e}}} := \begin{cases} \underline{\gamma}^{\mathrm{v}}_{\widetilde{\mathrm{e}}}, & \widetilde{\mathrm{e}} \in \mathrm{E}^{\mathrm{v}} \setminus \{(\alpha, \mathrm{rt}^{\mathrm{v}})\}, \\ \underline{\gamma}^{\mathrm{w}}_{\widetilde{\mathrm{e}}}, & \widetilde{\mathrm{e}} \in \mathrm{E}^{\mathrm{w}} \setminus \{(\mathrm{lf}_i^{\mathrm{w}}, \beta)\}, \\ \left(\underline{\gamma}^{\mathrm{v}}_{(\alpha,\mathrm{rt}^{\mathrm{v}})}, \underline{\gamma}^{\mathrm{w}}_{(\mathrm{lf}_i^{\mathrm{w}},\beta)}\right), & \widetilde{\mathrm{e}} = (\alpha, \beta). \end{cases}$$

This makes sense since the coincidence condition in Definition 11.4 implies that the critical points corresponding to rt^{v} and $\mathrm{lf}_i^{\mathrm{w}}$ agree. Lastly, the tuple of boundary marked points and disk maps are the product of those of T^{v} and T^{w}, i.e.,

$$(\underline{\widetilde{x}}_{\widetilde{\mathrm{v}}}, \underline{\widetilde{u}}_{\widetilde{\mathrm{v}}}) := \begin{cases} (\underline{x}^{\mathrm{v}}_{\widetilde{\mathrm{v}}}, \underline{u}^{\mathrm{v}}_{\widetilde{\mathrm{v}}}), & \widetilde{\mathrm{v}} \in (\mathrm{V}^{\mathrm{v}})^{\mathrm{m}}, \\ (\underline{x}^{\mathrm{w}}_{\widetilde{\mathrm{v}}}, \underline{u}^{\mathrm{w}}_{\widetilde{\mathrm{v}}}), & \widetilde{\mathrm{v}} \in (\mathrm{V}^{\mathrm{w}})^{\mathrm{m}}, \end{cases}$$

where $(\mathrm{V}^{\mathrm{v}})^{\mathrm{m}}$ and $(\mathrm{V}^{\mathrm{w}})^{\mathrm{m}}$ denote the set of main vertices of the trees T^{v} and T^{w}, respectively.

For a more general concatenation tree, we define the concatenated type $\circ_{\mathcal{T}}(\underline{z})$ inductively.

We can view $\circ_{\mathcal{T}}$ as a map as follows.

Proposition 11.10. *Given a concatenation tree $\mathcal{T} = (\mathrm{T}, \underline{Z}, \underline{\iota})$, the concatenation*

$$\circ_{\mathcal{T}} : \prod_{\mathrm{v}\in\mathrm{V}} \mathfrak{X}(Z^{\mathrm{v}}) \to \mathfrak{X}(\circ_{\mathcal{T}}(\underline{Z}))$$
$$\underline{z} \mapsto \circ_{\mathcal{T}}(\underline{z})$$

is an injective sc^{∞} map.

The injectivity and the smoothness follows from that of concatenating generalized Morse trajectories. Moreover, for each disk tree $z = [\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}]$ we define its pairing with the symplectic form ω to be the sum of pairings of all disk maps

$$\omega(z) = \sum_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}} \langle [u_{\mathrm{v}}]_{H_2}, [\omega]_{H^2} \rangle, \tag{11.13}$$

analogous to (2.6). Then the pairing between the concatenated disk trees and the symplectic form is given by the sum

$$\omega(\circ_{\mathcal{T}}(\underline{z})) = \sum_{\mathrm{v} \in \mathrm{V}} \omega(z^{\mathrm{v}}). \tag{11.14}$$

The concatenation map is closely related to the boundary and corners structure of the M-polyfold $\mathfrak{X}$ of disk trees $\mathfrak{X}$ (see Definition 10.2). The boundary $\partial\mathfrak{X}$ corresponds to the boundary component of the Morse trajectory space, which is the set of all broken flow lines. For $x \in \mathfrak{X}$, $d(x)$ is equal to the total number of broken flow lines in x. In other words, let $\mathcal{T}$ be the *maximal concatenation tree* of x, where $x = \circ_{\mathcal{T}}(\underline{z})$ and each disk tree z^{v} has no broken flow lines. Then we have

$$d(x) = |\mathrm{E}|.$$

In general, there is an equation for degeneracy index similar to that of Fredholm index (11.10),

$$d(\circ_{\mathcal{T}}(\underline{z})) = \sum_{\mathrm{v} \in \mathrm{V}} d(z^{\mathrm{v}}) + |\mathrm{E}|. \tag{11.15}$$

11.5. **Compatible Section and A_∞ Algebra.**

One can define the concatenation map $\circ^{\pi}_{\mathcal{T}}$ for the bundles of complex anti-linear sections $\pi : \mathfrak{Y} \to \mathfrak{X}$ as well. Here $\mathfrak{Y}$ composes of elements of the form $[\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}, \underline{\lambda}]$. The fiber $\mathfrak{Y}_{[\mathrm{T}, \underline{\gamma}, \underline{x}, \underline{u}]}$ consists of all tuples $(\lambda_{\mathrm{v}})_{\mathrm{v} \in \mathrm{V}^{\mathrm{m}}}$ where each λ_{v} is a complex anti-linear section of the bundle $(u_{\mathrm{v}}^* TM, J \circ u_{\mathrm{v}}) \to (D, i)$.

For $\mathcal{T} = (\mathrm{T}, \underline{Z}, \underline{\iota})$ and $\underline{z} \in \prod_{\mathrm{v} \in \mathrm{V}} \mathfrak{X}(Z^{\mathrm{v}})$, the observation below follows from the definition of concatenated disk tree.

$$\mathfrak{Y}_{\circ_{\mathcal{T}}(\underline{z})} = \prod_{\mathrm{v} \in \mathrm{V}} \mathfrak{Y}_{z^{\mathrm{v}}}, \tag{11.16}$$

i.e., the fiber at the concatenated disk tree is the product of the each fiber. Moreover, we denote the projection to each vertex component by

$$pr^{\mathrm{v}} : \mathfrak{Y}_{\circ_{\mathcal{T}}(\underline{z})} \to \mathfrak{Y}_{z^{\mathrm{v}}}. \tag{11.17}$$

Definition 11.11. Given $\mathcal{T} = (\mathrm{T}, \underline{Z}, \underline{\iota})$, let $\underline{y} \in \prod_{\mathrm{v} \in \mathrm{V}} \mathfrak{Y}(Z^{\mathrm{v}})$ be a tuple of bundle elements. We define the concatenated bundle element $\circ^{\pi}_{\mathcal{T}}(\eta)$ as follows. First write $\underline{y}$ as $y^{\mathrm{v}} = (z^{\mathrm{v}}, \zeta^{\mathrm{v}})$, where $z^{\mathrm{v}} \in \mathfrak{X}(Z^{\mathrm{v}})$ is the base component and ζ^{v} is the fiber component. We define

$$\circ^{\pi}_{\mathcal{T}}(\underline{y}) := (\circ_{\mathcal{T}}(\underline{z}), (\zeta^{\mathrm{v}})_{\mathrm{v} \in \mathrm{V}}),$$

i.e., the fiber component of the concatenated bundle element is simply the product $\underline{\zeta} \in \prod_{\mathrm{v}\in\mathrm{V}} \mathfrak{Y}_{z^{\mathrm{v}}}$.

We now define the notion of a $\circ$-compatible section, which as we shall see relates back to the A_∞ equation.

Definition 11.12. A section $f : \mathfrak{X} \to \mathfrak{Y}$ is **$\circ$-compatible** if for any concatenation tree $\mathcal{T} = (\mathrm{T}, \underline{Z}, \underline{\iota})$ and tuple $\underline{z} \in \prod_{\mathrm{v}\in\mathrm{V}} \mathfrak{X}(Z^{\mathrm{v}})$, we have

$$f(\circ_{\mathcal{T}}(\underline{z})) = \circ^{\pi}_{\mathcal{T}}(f(\underline{z})).$$

Example 11.13. The Cauchy-Riemann section $\bar{\partial}_J$ is $\circ$-compatible.

A related notion is a $\circ$-invariant subset. Given a subset $\mathfrak{K} \subset \mathfrak{X}$, we denote

$$\partial\mathfrak{K} := \partial\mathfrak{X} \cap \mathfrak{K}, \quad \mathfrak{K} \circ \mathfrak{K} := \{\circ_{\mathcal{T}}(\underline{z}) \,|\, |\mathrm{E}| = 1 \text{ and each } z^{\mathrm{v}} \in \mathfrak{K}\}. \tag{11.18}$$

Definition 11.14. A subset $\mathfrak{K} \subset \mathfrak{X}$ is **$\circ$-invariant** if we have $\partial\mathfrak{K} = \mathfrak{K} \circ \mathfrak{K}$.

Clearly, we have $\partial\mathfrak{X} = \mathfrak{X} \circ \mathfrak{X}$. Also if a section $f : \mathfrak{X} \to \mathfrak{Y}$ is $\circ$-compatible, then its moduli space $\mathfrak{M}_f = f^{-1}(0)$ satisfies $\partial\mathfrak{M}_f = \mathfrak{M}_f \circ \mathfrak{M}_f$.

Recall that by Theorem 9.10, the Cauchy-Riemann section $\bar{\partial}_J$ is sc-Fredholm. We now state the following result regarding the existence of a $\circ$-compatible sc^+ perturbation (see Section 1.4 of [7]) of $\bar{\partial}_J$ that achieves general position. (See Definition 10.8 for the notion of general position.)

Theorem 11.15. *Let $\mathfrak{X}$ be the quotient space of disk trees and $\pi : \mathfrak{Y} \to \mathfrak{X}$ the bundle of complex anti-linear sections. Then for each type Z and $\nu \geq 0$, the solution set $\bar{\partial}_J^{-1}(0) \cap \mathfrak{X}(Z, \nu)$ is compact. Moreover, there exists a $\circ$-compatible sc^+ section s such that $f = \bar{\partial}_J + s$ is in general position at the solution set $f^{-1}(0)$, with each $f^{-1}(0) \cap \mathfrak{X}(Z, \nu)$ compact.*

The compactness of $\bar{\partial}_J^{-1}(0) \cap \mathfrak{X}(Z, \nu)$ follows from Gromov compactness ([4]). Note that Theorem 11.15 is a $\circ$-compatible version of Theorem 5.5 of [9], except for simplicity we omit the auxiliary norm here (see Definition 5.1 of [9] for auxiliary norm). The full version of Theorem 11.15 can be found in [6].

Then Theorem 10.9 implies that each $f^{-1}(0) \cap \mathfrak{X}(Z, \nu)$ is a manifold with boundary and corners.

Theorem 11.16. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a section obtained by adding a $\circ$-compatible sc^+ perturbation to $\bar{\partial}_J$ that achieves general position. Then the k-th multiplication $m^k : \bigotimes^k C \to C$ defined for any $R = r_k \otimes \cdots \otimes r_1$ by*

$$\langle m^k(R), s\rangle := \sum_{z\in\mathfrak{M}_f([R,s])} e^{\omega(z)},$$

yields an A_∞ algebra.

Proof. First of all we note that another way of writing the k-th multiplication is

$$m^k(R, s) = \sum_{\nu\geq 0} n_{\mathbb{Z}_2}(\mathfrak{M}_f \cap \mathfrak{X}([R, s], \nu))\, e^{\nu},$$

where $n_{\mathbb{Z}_2}$ is the cardinality in $\mathbb{Z}_2$. By Theorem 11.15, $\mathfrak{M}_f \cap \mathfrak{X}([R,s],\nu)$ is a compact 0-dimensional manifold, so it has finite cardinality and therefore m^k is well-defined. In order to show $\widetilde{m} \circ \widetilde{m} = 0$, by Lemma 11.1 it suffices to show $\langle \widetilde{m} \circ \widetilde{m}(R), s\rangle = 0$. We have

$$\langle \widetilde{m} \circ \widetilde{m}(R), s\rangle = \sum_{R=P_1\otimes R'\otimes P_2} \langle \widetilde{m}(R'), s'\rangle \, \langle \widetilde{m}(P_1 \otimes s' \otimes P_2), s\rangle.$$

Let μ be the Maslov index such that $\operatorname{ind}([R,s;\mu]) = 1$ and we denote $A := [R,s;\mu]$. Then the above term is equal to

$$\begin{aligned} &\sum_{\substack{A=\circ_{\mathcal{T}}(\underline{Z}),\ |\mathrm{E}|=1 \\ \operatorname{ind}(Z^-)=\operatorname{ind}(Z^+)=0}} \left(\sum_{z^-\in\mathfrak{M}_f(Z^-)} e^{\omega(z^-)} \right) \left(\sum_{z^+\in\mathfrak{M}_f(Z^+)} e^{\omega(z^+)} \right) \\ &= \sum_{\nu\geq 0} n_{\mathbb{Z}_2}((\mathfrak{M}_f \circ \mathfrak{M}_f) \cap \mathfrak{X}(A,\nu))\, e^{\nu}. \end{aligned}$$

Since $\mathfrak{M}_f$ is $\circ$-invariant, we have $(\mathfrak{M}_f \circ \mathfrak{M}_f) \cap \mathfrak{X}(A,\nu) = \partial\mathfrak{M}_f \cap \mathfrak{X}(A,\nu)$. Again by Theorem 11.15, $\mathfrak{M}_f \cap \mathfrak{X}(A,\nu)$ is a compact 1-dimensional manifold with boundary, hence in $\mathbb{Z}_2$-counting $n_{\mathbb{Z}_2}(\partial\mathfrak{M}_f \cap \mathfrak{X}(A,\nu)) = 0$. This shows $\langle \widetilde{m} \circ \widetilde{m}(R), s\rangle = 0$, and thus $(C, \widetilde{m})$ is an A_∞ algebra.

□

11.6. **Gluing Near Boundary and Corners.**

Having understood the boundary and corners structure of $\mathfrak{X}$ the M-polyfold of disk trees, we try to study neighborhoods around $x \in \partial\mathfrak{X}$.

Any boundary point $x \in \partial\mathfrak{X}$ can be written as $x = \circ_{\mathcal{T}}(\underline{z})$, where $\mathcal{T}$ is the *maximal concatenation tree* of x, i.e., for each vertex v we have $d(z^{\mathrm{v}}) = 0$. Firstly, we fix an M-polyfold chart Θ on a neighborhood $\prod_{\mathrm{v}\in\mathrm{V}} \mathfrak{U}(z^{\mathrm{v}})$, where each $\mathfrak{U}(z^{\mathrm{v}})$ is a neighborhood of z^{v} in $\mathfrak{X}(Z^{\mathrm{v}})$. (See (6.12) for charts of the M-polyfolds of disk trees.) Then for a tuple of gluing parameters $r^{\mathrm{e}} \in [0,\varepsilon)$ and a tuple of disk trees $y^{\mathrm{v}} \in \mathfrak{U}(z^{\mathrm{v}})$, we define the **glued disk tree** $\sharp_{\mathcal{T},\underline{r}}(\underline{y})$ as follows.

Recall the definition of $\circ_{\mathcal{T}}(\underline{y})$ (Definition 11.9). For each edge $(\mathrm{v},\mathrm{w}) \in \mathrm{E}$, we denote

$$[\mathrm{T}^{\mathrm{v}}, \underline{\gamma}^{\mathrm{v}}, \underline{x}^{\mathrm{v}}, \underline{u}^{\mathrm{v}}] := y^{\mathrm{v}}, \quad [\mathrm{T}^{\mathrm{w}}, \underline{\gamma}^{\mathrm{w}}, \underline{x}^{\mathrm{w}}, \underline{u}^{\mathrm{w}}] := y^{\mathrm{w}}, \quad i := \iota^{(\mathrm{v},\mathrm{w})}.$$

As before, let rt^{v} be the root of T^{v}, and $\mathrm{lf}_i^{\mathrm{w}}$ the $\iota^{(\mathrm{v},\mathrm{w})}$-th leaf of T^{w}. Also, let $(\alpha, \mathrm{rt}^{\mathrm{v}})$ be the incoming edge of rt^{v}, and $(\mathrm{lf}_i^{\mathrm{w}}, \beta)$ the outgoing edge of $\mathrm{lf}_i^{\mathrm{w}}$. To construct the concatenated disk tree $\circ_{\mathcal{T}}(\underline{y}) = [\widetilde{\mathrm{T}}, \widetilde{\underline{\gamma}}, \widetilde{\underline{x}}, \widetilde{\underline{u}}]$, we delete the edges $(\alpha, \mathrm{rt}^{\mathrm{v}})$ from T^{v} and $(\mathrm{lf}_i^{\mathrm{w}}, \beta)$ from T^{w}, and connecting them by the edge (α,β). The generalized Morse trajectories at edge (α,β) are gotten by concatenating the unbroken flow line at $(\alpha, \mathrm{rt}^{\mathrm{v}})$ with the one at $(\mathrm{lf}_i^{\mathrm{w}}, \beta)$,

$$\widetilde{\underline{\gamma}}_{(\alpha,\beta)} := \left(\gamma^{\mathrm{v}}_{(\alpha,\mathrm{rt}^{\mathrm{v}})}, \gamma^{\mathrm{w}}_{(\mathrm{lf}_i^{\mathrm{w}},\beta)} \right).$$

Note that both are unbroken because $d(y^{\mathrm{v}}) = d(y^{\mathrm{w}}) = 0$. Subsequently we apply *Morse trajectories gluing* (2.4) at edge (α, β), i.e., replace $\widetilde{\underline{\gamma}}_{(\alpha,\beta)}$ by $\sharp_{r(\mathrm{v},\mathrm{w})}\left(\widetilde{\underline{\gamma}}_{(\alpha,\beta)}\right)$. If either α or β is a main vertex, then the Morse trajectories gluing changes end points on half-infinite trajectories. Hence we displace either the map u^{v}_{α} or u^{w}_{β} in the same way as (6.17) using the M-polyfold chart Θ to ensure coincidence condition.

Thus given a chart Θ, we have the **disk trees gluing map**, which is an sc-diffeomorphism onto a neighborhood of $\circ_{\mathcal{T}}(\underline{z})$,

$$
\begin{aligned}
[0,\varepsilon)^{\mathrm{E}} \times \prod_{\mathrm{v}\in\mathrm{V}} \mathfrak{U}(z^{\mathrm{v}}) &\to \mathfrak{X}(\circ_{\mathcal{T}}(\underline{Z})) \\
(\underline{r}, \underline{y}) &\mapsto \sharp_{\mathcal{T},\underline{r}}(\underline{y}).
\end{aligned}
\tag{11.19}
$$

Clearly when $\underline{r} = \underline{0}$, we have $\sharp_{\mathcal{T},\underline{0}}(\underline{y}) = \circ_{\mathcal{T}}(\underline{y})$. Consequently, the tangent space at $\circ_{\mathcal{T}}(\underline{z})$ admits the following isomorphism

$$
T_{\circ_{\mathcal{T}}(\underline{z})}\mathfrak{X} \simeq \mathbb{R}^{\mathrm{E}} \times \prod_{\mathrm{v}\in\mathrm{V}} T_{z^{\mathrm{v}}}\mathfrak{X}. \tag{11.20}
$$

The nearby neighborhood structure around $x \in \partial\mathfrak{X}$ becomes apparent using this map, and in the following section, we shall use this map to compute the linearized section at a concatenated disk tree.

12. Independence of the Almost-Complex Structures

Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be the bundle of complex anti-linear sections over the quotient space of disk trees. In Theorem 11.16, we construct an A_∞ algebra $(\widetilde{C}, \widetilde{m})$ by choosing any compatible almost-complex structure J and some sc^+ perturbation s. In order for $(\widetilde{C}, \widetilde{m})$ to be a symplectic invariant, we would like to show that the A_∞ algebra is independent of the choice of pairs (J, s). More precisely, given two such pairs (J_{-1}, s_{-1}) and (J_1, s_1) and let $(\widetilde{C}, \widetilde{m}_{-1})$ and $(\widetilde{C}, \widetilde{m}_1)$ be their respective A_∞ algebra, we need to construct an A_∞ isomorphism between $(\widetilde{C}, \widetilde{m}_{-1})$ and $(\widetilde{C}, \widetilde{m}_1)$. [6] deals with a similar problem in SFT. We also refer the readers to [12] and [11] for the case in generalized Morse theory, where the former uses an algebraic technique and the latter uses the obstruction bundle gluing.

To prove the above result, we first find a smooth 1-parameter family of almost-complex structures $(J_t)_{t\in[-1,1]}$ between J_{-1} and J_1. We can do this since the space of almost-complex structures compatible with ω is connected. The difficult part is finding a 1-parameter family of $\circ$-compatible perturbations $(s_t)_{t\in[-1,1]}$ so that the family of sections

$$
F : [-1,1] \times \mathfrak{X} \to \mathfrak{Y}, \quad (t, x) \mapsto (\bar{\partial}_{J_t} + s_t)(x)
$$

achieves a certain special transversality condition. The transversality problem is complicated by the presence of index -1 solutions: at certain irregular $t \in (-1, 1)$, the general position (Definition 10.8) breaks down, and at times even the good position (Definition 10.12) fails as well (see Proposition 12.14).

As a result, constructing an A_∞ isomorphism across those irregular times proves to be non-trivial. This section mainly deals with the A_∞ algebra at those irregularities.

12.1. **Index -1 and Index (-1, 0) Concatenation Trees.**

We first set up the terminologies for the index -1 type irregularity that occurs at certain $t \in (-1, 1)$.

Definition 12.1. An **index -1 concatenation tree of type A** is a concatenation tree $\mathcal{T} = (\mathrm{T}, \underline{Z}, \underline{\iota})$ such that $Z^{\mathrm{v}} = A$ with $\mathrm{ind}(A) = -1$ for every vertex v. In this case, we abbreviate the concatenated type as $\circ_{\mathcal{T}}(A)$.

Definition 12.2. An **index (-1,0) concatenation tree of type (A,B)** is a concatenation tree $\mathcal{T} = (\mathrm{T}, \underline{Z}, \underline{\iota})$ with a *center vertex* $\mathrm{ctr} \in \mathrm{V}$ such that $Z^{\mathrm{v}} = \begin{cases} A, & \mathrm{v} \neq \mathrm{ctr}, \\ B, & \mathrm{v} = \mathrm{ctr}, \end{cases}$ with $\mathrm{ind}(A) = -1$ and $\mathrm{ind}(B) = 0$. In this case, we abbreviate the concatenated type as $\circ_{\mathcal{T}}(A, B)$.

By the index formula (11.10), we have

$$\mathrm{ind}(\circ_{\mathcal{T}}(A)) = -1 \text{ and } \mathrm{ind}(\circ_{\mathcal{T}}(A, B)) = 0.$$

We adopt a similar notation for their corresponding concatenated disk trees, i.e., for $a \in \mathfrak{X}(A)$ and $b \in \mathfrak{X}(B)$, we abbreviate their concatenated disk tree as

$$\circ_{\mathcal{T}}(a) \text{ and } \circ_{\mathcal{T}}(a, b). \tag{12.1}$$

Definition 12.3. An index (-1,0) tree $\mathcal{T}'$ is a **sub-tree** of an index (-1,0) tree $\mathcal{T}$ if $\mathrm{ctr}' = \mathrm{ctr}$ and $\mathcal{T}'$ is a sub-tree of $\mathcal{T}$ as concatenation trees. Given an index (-1,0) tree $\mathcal{T}$, we define the **incoming sub-tree** $\mathcal{T}^-$ to be the maximal sub-tree containing $\mathrm{E}^{\mathrm{in}}(\mathrm{ctr})$ and not $\mathrm{E}^{\mathrm{out}}(\mathrm{ctr})$. Similarly, the **outgoing sub-tree** $\mathcal{T}^+$ is the maximal sub-tree containing $\mathrm{E}^{\mathrm{out}}(\mathrm{ctr})$ and not $\mathrm{E}^{\mathrm{in}}(\mathrm{ctr})$.

Note that the convention for naming $\mathcal{T}^-$ and $\mathcal{T}^+$ is consistent with the direction of the Morse trajectories.

Definition 12.4. We call an index -1 type $A = [P, q; \mu]$ **self-concatenating** if for $P = p_1 \otimes \cdots \otimes p_k$ we have $p_i = q$ for some i. We call A **singly self-concatenating** if there is exactly one such i, and **multiply self-concatenating** if there is more than one.

12.2. **Transversality of 1-family of Sc-Fredholm Sections.**

In order to construct an A_∞ isomorphism between A_∞ algebras arising from (J_{-1}, s_{-1}) and (J_1, s_1), we need a family of $\circ$-compatible perturbed sections that satisfies a certain transversality condition.

Definition 12.5. We say a 1-parameter family of sc-Fredholm sections

$$F : [-1, 1] \times \mathfrak{X} \to \mathfrak{Y}, \quad (t, x) \mapsto f_t(x)$$

is **$\circ$-compatible** if each f_t is $\circ$-compatible (see Definition 11.12).

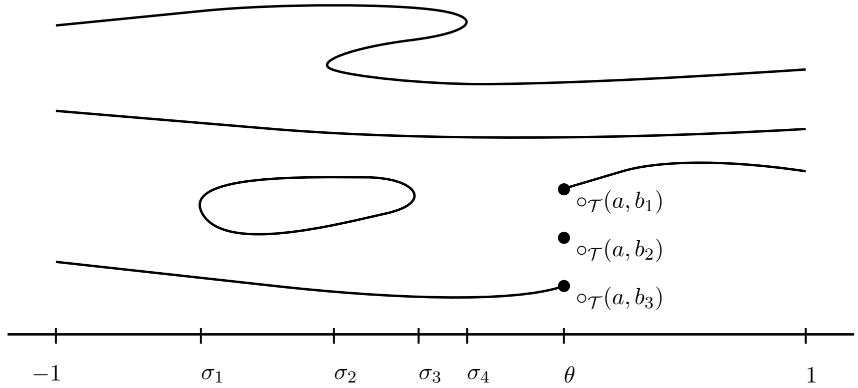


FIGURE 5. Cobordism $F^{-1}(0) \cap ([-1,1] \times \mathfrak{X}(Z,\nu))$

In order to study the relation between $(\widetilde{C}, \widetilde{m}_{-1})$ and $(\widetilde{C}, \widetilde{m}_1)$, we need to study the A_∞ algebra $(\widetilde{C}, \widetilde{m}_t)$ for t in between. Since by Theorem 11.16 $\widetilde{m}_t$ is defined by counting moduli spaces, here we denote the moduli space at time t by

$$\mathfrak{M}_t := f_t^{-1}(0) \text{ and } \mathfrak{M}_t(Z) := f_t^{-1}(0) \cap \mathfrak{X}(Z). \tag{12.2}$$

We now state the desired transversality condition for a 1-parameter family of sc-Fredholm sections.

As a primer, given a section $f : \mathfrak{X} \to \mathfrak{Y}$, suppose a concatenated disk tree $\circ_{\mathcal{T}}(\underline{z})$ with $|\mathrm{E}| = 1$ satisfies $f(\circ_{\mathcal{T}}(\underline{z}))$=0. Then the partial derivative with respect to the gluing parameter $\partial_r|_{r=0} f(\sharp_{\mathcal{T},r}(\underline{z}))$ lies in the product $\prod_{\mathrm{v}\in\mathrm{V}} \mathfrak{Y}_{z^{\mathrm{v}}}$. The projection to each component $pr^{\mathrm{v}}(\partial_r|_{r=0} f(\sharp_{\mathcal{T},r}(\underline{z})))$ plays an essential role in the following transversality condition.

Definition 12.6. We say a $\circ$-compatible 1-parameter family of sc-Fredholm sections $F : (t,x) \mapsto f_t(x)$ is **$\circ$-transverse** if there exists a *meager* set of **singular times** $\Sigma \subset (-1,1)$ and a *finite* set of **accident times** $\Theta \subset (-1,1)$ such that Σ and Θ are disjoint, and

1. $F|_{([-1,1]\setminus\Theta)\times\mathfrak{X}}$ is in general position,
2. for each index 0 type Z and $\nu \geq 0$, there exists a finite subset $\Sigma(Z,\nu) \subset (-1,1)$ with union $\bigcup_{Z,\nu} \Sigma(Z,\nu) = \Sigma$, such that
   (a) for $t \notin \Sigma(Z,\nu) \cup \Theta$, the section $f_t|_{\mathfrak{X}(Z,\nu)}$ is in general position,
   (b) for $\theta \in \Theta$, the section $f_\theta|_{\mathfrak{X}\setminus\partial\mathfrak{X}}$ is in general position,
3. for $\theta \in \Theta$, there is a unique index -1 solution $a \in \mathfrak{M}_\theta(A)$ with $d(a) = 0$. We call a an **accident solution** and A an **accident type**. Further assume that A is either *not self-concatenating* or *singly self-concatenating*, and moreover,
   (a) $Df_\theta(a)$ is injective and $\partial_t f_\theta(a) \notin \mathrm{im}(Df_\theta(a))$,

(b) for index -1 concatenation tree $\mathcal{T}$ of type A with a single edge $(\mathrm{v}^-, \mathrm{v}^+)$, let $\delta^\pm := pr^{\mathrm{v}^\pm}(\partial_r|_{r=0}\, f_\theta(\sharp_{\mathcal{T},r}(a)))$. We have

$$\delta^-, \delta^+, \delta^- - \delta^+ \notin \mathrm{im}(Df_\theta(a)),$$

(c) for each index (-1,0) concatenation tree $\mathcal{T}$ of type (A, B) with a single edge, and each index 0 solution $b \in \mathfrak{M}_\theta(B)$ with $d(b) = 0$, let v be the vertex of a and $\alpha := pr^{\mathrm{v}}(\partial_r|_{r=0}\, f_\theta(\sharp_{\mathcal{T},r}(a, b)))$. We have

$$\alpha \notin \mathrm{im}(Df_\theta(a)).$$

Remark 12.7. Assume F is $\circ$-transverse, and fix a sub-M-polyfold $\mathfrak{X}(Z, \nu)$ with index 0 type Z and $\nu \geq 0$. In Definition 12.6 (2), $\sigma \in \Sigma(Z, \nu)$ corresponds to a time when there is some solution $z \in \mathfrak{X}(Z, \nu)$ such that $Df_\sigma(z)$ has a one dimensional kernel. In this case, $\ker(DF(\sigma, z)) = \{0\} \oplus \ker(Df_\sigma(z))$. Pictorially, in Figure 5, $\Sigma(Z, \nu) = \{\sigma_i\}$ are the times when $F^{-1}(0)$ has a vertical tangent.

In Definition 12.6 (3), $\theta \in \Theta$ corresponds to a time when there is some boundary solution $\circ_{\mathcal{T}}(a, b) \in \partial\mathfrak{X}(Z)$ where $d(b) = 0$. Conversely, since a is the unique accident solution at θ, all boundary solutions $z \in \partial\mathfrak{X}(Z)$ with $\mathrm{ind}(Z) = 0$ is of the form $\circ_{\mathcal{T}}(a, b)$. Such a boundary solution is represented by a dot in Figure 5. Note that the degeneracy index $d(\circ_{\mathcal{T}}(a, b)) = |\mathrm{E}|$. If A is self-concatenating, then there exists index (-1,0) concatenation tree $\mathcal{T}$ with any number of edges. By Remark 10.10, F fails to be in general position at $(\theta, \circ_{\mathcal{T}}(a, b))$ when $|\mathrm{E}| > 1$ since the Fredholm index ind_F at this point is 1.

Remark 12.8. Definition 12.6 (3) is a simplistic assumption that enables our construction of A_∞ isomorphism. In general, an accident type A could be multiply self-concatenating, and the condition in Definition 12.6 (3) (b) would be more complicated due to the intricate interplay between different ways of a concatenating with itself. We believe by a similar construction in [6] there is a $\circ$-compatible 1-family of perturbations s_t so that $\bar{\partial}_{J_t} + s_t$ achieves the full transversality including multiply self-concatenating accident types.

By Definition 12.6 (2), f_t is in general position for $t \in [-1, 1] \setminus (\Sigma \cup \Theta)$, so it follows from Theorem 11.16 that $(\widetilde{C}, \widetilde{m}_t)$ is an A_∞ algebra. Assuming the transversality condition in Definition 12.6, we have the A_∞ isomorphism result.

Theorem 12.9. *Suppose there exists a 1-parameter family of $\circ$-compatible sc^+ perturbations so that the family $F(t, x) = (\bar{\partial}_{J_t} + s_t)(x)$ is $\circ$-transverse, then there exists an A_∞ isomorphism $\widetilde{\varphi} : (\widetilde{C}, \widetilde{m}_{-1}) \to (\widetilde{C}, \widetilde{m}_1)$.*

We shall prove this result by constructing an A_∞ isomorphism across each accident time, and then compose all A_∞ isomorphisms. The following section states the A_∞ isomorphism result for each accident time, and carries out this process in detail.

12.3. A_∞ Isomorphism Across Accident Times.

In this section, we address the major difficulty in Theorem 12.9: construct an A_∞ isomorphism across an accident time $\theta \in \Theta$. More precisely, we construct an A_∞ isomorphism between $(\widetilde{C}, \widetilde{m}_{\theta^-})$ and $(\widetilde{C}, \widetilde{m}_{\theta^+})$, where $\theta^-, \theta^+ \in [-1,1] \setminus (\Sigma \cup \Theta)$ are such that θ is the only accident time that lies in (θ^-, θ^+).

Suppose there exists a 1-parameter family of $\circ$-compatible sc^+ perturbations so that $F(t,x) = (\bar{\partial}_{J_t} + s_t)(x)$ is $\circ$-transverse. For a fixed accident time $\theta \in \Theta$, let

$$A = [P, q; \mu]$$

be the accident type. Moreover let a be the accident solution, and

$$\nu = \omega(a)$$

the pairing of a with the symplectic form. Then given $k \geq 0$ and an index -1 concatenation tree $\mathcal{T}$ of type A, we define a Λ-module homomorphism $\psi^k_{\mathcal{T}} : \bigotimes^k C \to C$ by defining on pure tensors,

$$\psi^k_{\mathcal{T}}(R) = \begin{cases} e^{|\mathrm{V}|\nu} q, & k = |\circ_{\mathcal{T}}(A)|^{\mathrm{in}},\ [R, q; |\mathrm{V}|\mu] = \circ_{\mathcal{T}}(A), \\ 0, & \text{otherwise.} \end{cases}$$

We reminder the reader that the identity Λ-module homomorphism $\mathrm{Id}^k : \bigotimes^k C \to C$ is of the form

$$\mathrm{Id}^k = \begin{cases} \mathrm{Id}, & k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The desired A_∞ isomorphism across the accident time θ can be expressed in terms of $\psi^k_{\mathcal{T}}$ and Id^k as follows.

Theorem 12.10. *Suppose a 1-parameter family of sc-Fredholm sections $F : (t,x) \mapsto f_t(x)$ is $\circ$-transverse. At an accident time θ, let a be the accident solution and and A the accident type. Define*

$$\varphi^k := \mathrm{Id}^k + \sum_{\mathcal{T} \text{ index -1 tree of type } A} \psi^k_{\mathcal{T}}.$$

Then the extension $\widetilde{\varphi} : \widetilde{C} \to \widetilde{C}$ given by (11.3) is a Λ-module isomorphism. Furthermore let $\theta^-, \theta^+ \in [-1,1] \setminus (\Sigma \cup \Theta)$ be such that θ is the only accident time that lies in (θ^-, θ^+). Then depending on properties of a, either $\widetilde{\varphi}$ or $\widetilde{\varphi}^{-1}$ is an A_∞ isomorphism from $(\widetilde{C}, \widetilde{m}_{\theta^-})$ to $(\widetilde{C}, \widetilde{m}_{\theta^+})$.

Assuming Theorem 12.10, it is straightforward to construct the desired A_∞ isomorphism in Theorem 12.9.

Proof of Theorem 12.9. The set of accident times Θ consists of $\theta_1 < \cdots < \theta_n$. We pick $-1 = \tau_1 < \cdots < \tau_{n+1} = 1$ such that $\tau_i < \theta_i < \tau_{i+1}$ and $\tau_i \notin \Sigma \cup \Theta$. We can do this since by Definition 12.6 the set Σ is meager. Now apply Theorem 12.10 we obtain an A_∞ isomorphism $\widetilde{\varphi}_i : (\widetilde{C}, \widetilde{m}_{\tau_i}) \to$

$(\widetilde{C}, \widetilde{m}_{\tau_{i+1}})$ for each i. Then $\widetilde{\varphi}_n \circ \cdots \circ \widetilde{\varphi}_1$ is an A_∞ isomorphism from $(\widetilde{C}, \widetilde{m}_{-1})$ to $(\widetilde{C}, \widetilde{m}_1)$. $\square$

The algebraic problem presented in Theorem 12.10 can be traced to the problem of finding a 1-parameter family of solutions to F near a solution of the form $(\theta, \circ_{\mathcal{T}}(a, b))$. Thus we study this 1-parameter gluing problem.

12.4. **1-Parameter Gluing Problem.**

We fix a $\circ$-transverse 1-parameter family of sc-Fredholm sections $F : (t, x) \mapsto f_t(x)$, and an accident time θ as in Definition 12.6 (3). Let a be the accident solution and A the accident type.

Let $\mathcal{T}$ be an index (-1,0) concatenation tree $\mathcal{T}$ of type (A, B), and $b \in \mathfrak{M}_\theta(B)$ a solution with $d(b) = 0$.

We denote by $\underline{z}(a, b)$ the tuple $z(a, b)^{\mathrm{v}} = \begin{cases} a, & \mathrm{v} \neq \mathrm{ctr}, \\ b, & \mathrm{v} = \mathrm{ctr}, \end{cases}$ and study the *1-parameter gluing problem*: solve the equation

$$F(t, \sharp_{\mathcal{T}, \underline{r}}(\underline{w})) = 0 \tag{12.3}$$

in a neighborhood of $(\theta, \circ_{\mathcal{T}}(a, b))$ in $[-1, 1] \times \mathfrak{X}$. (See Section 11.6 for disk trees gluing and a characterization of neighborhoods of $\circ_{\mathcal{T}}(a, b)$.) More precisely, we need to determine whether there exists a 1-parameter family

$$\big(t, \sharp_{\mathcal{T}, \underline{r}(t)}(\underline{w}(t))\big) \in F^{-1}(0), \quad \text{with } \underline{r}(\theta) = \underline{0},\ \underline{w}(\theta) = \underline{z}(a, b) \tag{12.4}$$

for $t < \theta$ or $t > \theta$, or not near θ.

We categorize all the possible outcomes of the 1-parameter gluing problem as follows.

Definition 12.11. Given an index (-1,0) concatenation tree $\mathcal{T}$ of type (A, B), we say the point $\circ_{\mathcal{T}}(a, b)$

- **glues to both left and right** if a 1-parameter family (12.4) exists for $t \in (\theta - \varepsilon, \theta + \varepsilon)$ for a small $\varepsilon > 0$,
- **glues to the left** if a 1-parameter family (12.4) exists for $t \in (\theta - \varepsilon, \theta]$ for a small $\varepsilon > 0$,
- **glues to the right** if a 1-parameter family (12.4) exists for $t \in [\theta, \theta + \varepsilon)$ for a small $\varepsilon > 0$,
- **fails to glue** if a 1-parameter family (12.4) does not exist near θ.

To encode all the cases, we define the **gluing directions** $c^{\pm}_{\mathcal{T}}(a, b)$ by

$$\big(c^-_{\mathcal{T}}(a, b), c^+_{\mathcal{T}}(a, b)\big) := \begin{cases} (1, 1), & \text{if } \circ_{\mathcal{T}}(a, b) \text{ glues to both left and right}, \\ (1, 0), & \text{if } \circ_{\mathcal{T}}(a, b) \text{ glues to the left}, \\ (0, 1), & \text{if } \circ_{\mathcal{T}}(a, b) \text{ glues to the right}, \\ (0, 0), & \text{if } \circ_{\mathcal{T}}(a, b) \text{ fails to glue}. \end{cases}$$

Example 12.12. Pictorially, in Figure 5, $\circ_{\mathcal{T}}(a, b_1)$ glues to the right, $\circ_{\mathcal{T}}(a, b_2)$ fails to glue, and $\circ_{\mathcal{T}}(a, b_3)$ glues to the left.

We now try to understand the transversality of F at $(\theta, \circ_{\mathcal{T}}(a,b))$. From Remark 12.7, we know F is *not* in general position at $(\theta, \circ_{\mathcal{T}}(a,b))$ if $|\mathrm{E}| > 1$. In the following text, we determine the transversality and gluing direction by studying the linearized equation of (12.3).

Recall from Section 11.6 that the disk trees gluing map (11.19)

$$(\underline{r}, \underline{w}) \mapsto \sharp_{\mathcal{T},\underline{r}}(\underline{w})$$

is an sc-diffeomorphism onto a neighborhood of $\circ_{\mathcal{T}}(a,b)$, and the tangent space at $\circ_{\mathcal{T}}(a,b)$ can be written as

$$T_{\circ_{\mathcal{T}}(a,b)}\mathfrak{X} \simeq \mathbb{R}^{\mathrm{E}} \times \prod_{\mathrm{v}\in\mathrm{V}} T_{z(a,b)^{\mathrm{v}}}\mathfrak{X}. \tag{12.5}$$

Under this identification, the tangent vectors are of the form

$$T_{\circ_{\mathcal{T}}(a,b)}\mathfrak{X} = \{(\underline{\rho}, \underline{\eta}) \,|\, \rho^{\mathrm{e}} \in \mathbb{R}, \eta^{\mathrm{v}} \in T_{z(a,b)^{\mathrm{v}}}\mathfrak{X}\},$$

and the partial cone $C_{\circ_{\mathcal{T}}(a,b)}\mathfrak{X}$ (see Definition 10.12) is identified with

$$C_{\circ_{\mathcal{T}}(a,b)}\mathfrak{X} = [0,\infty)^{\mathrm{E}} \times \prod_{\mathrm{v}\in\mathrm{V}} T_{z(a,b)^{\mathrm{v}}}\mathfrak{X}.$$

The following statement on the linearized section $Df_\theta(\circ_{\mathcal{T}}(a,b))$ is a partial result of Proposition 12.21, which we shall prove in the next section.

Proposition 12.13. *Let $\mathcal{T}$ be an index (-1,0) concatenation tree of type (A,B) and a solution $b \in \mathfrak{M}_\theta(B)$ with $d(b) = 0$. Then the linearized section $Df_\theta(\circ_{\mathcal{T}}(a,b))$ is an isomorphism, and the solution $(\underline{\rho}, \underline{\eta})$ to the equation*

$$-\partial_t f_\theta(\circ_{\mathcal{T}}(a,b)) = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta})$$

satisfies $\rho^{\mathrm{e}} \neq 0$ for all $\mathrm{e} \in \mathrm{E}$.

Then it follows that the linearized section $DF(\theta, \circ_{\mathcal{T}}(a,b))$ is surjective. To understand if F is in good position at $(\theta, \circ_{\mathcal{T}}(a,b))$, it suffices to study the 1-dimensional kernel of $DF(\theta, \circ_{\mathcal{T}}(a,b))$ in relation to the partial cone at $(\theta, \circ_{\mathcal{T}}(a,b))$.

Proposition 12.14. *Let $\mathcal{T}$ be an index (-1,0) concatenation tree of type (A,B) and a solution $b \in \mathfrak{M}_\theta(B)$ with $d(b) = 0$. Let $\rho^{\mathrm{e}} \in \mathbb{R}$ and $\eta^{\mathrm{v}} \in T_{z(a,b)^{\mathrm{v}}}\mathfrak{X}$ be tangent vectors such that the equation*

$$-\partial_t f_\theta(\circ_{\mathcal{T}}(a,b)) = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta})$$

is satisfied. Then we can conclude the following about the gluing direction at $\circ_{\mathcal{T}}(a,b)$.

(1) *For $|\mathrm{E}| = 0$, F is in general position at $(\theta, \circ_{\mathcal{T}}(a,b))$ and $\circ_{\mathcal{T}}(a,b) = b$ glues to both left and right.*
(2) *For $|\mathrm{E}| \geq 1$, we have*
   - *if $\underline{\rho} \in (-\infty, 0)^{\mathrm{E}}$, then F is in good position at $(\theta, \circ_{\mathcal{T}}(a,b))$ and $\circ_{\mathcal{T}}(a,b)$ glues to the left,*
   - *if $\underline{\rho} \in (0, \infty)^{\mathrm{E}}$, then F is in good position at $(\theta, \circ_{\mathcal{T}}(a,b))$ and $\circ_{\mathcal{T}}(a,b)$ glues to the right, or*

- *if* $\underline{\rho} \in (\mathbb{R} \setminus \{0\})^{\mathrm{E}}$ *but not in the above two scenarios, then* F *is not in good position at* $(\theta, \circ_{\mathcal{T}}(a,b))$ *and* $\circ_{\mathcal{T}}(a,b)$ *fails to glue.*

Proof. In order to determine whether a 1-parameter family $\big(t, \sharp_{\mathcal{T},\underline{r}(t)}(\underline{w}(t))\big)$ in $F^{-1}(0)$ exists for $t < \theta$ or $t > \theta$, we study the linearized equation of $F(t, \sharp_{\mathcal{T},\underline{r}}(\underline{w})) = 0$ at $(\theta, \circ_{\mathcal{T}}(a,b))$. In other words we find the set of tangent vectors $(h, (\underline{\rho}, \underline{\eta}))$ such that

$$DF(\theta, \circ_{\mathcal{T}}(a,b))(h, (\underline{\rho}, \underline{\eta})) = 0.$$

We re-write the above equation as

$$-\partial_t f_\theta(\circ_{\mathcal{T}}(a,b))\, h = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta}).$$

Suppose $h = 0$, then by Proposition 12.13, $(\underline{\rho}, \underline{\eta}) = \underline{0}$. Suppose $h \neq 0$, without loss of generality, assume $h = 1$. Then it follows from Proposition 12.13 that there exists a *unique* solution $(\underline{\rho}, \underline{\eta})$. Thus

$$\ker(DF(\theta, \circ_{\mathcal{T}}(a,b))) = \mathrm{span}\{(1, (\underline{\rho}, \underline{\eta}))\}.$$

Moreover, under the identification of (12.5), the partial cone at $(\theta, \circ_{\mathcal{T}}(a,b))$ is of the form

$$C_{(\theta, \circ_{\mathcal{T}}(a,b))}\left([-1,1] \times \mathfrak{X}\right) = \mathbb{R} \times \left([0,\infty)^{\mathrm{E}} \times \prod_{\mathrm{v} \in \mathrm{V}} T_{z(a,b)^{\mathrm{v}}} \mathfrak{X}\right),$$

where $\mathbb{R}$ is the tangent space at θ.

By Proposition 12.13, the linearized section $DF(\theta, \circ_{\mathcal{T}}(a,b))$ is surjective. Furthermore, $\ker(DF(\theta, \circ_{\mathcal{T}}(a,b)))$ is in good position to the partial cone if and only if $\underline{\rho} \in (-\infty, 0)^{\mathrm{E}}$ or $\underline{\rho} \in (0,\infty)^{\mathrm{E}}$. Suppose $\underline{\rho} \in (0,\infty)^{\mathrm{E}}$, then $(1, (\underline{\rho}, \underline{\eta}))$ lies in the partial cone, and by a local version of Theorem 10.14, there exists a 1-parameter family of solution for $t \in [\theta, \theta + \varepsilon)$. On the other hand, suppose $\underline{\rho} \in (-\infty, 0)^{\mathrm{E}}$, then $(-1, (\underline{\rho}, \underline{\eta}))$ lies in the partial cone, and there exists a 1-parameter family of solution for $t \in (\theta - \varepsilon, \theta]$. Finally, the very last scenario in Proposition 12.14 is obvious. $\square$

With our knowledge of 1-parameter gluing problem, we now prove Theorem 12.10.

12.5. **Proof of A_∞ Isomorphism.**

In this section, we prove the A_∞ isomorphism result in Theorem 12.10. We fix an accident time θ, and let a be the accident solution and A the accident type.

We start by showing the map $\widetilde{\varphi} : \widetilde{C} \to \widetilde{C}$ in Theorem 12.10 is a Λ-module isomorphism.

Lemma 12.15. *The Λ-module homomorphism $\widetilde{\varphi} : \widetilde{C} \to \widetilde{C}$ in Theorem 12.10 is an isomorphism. In particular, if the accident type A is* not self-concatenating*, then* $\widetilde{\varphi}^{-1} = \widetilde{\varphi}$.

Proof. Let $\mathcal{T}_1$ be the index -1 concatenation tree of type A with a single vertex. Define an Λ-module homomorphism $\xi^k : \bigotimes^k C \to C$ by $\xi^k = \mathrm{Id}^k + \psi^k_{\mathcal{T}_1}$. One can verify that $\widetilde{\xi}$ is the inverse of $\widetilde{\varphi}$. In the case when A not self-concatenating, $\varphi^k = \mathrm{Id}^k + \psi^k_{\mathcal{T}_1}$ and therefore $\widetilde{\varphi}$ is its own inverse. $\square$

Note that the above proof for Λ-module isomorphism does not exclude multiply self-concatenating A.

We now prove the A_∞ homomorphism result of Theorem 12.10. From here on in, we shall always assume that the 1-parameter family of sc-Fredholm sections $F : (t, x) \mapsto f_t(x)$ is $\circ$-transverse.

By Definition 12.6 (3), the linearized section $Df_\theta(a)$ is injective, so its image $\mathrm{im}(Df_\theta(a))$ has codimension 1 in the fiber $\mathfrak{Y}_a$. Thus we denote the quotient map by

$$[\cdot] : \mathfrak{Y}_a \to \mathfrak{Y}_a/\mathrm{im}(Df_\theta(a)) \simeq \mathbb{R}.$$

Suppose $\gamma, \gamma' \in \mathfrak{Y}_a$ satisfies $\gamma' \notin \mathrm{im}(Df_\theta(a))$, then the ratio $\frac{[\gamma]}{[\gamma']}$ is a well-defined real number. Indeed, we can define it by choosing an identification $\mathfrak{Y}_a/\mathrm{im}(Df_\theta(a)) \simeq \mathbb{R}$, and one can show that the ratio is independent of the choice of identification.

We now define an important quantity, which as we shall is closely related to the map $\widetilde{\varphi}$ in Theorem 12.10.

Definition 12.16. Suppose A is *singly self-concatenating*, define the **self-concatenation factor** $\tau(a) := \left(1 - \frac{[\delta^+]}{[\delta^-]}\right) \frac{[-\partial_t f_\theta(a)]}{[\delta^-]}$. (See Definition 12.6 (3) for δ^-, δ^+.)

We now state the A_∞ homomorphism property in Theorem 12.10.

Theorem 12.17. *Let $\theta^-, \theta^+ \in [-1, 1] \setminus (\Sigma \cup \Theta)$ be such that θ is the only accident time that lies in (θ^-, θ^+).*

(1) *Suppose A is either* not self-concatenating *or* singly self-concatenating *with $\tau(a) < 0$. Then $\widetilde{\varphi} : (\widetilde{C}, \widetilde{m}_{\theta^-}) \to (\widetilde{C}, \widetilde{m}_{\theta^+})$ is an A_∞ homomorphism.*

(2) *Suppose A is either* not self-concatenating *or* singly self-concatenating *with $\tau(a) > 0$. Then $\widetilde{\varphi} : (\widetilde{C}, \widetilde{m}_{\theta^+}) \to (\widetilde{C}, \widetilde{m}_{\theta^-})$ is an A_∞ homomorphism.*

Proof of Theorem 12.10. The A_∞ homomorphism property of Theorem 12.17 and the module isomorphism result of Lemma 12.15 imply the desired A_∞ isomorphism property. $\square$

We now prepare to prove the A_∞ homomorphism property of $\widetilde{\varphi}$. To that end, it is often convenient to use the language of types in order to express the multiplication $\widetilde{m}_t$. Hence we define the *type counting function* as follows.

Definition 12.18. Given an index 0 type Z and $t \in [-1,1] \setminus (\Sigma \cup \Theta)$, we define the **type counting function** $H_t(Z)$ by

$$H_t(Z) := \sum_{z \in \mathfrak{M}_t(Z)} e^{\omega(z)},$$

where $\mathfrak{M}_t(Z)$ is equal to $f_t^{-1}(0) \cap \mathfrak{X}(Z)$, the moduli space of type Z at time t.

This definition is simply rephrasing the definition of the multiplication $\widetilde{m}_t$: for each pure tensor $R = r_l \otimes \cdots \otimes r_1$ and s, where r_i and s are Morse critical points, we have by definition

$$\langle \widetilde{m}_t(R), s \rangle = H_t([R, s]). \tag{12.6}$$

Around an accident time θ, we can express the type counting function at θ^-, θ^+ in terms of gluing directions as follows. As we shall see, it proves to be useful to translate the A_∞ homomorphism property of $\widetilde{\varphi}$ in Theorem 12.17 into equations of gluing directions.

Lemma 12.19. *Let $\theta^-, \theta^+ \in [-1,1] \setminus (\Sigma \cup \Theta)$ be such that θ is the only accident time that lies in (θ^-, θ^+). Then for an index 0 type Z, we can express $H_{\theta^\pm}(Z)$ in terms of gluing direction at θ as follows.*

$$H_{\theta^\pm}(Z) = \sum_{\substack{\{(B,\mathcal{T}) \,|\, Z = \circ_{\mathcal{T}}(A,B)\} \\ b \in \mathfrak{M}_\theta(B),\, d(b)=0}} e^{|\mathrm{E}|\nu + \omega(b)} c^{\pm}_{\mathcal{T}}(a,b).$$

We recall that $\nu = \omega(a)$ is the pairing of the accident solution a with the symplectic form ω.

Proof. We shall prove the lemma for θ^+, as the case for θ^- is completely analogous. By definition, it suffices to show that for each index 0 type Z we have

$$\sum_{z \in \mathfrak{M}_{\theta^+}(Z)} e^{\omega(z)} = \sum_{\substack{\{(B,\mathcal{T}) \,|\, Z = \circ_{\mathcal{T}}(A,B)\} \\ b \in \mathfrak{M}_\theta(B),\, d(b)=0}} e^{|\mathrm{E}|\nu + \omega(b)} c^{+}_{\mathcal{T}}(a,b). \tag{12.7}$$

For each index 0 type Z and $\xi \geq 0$, we define $\mathfrak{N}(Z, \xi)$ to be the set of tuples $(b, B, \mathcal{T})$ such that

- $Z = \circ_{\mathcal{T}}(A, B)$,
- $b \in \mathfrak{M}_\theta(B)$, $d(b) = 0$, and $\circ_{\mathcal{T}}(a, b)$ glues to the right, and
- $|\mathrm{E}|\nu + \omega(b) = \xi$.

Thus to prove (12.7), it suffices to show that for fixed index 0 type Z and $\zeta \geq 0$, we have the same $\mathbb{Z}_2$ counting

$$n_{\mathbb{Z}_2}(\mathfrak{M}_{\theta^+} \cap \mathfrak{X}(Z, \xi)) = n_{\mathbb{Z}_2}(\mathfrak{N}(Z, \xi)). \tag{12.8}$$

This above identity follows from a standard cobordism argument. $\square$

Recall from Definition 12.3, given an index (-1,0) concatenation tree $\mathcal{T}$, the sub-trees $\mathcal{T}^-$ and $\mathcal{T}^+$ are the incoming and outgoing sub-tree, respectively. The following result shows that the A_∞ homomorphism property of $\widetilde{\varphi}$ in Theorem 12.17 follows from an equation of gluing directions involving $\mathcal{T}^-$ and $\mathcal{T}^+$.

Lemma 12.20. *Let $\theta^-, \theta^+ \in [-1,1] \setminus (\Sigma \cup \Theta)$ be such that θ is the only accident time that lies in (θ^-, θ^+).*

(1) *Suppose for all index (-1,0) concatenation tree $\mathcal{T}$ of type (A, B) and all $b \in \mathfrak{M}_\theta(B)$ with $d(b) = 0$, we have*

$$\sum_{\mathcal{T}^- \le \mathcal{T}' \le \mathcal{T}} c^-_{\mathcal{T}'}(a,b) = \sum_{\mathcal{T}^+ \le \mathcal{T}' \le \mathcal{T}} c^+_{\mathcal{T}'}(a,b).$$

Then $\widetilde{\varphi} : (\widetilde{C}, \widetilde{m}_{\theta^-}) \to (\widetilde{C}, \widetilde{m}_{\theta^+})$ is an A_∞ homomorphism.

(2) *Suppose for all index (-1,0) concatenation tree $\mathcal{T}$ of type (A, B) and all $b \in \mathfrak{M}_\theta(B)$ with $d(b) = 0$, we have*

$$\sum_{\mathcal{T}^- \le \mathcal{T}' \le \mathcal{T}} c^+_{\mathcal{T}'}(a,b) = \sum_{\mathcal{T}^+ \le \mathcal{T}' \le \mathcal{T}} c^-_{\mathcal{T}'}(a,b).$$

Then $\widetilde{\varphi} : (\widetilde{C}, \widetilde{m}_{\theta^+}) \to (\widetilde{C}, \widetilde{m}_{\theta^-})$ is an A_∞ homomorphism.

Note that this lemma does not assume A is either not self-concatenating or singly self-concatenating.

Proof. We only prove (1) because (2) is analogous. In order to show $\widetilde{\varphi} : (\widetilde{C}, \widetilde{m}_{\theta^-}) \to (\widetilde{C}, \widetilde{m}_{\theta^+})$ is an A_∞ homomorphism, it suffices to prove the following equation holds for each pure tensor $R = r_l \otimes \cdots \otimes r_1$ and s, where r_i and s are Morse critical points,

$$\langle \widetilde{\varphi} \circ \widetilde{m}_{\theta^-}(R), s\rangle = \langle \widetilde{m}_{\theta^+} \circ \widetilde{\varphi}(R), s\rangle. \tag{12.9}$$

By the form of $\widetilde{\varphi}$ in Theorem 12.10, we can re-write (12.9) in terms of type counting functions, and incoming sub-trees $\mathcal{T}^-$ and outgoing sub-trees $\mathcal{T}^+$ (Definition 12.3),

$$\sum_{\substack{\{(B,\mathcal{T}) \,|\, \mathcal{T}=\mathcal{T}^+, \\ [R,s]=\circ_{\mathcal{T}}(A,B)\}}} e^{|\mathrm{E}|\nu} H_{\theta^-}(B) = \sum_{\substack{\{(B,\mathcal{T}) \,|\, \mathcal{T}=\mathcal{T}^-, \\ [R,s]=\circ_{\mathcal{T}}(A,B)\}}} e^{|\mathrm{E}|\nu} H_{\theta^+}(B). \tag{12.10}$$

We first analyze the right-hand side of (12.10). By Lemma 12.19, it is given by

$$\begin{aligned}
&\sum_{\substack{\{(B,\mathcal{T})\,|\,\mathcal{T}=\mathcal{T}^-,\\ [R,s]=\circ_{\mathcal{T}}(A,B)\}}} e^{|\mathrm{E}|\nu} H_{\theta+}(B) \\
&= \sum_{\substack{\{(B',\widetilde{\mathcal{T}})\,|\,\widetilde{\mathcal{T}}=\widetilde{\mathcal{T}}^-,\\ [R,s]=\circ_{\widetilde{\mathcal{T}}}(A,B')\}}} e^{|\mathrm{E}|\nu} \sum_{\substack{\{(B,\mathcal{T}')\,|\,B'=\circ_{\mathcal{T}'}(A,B)\}\\ b\in\mathfrak{M}_\theta(B),\, d(b)=0}} e^{|\mathrm{E}'|\nu+\omega(b)} c^{+}_{\mathcal{T}'}(a,b) \\
&= \sum_{\substack{(B,\widetilde{\mathcal{T}},\mathcal{T}')\in\widetilde{\mathcal{S}}^-\\ b\in\mathfrak{M}_\theta(B),\, d(b)=0}} e^{(|\widetilde{\mathrm{E}}|+|\mathrm{E}'|)\nu+\omega(b)} c^{+}_{\mathcal{T}'}(a,b),
\end{aligned}\tag{12.11}$$

where $\widetilde{\mathcal{S}}^- = \{(B,\widetilde{\mathcal{T}},\mathcal{T}')\,|\,\widetilde{\mathcal{T}}=\widetilde{\mathcal{T}}^-,\ [R,s]=\circ_{\widetilde{\mathcal{T}}}(A,\circ_{\mathcal{T}'}(A,B))\}$. The expression (12.11) is equal to

$$\sum_{\substack{(B,\mathcal{T},\mathcal{T}')\in\mathcal{S}^+\\ b\in\mathfrak{M}_\theta(B),\, d(b)=0}} e^{|\mathrm{E}|\nu+\omega(b)} c^{+}_{\mathcal{T}'}(a,b),\tag{12.12}$$

where $\mathcal{S}^+ = \{(B,\mathcal{T},\mathcal{T}')\,|\,\mathcal{T}^+\le\mathcal{T}'\le\mathcal{T},\ [R,s]=\circ_{\mathcal{T}}(A,B)\}$. This is due to the associativity $\circ_{\mathcal{T}}(A,B)=\circ_{\widetilde{\mathcal{T}}}(A,\circ_{\mathcal{T}'}(A,B))$ as discussed in Remark 11.8, and the observation that $\widetilde{\mathcal{T}}=\widetilde{\mathcal{T}}^-$ iff $\mathcal{T}^+\le\mathcal{T}'$. Moreover, it is easy to see $|\mathrm{E}|=|\widetilde{\mathrm{E}}|+|\mathrm{E}'|$. Also we can conclude the same about the left-hand side of (12.10). In summary, we have

$$\sum_{\substack{\{(B,\mathcal{T})\,|\,\mathcal{T}=\mathcal{T}^{\mp},\\ [R,s]=\circ_{\mathcal{T}}(A,B)\}}} e^{|\mathrm{E}|\nu} H_{\theta\pm}(B) = \sum_{\substack{(B,\mathcal{T},\mathcal{T}')\in\mathcal{S}^{\pm}\\ b\in\mathfrak{M}_\theta(B),\, d(b)=0}} e^{|\mathrm{E}|\nu+\omega(b)} c^{\pm}_{\mathcal{T}'}(a,b),$$

where $\mathcal{S}^{\pm} = \{(B,\mathcal{T},\mathcal{T}')\,|\,\mathcal{T}^{\pm}\le\mathcal{T}'\le\mathcal{T},\ [R,s]=\circ_{\mathcal{T}}(A,B)\}$.

Thus to prove (12.10), it suffices to show that for all fixed index (-1,0) concatenation tree $\mathcal{T}$ of type (A,B) and all $b\in\mathfrak{M}_\theta$ with $d(b)=0$, the following equation holds,

$$\sum_{\mathcal{T}^-\le\mathcal{T}'\le\mathcal{T}} c^{-}_{\mathcal{T}'}(a,b) = \sum_{\mathcal{T}^+\le\mathcal{T}'\le\mathcal{T}} c^{+}_{\mathcal{T}'}(a,b).$$

This proves Lemma 12.20. $\square$

Having translated the A_∞ homomorphism property into an equation of gluing directions, it suffices to prove the gluing directions equation is satisfied.

Proposition 12.21. *Assume that A is not multiply self-concatenating. Let $\mathcal{T}$ be an index (-1,0) concatenation tree of type (A,B) and a solution $b\in\mathfrak{M}_\theta(B)$ with $d(b)=0$. Then the linearized section $Df_\theta(\circ_{\mathcal{T}}(a,b))$ is an isomorphism, and the solution $(\underline{\rho},\underline{\eta})$ to the equation*

$$-\partial_t f_\theta(\circ_{\mathcal{T}}(a,b)) = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho},\underline{\eta})$$

satisfies $\rho^{\mathrm{e}} \neq 0$ *for all* $\mathrm{e} \in \mathrm{E}$.

(1) *Suppose A is either* not self-concatenating *or* singly self-concatenating *with* $\tau(a) < 0$. *Then we have*

$$\sum_{\mathcal{T}^- \leq \mathcal{T}' \leq \mathcal{T}} c^-_{\mathcal{T}'}(a,b) = \sum_{\mathcal{T}^+ \leq \mathcal{T}' \leq \mathcal{T}} c^+_{\mathcal{T}'}(a,b).$$

(2) *Suppose A is either* not self-concatenating *or* singly self-concatenating *with* $\tau(a) > 0$. *Then we have*

$$\sum_{\mathcal{T}^- \leq \mathcal{T}' \leq \mathcal{T}} c^+_{\mathcal{T}'}(a,b) = \sum_{\mathcal{T}^+ \leq \mathcal{T}' \leq \mathcal{T}} c^-_{\mathcal{T}'}(a,b).$$

Proof of Theorem 12.17. Proposition 12.21 shows that the equations of gluing directions are satisfied when A is not multiply self-concatenating. Lemma 12.20 then shows that the desired A_∞ homomorphism property is satisfied. □

In order to prove Proposition 12.21, we start by proving the desired result for a special class of concatenation trees.

Lemma 12.22. *Proposition 12.21 is true while assuming that the center vertex* ctr *of* $\mathcal{T}$ *is either a leaf or the root with a single edge.*

Proof. Fix an index (-1,0) concatenation tree $\mathcal{T}$ of type (A,B) and fix $b \in \mathfrak{M}_\theta(B)$ with $d(b) = 0$. Assume that ctr is the root and it has a single edge.

Since A is not multiply self-concatenating, $\mathcal{T}$ has a linear structure. More precisely, let n be the number of edges of $\mathcal{T}$. Then its set of vertices is of the form $\mathrm{V} := \{\mathrm{v}_0 = \mathrm{ctr}, \mathrm{v}_1, \ldots, \mathrm{v}_n\}$, and its set of edges $\mathrm{E} := \{\mathrm{e}_1 = (\mathrm{v}_1, \mathrm{v}_0), \ldots, \mathrm{e}_n = (\mathrm{v}_n, \mathrm{v}_{n-1})\}$. In order to show that $Df_\theta(\circ_{\mathcal{T}}(a,b))$ is an isomorphism, we prove that the equation

$$\underline{u} = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta}) \tag{12.13}$$

has a unique solution $(\underline{\rho}, \underline{\eta})$, and we shall do so by studying each component

$$pr^{\mathrm{v}}(\underline{u}) = pr^{\mathrm{v}}\left(Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta})\right) \tag{12.14}$$

for each $\mathrm{v} \in \mathrm{V}$.

If A is singly self-concatenating, then for an index -1 sub-tree $\mathcal{T}'$ of type A with a single edge $(\mathrm{v}^-, \mathrm{v}^+)$, we denote

$$\delta^{\pm} := pr^{\mathrm{v}^{\pm}}\left(\partial_r|_{r=0}\, f_\theta(\sharp_{\mathcal{T}',r}(a))\right),$$

and recall from Definition 12.6 (3) that $\delta^-, \delta^+, \delta^- - \delta^+ \notin \mathrm{im}(Df_\theta(a))$. Also for the index (-1,0) sub-tree $\mathcal{T}'$ of type (A,B) with a single edge $(\mathrm{v}, \mathrm{ctr})$, where v, ctr corresponds to a, b respectively, we denote

$$\alpha := pr^{\mathrm{v}}(\partial_r|_{r=0}\, f_\theta(\sharp_{\mathcal{T}',r}(a,b))), \;\; \beta := pr^{\mathrm{ctr}}(\partial_r|_{r=0}\, f_\theta(\sharp_{\mathcal{T}',r}(a,b))),$$

and we have $\alpha \notin \mathrm{im}(Df_\theta(a))$.

Abbreviating each ρ^{e_k} as ρ^k and η^{v_k} as η^k, we express (12.14) componentwise as follows.

For $n = 0$,

$$u^0 = Df_\theta(b)(\eta^0).$$

For $n = 1$,

$$\begin{aligned} u^1 &= Df_\theta(a)(\eta^1) + \rho^1\alpha, \\ u^0 &= Df_\theta(b)(\eta^0) + \rho^1\beta. \end{aligned}$$

For $n \geq 2$,

$$\begin{aligned} u^n &= Df_\theta(a)(\eta^n) + \rho^n\delta^-, \\ &\dots \\ u^k &= Df_\theta(a)(\eta^k) + \rho^k\delta^- + \rho^{k+1}\delta^+, \quad 2 \leq k \leq n-1 \\ &\dots \\ u^1 &= Df_\theta(a)(\eta^1) + \rho^1\alpha + \rho^2\delta^+, \\ u^0 &= Df_\theta(b)(\eta^0) + \rho^1\beta. \end{aligned}$$

Note that if A is not self-concatenating, then $n = 0, 1$.

In the following computation, we keep in mind that Definition 12.6 (2)(c) implies that $Df_\theta(b)$ is an isomorphism since $d(b) = 0$. Thus for $n = 0$, we can solve $u^0 = Df_\theta(b)(\eta^0)$. For $n = 1$, we apply the quotient map

$$[\cdot] : \mathfrak{Y}_a \to \mathfrak{Y}_a/\mathrm{im}(Df_\theta(a))$$

on both sides of the top equation and get $[u^1] = \rho^1[\alpha]$. Since by assumption $[\alpha] \neq 0$, we have $\rho^1 = [u^1]/[\alpha]$. This implies $u^1 - \rho^1\alpha \in \mathrm{im}(Df_\theta(a))$, so there is a unique η^1 that satisfies $u^1 = Df_\theta(a)(\eta^1) + \rho^1\alpha$. Lastly, plug in ρ_1 in the bottom equation and we solve η^0.

For $n \geq 2$, we proceed similar as in the case of $n = 1$. Apply the quotient map $[\cdot]$ to the top equation and get $\rho^1 = [u^1]/[\delta^-]$, and then solve for η^n. Inductively for $2 \leq k \leq n-1$, suppose we know (ρ^{k+1}, η^{k+1}). Then we can solve for ρ^k by using $[u^k] = \rho^k[\delta^-] + \rho^{k+1}[\delta^+]$, and we subsequently solve for η^k. Similarly we find (ρ^1, η^1) and (ρ^0, η^0). This finishes proving $Df_\theta(\circ_{\mathcal{T}}(a, b))$ is an isomorphism.

Now the equation

$$-\partial_t f_\theta(\circ_{\mathcal{T}}(a,b)) = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta}) \tag{12.15}$$

is a special case of (12.13), where $u^k = -\partial_t f_\theta(a)$ for $k \geq 1$ and $u^0 = -\partial_t f_\theta(b)$. Thus by studying the component-wise equation (12.14), we find for $n \geq 1$,

(12.16)

$$\begin{aligned} \rho^1 &= \frac{[-\partial_t f_\theta(a)]}{[\alpha]} \left(1 + \left(-\frac{[\delta^+]}{[\delta^-]} \right) + \cdots + \left(-\frac{[\delta^+]}{[\delta^-]} \right)^{n-1} \right), \\ \rho^k &= \frac{[-\partial_t f_\theta(a)]}{[\delta^-]} \left(1 + \left(-\frac{[\delta^+]}{[\delta^-]} \right) + \cdots + \left(-\frac{[\delta^+]}{[\delta^-]} \right)^{n-k} \right), \text{ for } 2 \leq k \leq n. \end{aligned}$$

Note that $1-\frac{[\delta^+]}{[\delta^-]}\neq 0$ because $\delta^- - \delta^+ \notin \mathrm{im}(Df_\theta(a))$. Moreover, the geometric sum

$$1+x+\cdots+x^m \tag{12.17}$$

is non-zero if $1+x\neq 0$. This along with the condition $\partial_t f_\theta(a)\notin \mathrm{im}(Df_\theta(a))$ shows $\rho^k\neq 0$.

We now study the gluing direction of sub-trees of $\mathcal{T}$. We observe that the set of all sub-trees of $\mathcal{T}$ is $\{\mathcal{T}^0,\cdots,\mathcal{T}^n=\mathcal{T}\}$, where each $\mathcal{T}^i$ has i edges. For $i=0$, by Proposition 12.14 we have $c^-_{\mathcal{T}^0}(a,b)=c^+_{\mathcal{T}^0}(a,b)=1$. And for $i\geq 1$, we use (12.16) and Proposition 12.14 to analyze the coefficients $c^\pm_{\mathcal{T}^i}(a,b)$. We discuss this problem by cases. (In the following text we omit the (a,b) in the $c^\pm_{\mathcal{T}}(a,b)$ notation because they are fixed.)

(1) Suppose A is either *not self-concatenating* or *singly self-concatenating* with self-concatenating factor $\tau(a)<0$. To prove Proposition 12.22 in this case, it suffices to show

$$\sum_{\mathcal{T}^-\leq\mathcal{T}'\leq\mathcal{T}} c^-_{\mathcal{T}'}=\sum_{\mathcal{T}^+\leq\mathcal{T}'\leq\mathcal{T}} c^+_{\mathcal{T}'},$$

which can be re-written as

$$c^-_{\mathcal{T}^n}=\sum_{i=0}^n c^+_{\mathcal{T}^i}. \tag{12.18}$$

For $n=0$, we have $c^-_{\mathcal{T}^0}=1$ and $c^+_{\mathcal{T}^0}=1$. Hence (12.18) is satisfied.

For $n=1$, we discuss the following cases.

i) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]}>0$, then $c^-_{\mathcal{T}^1}=0$, and $c^+_{\mathcal{T}^i}=1$ for $i=0,1$.

ii) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]}<0$, then $c^-_{\mathcal{T}^1}=1$, and $c^+_{\mathcal{T}^i}=1$ for $i=0$ only. In both cases, (12.18) is satisfied.

For $n\geq 2$, note that A must be singly self-concatenating with

$$\tau(a)=\left(1-\frac{[\delta^+]}{[\delta^-]}\right)\frac{[-\partial_t f_\theta(a)]}{[\delta^-]}<0.$$

We discuss the following cases.

I) $1-\frac{[\delta^+]}{[\delta^-]}>0$, $\frac{[-\partial_t f_\theta(a)]}{[\delta^-]}<0$.

i) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]}>0$, then $c^-_{\mathcal{T}^n}=0$, and $c^+_{\mathcal{T}^i}=1$ for $i=0,1$ only.

ii) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]}<0$, then $c^-_{\mathcal{T}^n}=1$, and $c^+_{\mathcal{T}^i}=1$ for $i=0$ only. In both sub-cases, (12.18) is satisfied.

II) $1-\frac{[\delta^+]}{[\delta^-]}<0$, $\frac{[-\partial_t f_\theta(a)]}{[\delta^-]}>0$.

i) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]}>0$, then $c^-_{\mathcal{T}^n}=0$, and $c^+_{\mathcal{T}^i}=1$ for $i=0,1$ only.

ii) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]}<0$, then $c^-_{\mathcal{T}^n}=0$, and $c^+_{\mathcal{T}^i}=1$ for $i=0,2$ only. In both sub-cases, (12.18) is satisfied.

(2) Suppose A is either *not self-concatenating* or *singly self-concatenating* with self-concatenating factor $\tau(a) > 0$. To prove Proposition 12.22 in this case, it suffices to show

$$\sum_{\mathcal{T}^- \leq \mathcal{T}' \leq \mathcal{T}} c^+_{\mathcal{T}'} = \sum_{\mathcal{T}^+ \leq \mathcal{T}' \leq \mathcal{T}} c^-_{\mathcal{T}'},$$

which can be re-written as

$$c^+_{\mathcal{T}^n} = \sum_{i=0}^{n} c^-_{\mathcal{T}^i}. \tag{12.19}$$

For $n = 0$, we have $c^+_{\mathcal{T}^0} = 1$ and $c^-_{\mathcal{T}^0} = 1$. Hence (12.19) is satisfied.

For $n = 1$, we discuss the following cases.

i) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]} > 0$, then $c^+_{\mathcal{T}^1} = 1$, and $c^-_{\mathcal{T}^i} = 1$ for $i = 0$ only.

ii) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]} < 0$, then $c^+_{\mathcal{T}^1} = 0$, and $c^-_{\mathcal{T}^i} = 1$ for $i = 0, 1$. In both cases, (12.19) is satisfied.

For $n \geq 2$, note that A must be singly self-concatenating with $\tau(a) > 0$. We discuss the following cases.

I) $1 - \frac{[\delta^+]}{[\delta^-]} > 0$, $\frac{[-\partial_t f_\theta(a)]}{[\delta^-]} > 0$.

i) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]} > 0$, then $c^+_{\mathcal{T}^n} = 1$, and $c^-_{\mathcal{T}^i} = 1$ for $i = 0$ only.

ii) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]} < 0$, then $c^+_{\mathcal{T}^n} = 0$, and $c^-_{\mathcal{T}^i} = 1$ for $i = 0, 1$ only. In both sub-cases, (12.19) is satisfied.

II) $1 - \frac{[\delta^+]}{[\delta^-]} < 0$, $\frac{[-\partial_t f_\theta(a)]}{[\delta^-]} < 0$.

i) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]} > 0$, then $c^+_{\mathcal{T}^n} = 0$, and $c^-_{\mathcal{T}^i} = 1$ for $i = 0, 2$ only.

ii) $\frac{[-\partial_t f_\theta(a)]}{[\alpha]} < 0$, then $c^+_{\mathcal{T}^n} = 0$, and $c^-_{\mathcal{T}^i} = 1$ for $i = 0, 1$ only. In both sub-cases, (12.19) is satisfied.

This finishes the proof of the case when ctr is the root with a single edge.

Now suppose the center vertex ctr of $\mathcal{T}$ is a leaf. Then its set of vertices is of the form $\mathrm{V} := \{\mathrm{v}_0 = \mathrm{ctr}, \mathrm{v}_1, \ldots, \mathrm{v}_n\}$, and its set of edges is given by $\mathrm{E} := \{\mathrm{e}_1 = (\mathrm{v}_0, \mathrm{v}_1), \ldots, \mathrm{e}_n = (\mathrm{v}_{n-1}, \mathrm{v}_n)\}$. The process of solving the equation

$$\underline{u} = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta})$$

is similar as above. And the solution to the equation

$$-\partial_t f_\theta(\circ_{\mathcal{T}}(a,b)) = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta})$$

for $n \geq 1$ is given by,

$$\rho^1 = \frac{[-\partial_t f_\theta(a)]}{[\alpha]}\left(1 + \left(-\frac{[\delta^-]}{[\delta^+]}\right) + \cdots + \left(-\frac{[\delta^-]}{[\delta^+]}\right)^{n-1}\right),$$

$$\rho^k = \frac{[-\partial_t f_\theta(a)]}{[\delta^-]}\left(1 + \left(-\frac{[\delta^-]}{[\delta^+]}\right) + \cdots + \left(-\frac{[\delta^-]}{[\delta^+]}\right)^{n-k}\right), \text{ for } 2 \leq k \leq n.$$

Similarly, we can discuss by cases according to the sign of $\left(1 - \frac{[\delta^-]}{[\delta^+]}\right) \frac{[-\partial_t f_\theta(a)]}{[\delta^+]}$, which is the opposite sign of $\tau(a)$ because

$$\left(1 - \frac{[\delta^-]}{[\delta^+]}\right) \frac{[-\partial_t f_\theta(a)]}{[\delta^+]} = -\left(\frac{[\delta^+]}{[\delta^-]}\right)^2 \tau(a).$$

This finishes proving Proposition 12.22.

□

In the rest of this section, we use the result in Lemma 12.22 for the special class of concatenation trees to prove the result for the general case in Proposition 12.21.

Let $\mathcal{T}$ be an index (-1,0) concatenation tree type (A, B) and a solution $b \in \mathfrak{M}_\theta(B)$ with $d(b) = 0$. Moreover, let $\mathrm{e}^1, \cdots, \mathrm{e}^k$ be the set of incoming edges of the center vertex ctr ordered clockwise, and e^0 the outgoing edge of ctr, if it has one. We denote by J the index set of all edges of ctr, i.e., $J := \{1, \ldots, k\}$ if ctr is the root, or $J := \{0, 1, \ldots, k\}$ otherwise.

For $j \in J$, let $\mathcal{T}_j$ be the maximal sub-tree of $\mathcal{T}$ that contains e^j and no other edges of ctr. We call such $\mathcal{T}_j$ a **main branch**. Note that each main branch $\mathcal{T}_j$ satisfies the assumption of Lemma 12.22. The following lemma shows that $Df_\theta(\circ_{\mathcal{T}}(a, b))$ is an isomorphism by using the knowledge that for each main branch $Df_\theta(\circ_{\mathcal{T}_j}(a, b))$ is an isomorphism. Furthermore, it expresses the gluing direction in terms of the gluing direction of each main branch.

Lemma 12.23. *Assume that A is not multiply self-concatenating. Let $\mathcal{T}$ be an index (-1,0) concatenation tree of type (A, B) and a solution $b \in \mathfrak{M}_\theta(B)$ with $d(b) = 0$. Then the linearized section $Df_\theta(\circ_{\mathcal{T}}(a, b))$ is an isomorphism, and the solution $(\underline{\rho}, \underline{\eta})$ to the equation*

$$-\partial_t f_\theta(\circ_{\mathcal{T}}(a, b)) = Df_\theta(\circ_{\mathcal{T}}(a, b))(\underline{\rho}, \underline{\eta})$$

satisfies $\rho^{\mathrm{e}} \neq 0$ for all $\mathrm{e} \in \mathrm{E}$.

Furthermore, let J be the index set of all edges of ctr*, and each $\mathcal{T}_j$ a main branch for $j \in J$. Then we have*

$$c^{\pm}_{\mathcal{T}}(a, b) = \prod_{j \in J} c^{\pm}_{\mathcal{T}_j}(a, b).$$

Proof. In order to show $Df_\theta(\circ_{\mathcal{T}}(a, b))$ is an isomorphism, it suffices to prove that the equation

$$\underline{u} = Df_\theta(\circ_{\mathcal{T}}(a, b))(\underline{\rho}, \underline{\eta}) \tag{12.20}$$

has a unique solution $(\underline{\rho}, \underline{\eta})$. We now solve its each component

$$pr^{\mathrm{v}}(\underline{u}) = pr^{\mathrm{v}}\left(Df_\theta(\circ_{\mathcal{T}}(a, b))(\underline{\rho}, \underline{\eta})\right) \tag{12.21}$$

for each $\mathrm{v} \in \mathrm{V}$. By the construction of main branch $\mathcal{T}_j$, it is straightforward to see $\mathrm{V} = \left(\bigsqcup_{j \in J} (\mathrm{V}_j \setminus \{\mathrm{ctr}\})\right) \sqcup \{\mathrm{ctr}\}$ and $\mathrm{E} = \bigsqcup_{j \in J} \mathrm{E}_j$. Hence we solve (12.21) by each main branch as follows.

For $j \in J$, it follows from Lemma 12.22 that there is a unique solution $(\underline{\rho}_j, \underline{\eta}_j)$ to the equation for each main branch

$$\underline{u}_j = Df_\theta(\circ_{\mathcal{T}_j}(a,b))(\underline{\rho}_j, \underline{\eta}_j).$$

Recall that the set of edges of ctr is $\mathrm{E}(\mathrm{ctr}) = \{\mathrm{e}^j\}_{j\in J}$. For each $j \in J$, let $\mathcal{T}_j'$ be the sub-tree of $\mathcal{T}_j$ with a single edge e^j. We denote

$$\beta_j := pr^{\mathrm{ctr}}(\partial_r|_{r=0}\, f_\theta(\sharp_{\mathcal{T}_j',r}(a,b))).$$

Hence for $\mathrm{v} = \mathrm{ctr}$, (12.21) is given by

$$u^{\mathrm{ctr}} = Df_\theta(b)(\eta^{\mathrm{ctr}}) + \sum_{j\in J} \rho_j^{\mathrm{e}^j} \beta_j.$$

The above equation has a unique solution η^{ctr} since Definition 12.6 (2)(c) implies that $Df_\theta(b)$ is an isomorphism.

Now let $\underline{\rho} = (\rho^{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$ and $\underline{\eta} = (\eta^{\mathrm{v}})_{\mathrm{v}\in\mathrm{V}}$ be tuples given by

$$\rho^{\mathrm{e}} := \rho_j^{\mathrm{e}} \text{ for } \mathrm{e} \in \mathrm{E}_j, \quad \eta^{\mathrm{v}} := \begin{cases} \eta_j^{\mathrm{v}}, & \mathrm{v} \in \mathrm{V}_j \setminus \{\mathrm{ctr}\}, \\ \eta^{\mathrm{ctr}}, & \mathrm{v} = \mathrm{ctr}. \end{cases}$$

Clearly $(\underline{\rho}, \underline{\eta})$ solves (12.20).

Now as a special case of (12.20), we carry out the above procedure to solve the equation

$$-\partial_t f_\theta(\circ_{\mathcal{T}}(a,b)) = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta}).$$

By the construction, each component of $\underline{\rho}$ being positive is equivalent to each component of $\underline{\rho}_j$ being positive for all $j \in J$. The corresponding statement is true for the negative case. Then it follows from Proposition 12.14 that $c^{\pm}_{\mathcal{T}}(a,b) = 1$ iff $c^{\pm}_{\mathcal{T}_j}(a,b) = 1$ for all $j \in J$. Hence

$$c^{\pm}_{\mathcal{T}}(a,b) = \prod_{j\in J} c^{\pm}_{\mathcal{T}_j}(a,b).$$

□

With the knowledge that $c^{\pm}_{\mathcal{T}}(a,b)$ can be expressed as a product of $c^{\pm}_{\mathcal{T}_j}(a,b)$ for each main branch, and the fact that each main branch $\mathcal{T}_j$ satisfies the assumption of Lemma 12.22, we now prove the desired gluing directions equation in Proposition 12.21.

Proof of Proposition 12.21. Lemma 12.23 shows that $Df_\theta(\circ_{\mathcal{T}}(a,b))$ is an isomorphism, and the solution $(\underline{\rho}, \underline{\eta})$ to $-\partial_t f_\theta(\circ_{\mathcal{T}}(a,b)) = Df_\theta(\circ_{\mathcal{T}}(a,b))(\underline{\rho}, \underline{\eta})$ satisfies $\rho^{\mathrm{e}} \neq 0$.

Now it suffices to prove the gluing directions equation in Proposition 12.21. Fix an index (-1,0) concatenation tree $\mathcal{T}$ of type (A,B) and fix $b \in \mathfrak{M}_\theta(B)$ with $d(b) = 0$. In the following computation, we abbreviate $c^{\pm}_{\mathcal{T}'}(a,b)$ as $c^{\pm}_{\mathcal{T}'}$ for all $\mathcal{T}'$.

Suppose A is either *not self-concatenating*, or *singly self-concatenating* with self-concatenating factor $\tau(a) < 0$. We shall prove the desired equation

$$\sum_{\mathcal{T}^- \leq \mathcal{T}' \leq \mathcal{T}} c^-_{\mathcal{T}'} = \sum_{\mathcal{T}^+ \leq \mathcal{T}' \leq \mathcal{T}} c^+_{\mathcal{T}'} \tag{12.22}$$

by first assuming $\mathcal{T} = \mathcal{T}^-$, and then prove it for $\mathcal{T}$ in general.

Suppose we have $\mathcal{T} = \mathcal{T}^-$. Then proving (12.22) is equivalent to proving

$$c^-_{\mathcal{T}} = \sum_{\mathcal{T}' \leq \mathcal{T}} c^+_{\mathcal{T}'}. \tag{12.23}$$

We now use Lemma 12.23 to express $c^-_{\mathcal{T}}$ as the product of gluing direction of each main branch

$$c^-_{\mathcal{T}} = \prod_{j \in J} c^-_{\mathcal{T}_j} = \prod_{j \in J} \sum_{\mathcal{T}'_j \leq \mathcal{T}_j} c^+_{\mathcal{T}'_j} = \sum_{\mathcal{T}' \leq \mathcal{T}} c^+_{\mathcal{T}'}.$$

The second equality follows from Lemma 12.22 since the center vertex ctr of each $\mathcal{T}_j$ is the root with a single edge. The third equality is by Lemma 12.23 again. Thus equation (12.23) is satisfied.

Now remove the $\mathcal{T} = \mathcal{T}^-$ restriction. The sub-tree $\mathcal{T}^+$ is a main branch of $\mathcal{T}$, and the center vertex ctr of $\mathcal{T}^+$ is a leaf. Hence by Lemma 12.22, we have

$$\sum_{\mathcal{T}' \leq \mathcal{T}^+} c^-_{\mathcal{T}'} = c^+_{\mathcal{T}^+}. \tag{12.24}$$

It follows from Lemma 12.23 and then (12.24) that

$$\sum_{\mathcal{T}^- \leq \mathcal{T}' \leq \mathcal{T}} c^-_{\mathcal{T}'} = c^-_{\mathcal{T}^-} \sum_{\mathcal{T}' \leq \mathcal{T}^+} c^-_{\mathcal{T}'} = c^-_{\mathcal{T}^-}\, c^+_{\mathcal{T}^+}.$$

On the other hand, by Lemma 12.23 we have

$$\sum_{\mathcal{T}^+ \leq \mathcal{T}' \leq \mathcal{T}} c^+_{\mathcal{T}'} = c^+_{\mathcal{T}^+} \sum_{\mathcal{T}' \leq \mathcal{T}^-} c^+_{\mathcal{T}'} = c^+_{\mathcal{T}^+}\, c^-_{\mathcal{T}^-},$$

where the last equality follows from (12.23) in the previous paragraph. Comparing the above two equations, we conclude equation (12.22) in general.

Now suppose A is either *not self-concatenating*, or *singly self-concatenating* with self-concatenating factor $\tau(a) > 0$. We can prove the corresponding equation

$$\sum_{\mathcal{T}^- \leq \mathcal{T}' \leq \mathcal{T}} c^+_{\mathcal{T}'} = \sum_{\mathcal{T}^+ \leq \mathcal{T}' \leq \mathcal{T}} c^-_{\mathcal{T}'}$$

in a similar way as above. This finishes proving Proposition 12.21. $\square$

13. Appendix

13.1. **Disk Automorphism.**

Let $\mathrm{Aut}(D)$ denote the group of holomorphic disk automorphisms. For $\eta \in \partial D$ and $a \in D^\circ$, we define the Möbius transformation

$$\psi_{\eta,a}(z) = \eta \frac{z-a}{\bar{a}z-1}. \tag{13.1}$$

The map

$$\begin{aligned} \partial D \times D^\circ &\to \mathrm{Aut}(D) \\ (\eta, a) &\mapsto \psi_{\eta,a} \end{aligned}$$

is bijective. We use this bijection to give $\mathrm{Aut}(D)$ a topology and a 3-dimensional Lie group structure.

The following result shows that marked points satisfying the stability condition (5.1) have no non-trivial automorphism.

Lemma 13.1. *Suppose we have boundary marked points and interior marked points $(\underline{x}, O)$ satisfying either $n(\underline{x}) \geq 3$ or $n(O) \geq 1$. Suppose a disk automorphism ψ satisfies $\psi(\underline{x}, O) = (\underline{x}, O)$, then $\psi = \mathrm{Id}$.*

Remark 13.2. The disk automorphism group $\mathrm{Aut}(D)$ admits a natural compactification $\overline{\mathrm{Aut}}(D)$ given by the (η, a) coordinates in (13.1)

$$\begin{aligned} \partial D \times D &\to \overline{\mathrm{Aut}}(D) \\ (\eta, a) &\mapsto \psi_{\eta,a}. \end{aligned}$$

For $a \in \partial D$, $\psi_{\eta,a}$ is not a map, but it arises naturally as the limit of a sequence of disk automorphisms.

We now examine the convergence property of elements of $\mathrm{Aut}(D)$.

Lemma 13.3. *Let ψ_{η_n,a_n} be a sequence in $\mathrm{Aut}(D)$, and suppose the sequence $(\eta_n, a_n) \in \partial D \times D^\circ$ converges to $(\eta, a) \in \partial D \times D$. Then the convergence of ψ_{η_n,a_n} is one of the following two cases.*

1. *If $a \in D^\circ$, then ψ_{η_n,a_n} converges to $\psi_{\eta,a}$ uniformly on D.*
2. *If $a \in \partial D$, then ψ_{η_n,a_n} converges uniformly on compact subsets of $D \setminus \{a\}$ to the constant map $z \mapsto \eta a$.*

Proof. If $a \in D^\circ$, then there is some $r \in (0,1)$ such that $|a_n| \leq r$. Hence

$$\begin{aligned} &|\psi_{\eta_n,a_n}(z) - \psi_{\eta,a}(z)| \\ &= \left| \eta_n \frac{z-a_n}{\bar{a}_n z - 1} - \eta \frac{z-a}{\bar{a}z-1} \right| \\ &= \frac{|\eta_n (z-a_n)(\bar{a}z-1) - \eta(z-a)(\bar{a}_n z - 1)|}{|(\bar{a}_n z - 1)(\bar{a}z-1)|}. \end{aligned}$$

The denominator of the last expression is bounded below by $(1-r)^2$ and the numerator converges to 0 uniformly on D.

If $a \in \partial D$, then write $\psi_{\eta_n,a_n}(z)$ as

$$\psi_{\eta_n,a_n}(z) = \eta_n \frac{z - a_n}{\bar{a}_n z - 1} = \left(\frac{\eta_n}{\bar{a}_n}\right) \frac{z - a_n}{z - \frac{1}{\bar{a}_n}}.$$

We note that $\frac{1}{\bar{a}_n} \to \frac{1}{\bar{a}} = a$. Therefore $\frac{\eta_n}{\bar{a}_n} \to \eta a$. Now it suffices to show that for any given compact subset K of $D \setminus \{a\}$, we have $\frac{z-a_n}{z-\frac{1}{\bar{a}_n}} \to 1$ uniformly on K. We have

$$\left|\frac{z - a_n}{z - \frac{1}{\bar{a}_n}} - 1\right| = \left|\frac{\frac{1}{\bar{a}_n} - a_n}{z - \frac{1}{\bar{a}_n}}\right| = \frac{1 - |a_n|^2}{|a_n|} \cdot \frac{1}{|z - \frac{1}{\bar{a}_n}|},$$

where $\frac{1-|a_n|^2}{|a_n|} \to 0$. Moreover, since $\frac{1}{\bar{a}_n} \to a$ and K is bounded away from a, we have $|z - \frac{1}{\bar{a}_n}|$ bounded away from 0 for n large enough. This finishes the proof. $\square$

Due to the compactness of $\partial D \times D$, every sequence $(\eta_n, a_n) \in \partial D \times D^\circ$ has a subsequence that converges to some $(\eta, a) \in \partial D \times D$. The following is an immediate corollary of the above lemma.

Corollary 13.4. *Given a sequence $\psi_n \in \mathrm{Aut}(D)$, there is a subsequence ψ_{n_j} that either*

(1) *converges uniformly on D to some $\psi \in \mathrm{Aut}(D)$, or*
(2) *there are $a, b \in \partial D$, such that ψ_{n_j} converges uniformly on compact subsets of $D \setminus \{a\}$ to the constant map $z \mapsto b$.*

The following discussion uses the notion of Hausdorff distance of two sets in a metric space, which we recall for reference.

Definition 13.5. Let X and Y be two non-empty compact subsets of a metric space (M, d). Then we define their **Hausdorff distance** by

$$d(X, Y) := \max\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\}.$$

Recall that the Hausdorff distance on the collection of non-empty compact subsets of M is a metric. Hausdorff convergent subsets behave well with uniformly convergent maps.

Lemma 13.6. *Let $\psi_n : X \to Y$ be a sequence of continuous maps that converges uniformly to a uniformly continuous map $\psi : X \to Y$. Suppose $W_n \subset X$ is a sequence of non-empty compact sets that converges to a non-empty compact set $V \subset X$ in the Hausdorff distance. Then we have $\psi_n(W_n) \to \psi(V)$ in the Hausdorff distance.*

The following proposition is crucial in Deligne-Mumford space.

Proposition 13.7. *Suppose we have boundary marked points and interior marked points $(\underline{\bar{x}}, \bar{O}), (\underline{\bar{x}}', \bar{O}')$ satisfying either $n(\underline{\bar{x}}) \geq 3$ or $n(\bar{O}) \geq 1$, and $(\underline{\bar{x}}', \bar{O}')$ satisfying the same cardinality condition. Suppose we have the following sequences.*

- $x_{n,0}, x'_{n,0}$ *two sequences of points on* ∂D *such that* $x_{n,0} \to \bar{x}_0$ *and* $x'_{n,0} \to \bar{x}'_0$,
- W_n, W'_n *two sequences of non-empty compact subsets of* D *such that* $W_n \to \{\bar{x}_i\}_{i>0} \cup \bar{O}$ *and* $W'_n \to \{\bar{x}'_i\}_{i>0} \cup \bar{O}'$ *in Hausdorff distance,*
- ψ_n *a sequence in* $\mathrm{Aut}(D)$ *such that* $\psi_n(x_{n,0}) = x'_{n,0}$ *and* $\psi_n(W_n) = W'_n$.

Then there is a subsequence ψ_{n_j} *such that* $\psi_{n_j} \to \bar{\psi}$, *where* $\bar{\psi}$ *is a disk automorphism such that* $\bar{\psi}(\underline{\bar{x}}, \bar{O}) = (\underline{\bar{x}}', \bar{O}')$.

In order to extract a subsequence that converges as disk automorphisms, it is important to have the convergence of a distinguished marked point $x_{n,0}, x'_{n,0}$, and the convergence of W_n, W'_n to marked points away from $\bar{x}_0, \bar{x}'_0$. Indeed, one can construct a sequence ψ_{η_n,a_n} with $a_n \to a \in \partial D$ such that $\psi_{\eta_n,a_n}(W_n) = W'_n$ with $W_n \to \underline{\bar{x}} \cup \bar{O}$ and $W'_n \to \underline{\bar{x}}' \cup \bar{O}'$.

Proof. Corollary 13.4 states that there is a subsequence ψ_{n_j} that converges either in the sense of (1) or in the sense of (2). Now we show that the type (2) convergence is not possible by deriving a contradiction.

Suppose there are $a, b \in \partial D$, and a subsequence (which we also call ψ_n for convenience) converges uniformly on compact subsets of $D \setminus \{a\}$ to the constant map $\phi : z \mapsto b$. Then we have the following two cases.

Case (i) $a = \bar{x}_0$.
Since $W_n \to \{\bar{x}_i\}_{i>0} \cup \bar{O}$, then for n large enough, W_n is contained in $D \setminus B(\bar{x}_0)$. Hence ψ_n convergs to the constant map $z \mapsto b$ uniformly on $D \setminus B(\bar{x}_0)$. By Lemma 13.6, $W'_n \to \{b\}$. However, $W'_n \to \{\bar{x}'_i\}_{i>0} \cup \bar{O}'$. This yields a contradiction since either $n(\underline{\bar{x}}') \geq 3$ or $n(\bar{O}') \geq 1$.

Case (ii) $a \neq \bar{x}_0$.
Then the sequence $x_{n,0}$ is contained in some compact subset of $D \setminus \{a\}$. By Lemma 13.6, $x'_{n,0} \to b$. Since $x'_{n,0} \to \bar{x}'_0$ by assumption, we have $b = \bar{x}'_0$.

We choose a neighborhood $B(a)$ of a in D such that $\overline{B(a)}$ is disjoint from $\bar{O}$ and contains *at most* one element of $\{\bar{x}_i\}_{i>0}$. Since either $n(\underline{\bar{x}}) \geq 3$ or $n(\bar{O}) \geq 1$, the set $W_n \setminus B(a) \neq \emptyset$ for large n. By Lemma 13.6, $\psi_n(W_n \setminus B(a)) \to \{\bar{x}'_0\}$. This contradicts with the assumption $W'_n \to \{\bar{x}'_i\}_{i>0} \cup \bar{O}'$. This proves that the type (2) convergence in Corollary 13.4 is not possible.

Therefore, there is a subsequence (also denote by ψ_n) that converges to some $\bar{\psi} \in \mathrm{Aut}(D)$. Applying Lemma 13.6, we conclude that $\bar{\psi}(\bar{x}_0) = \bar{x}'_0$ and $\bar{\psi}\left(\{\bar{x}_i\}_{i>0} \cup \bar{O}\right) = \{\bar{x}'_i\}_{i>0} \cup \bar{O}'$. Hence $\bar{\psi}(\underline{\bar{x}}, \bar{O}) = (\underline{\bar{x}}', \bar{O}')$. □

We now provide an alternative way to parametrize the disk automorphism group $\mathrm{Aut}(D)$ besides the (η, a) coordinates in (13.1).

Proposition 13.8. *The disk automorphism group* $\mathrm{Aut}(D)$ *admits the following parametrizations.*

(1) *Fix a boundary marked point and an interior marked point* $(\hat{x}, \hat{o})$. *Then for any boundary marked point and an interior marked point*

(x,o), *there is a unique disk automorphism* $\psi_{(x,o)}$ *such that*

$$\psi_{(x,o)}(\hat{x},\hat{o}) = (x,o).$$

(2) *Fix boundary marked points* $\underline{\hat{x}} = (\hat{x}_0, \hat{x}_1, \hat{x}_2)$*. Then for any boundary marked points* $\underline{x} = (x_0, x_1, x_2)$*, there is a unique disk automorphism* $\psi_{\underline{x}}$ *such that*

$$\psi_{\underline{x}}(\underline{\hat{x}}) = \underline{x}.$$

This result is an elementary exercise in complex analysis.

13.2. **Exponential and Logarithm Gluing Profiles.**

Let $\varphi(r) = e^{1/r} - e$ be the exponential gluing profile, and $\rho(r) = -\ln(r)$ the logarithmic gluing profile. Define

$$\eta := \rho^{-1} \circ \varphi. \tag{13.2}$$

The function η and its inverse are given by

$$\eta(r) = \begin{cases} \exp(-\exp(1/r) + e), & r > 0, \\ 0, & r = 0. \end{cases} \quad \eta^{-1}(r) = \begin{cases} \frac{1}{\ln(-\ln(r)+e)}, & r > 0, \\ 0, & r = 0. \end{cases} \tag{13.3}$$

Note that both η and η^{-1} are continuous; η is smooth with $\eta'(0) = 0$ and η^{-1} is smooth away from 0.

Let $g : (r,y) \mapsto r'$ be the transition function of gluing parameters under logarithmic gluing profile for $r, r' \geq 0$ with $g(0,y) = 0$. Then the transition function of gluing parameters under the exponential gluing profile is given by $\eta^{-1}(g(\eta(r),y))$. The following result shows that it extends smoothly by identity for $r < 0$.

Theorem 13.9. *Let* $g : [0,\infty) \times \mathbb{R} \to [0,\infty)$ *be a smooth function with* $g(0,y) = 0$ *and* $\partial_x g(0,y) > 0$*. Then the composite function* $f(r,y) := \eta^{-1}(g(\eta(r),y))$ *is smooth at* $(0,y)$ *and its derivatives are*

$$\partial_r^m \partial_y^n f(0,y) = \begin{cases} 1, & m = 1, n = 0, \\ 0, & \textit{otherwise}. \end{cases}$$

We shall use the following limits throughout the proof of Theorem 13.9.

Lemma 13.10. *For any* $k, l \geq 0$*, we have*

(1) $\lim_{x\to 0} x \ln(-\ln(x) + e)^l = 0$.
(2) $\lim_{x\to 0} \frac{\ln(-\ln(x)+e)^l}{-\ln(x)+e} = 0$.
(3) $\lim_{r\to 0} \frac{\eta^{(k)}(r)}{r^l} = 0$.

Proof. We can compute the above limits by L'Hôpital's rule. □

Proof of Theorem 13.9. To compute the derivative of f, we shall use the factorization

$$\partial_r f(r,y) = V(r,y) W(r,y)^2, \quad W(r,y) = \frac{f(r,y)}{r}, \tag{13.4}$$

with

$$V(r,y) := v(\eta(r),y), \quad v(x,y) := \frac{\partial_x g(x,y)x}{g(x,y)} \frac{-\ln(x)+e}{-\ln(g(x,y))+e},$$

$$W(r,y) := w(\eta(r),y), \quad w(x,y) := \frac{\ln(-\ln(x)+e)}{\ln(-\ln(g(x,y))+e)}.$$

We now examine the derivatives of V and W as follows. Here we adopt the multi-index notation $\alpha = (m,n)$ with

$$|\alpha| := m+n, \quad \partial^\alpha := \partial_x^m \partial_y^n.$$

We shall prove that for all multi-index α with $|\alpha| \geq 1$ and $l \geq 0$, we have

$$\lim_{(r,y)\to(0,\hat{y})} \frac{V(r,y)-1}{r^l} = 0, \quad \lim_{(r,y)\to(0,\hat{y})} \frac{\partial^\alpha V(r,y)}{r^l} = 0, \tag{13.5}$$

$$\lim_{(r,y)\to(0,\hat{y})} \frac{W(r,y)-1}{r^l} = 0, \quad \lim_{(r,y)\to(0,\hat{y})} \frac{\partial^\alpha W(r,y)}{r^l} = 0. \tag{13.6}$$

We show that the above limits imply the theorem. By (13.4), we have $\partial_r f(0,y) = \lim_{r\to 0} \frac{f(r,y)}{r} = \lim_{r\to 0} W(r,y) = 1$; we have $\partial_y f(0,y) = 0$ because of $f(0,y) \equiv 0$. For continuity, we have $\lim_{(r,y)\to(0,\hat{y})} \partial_r f(r,y) = \lim_{(r,y)\to(0,\hat{y})} V(r,y)W(r,y)^2 = 1$; moreover we have $\lim_{(r,y)\to(0,\hat{y})} \partial_y f(r,y) = \lim_{(r,y)\to(0,\hat{y})} \partial_y W(r,y) r = 0$. We can compute the higher derivatives and show their continuity by a combination of (13.4) and the above limits. This proves Theorem 13.9.

In order to prove (13.5), we write V as $V = T \cdot U$, with

$$T(r,y) := t(\eta(r),y), \quad t(x,y) := \frac{\partial_x g(x,y)x}{g(x,y)},$$

$$U(r,y) := u(\eta(r),y), \quad u(x,y) := \frac{-\ln(x)+e}{-\ln(g(x,y))+e}.$$

Step 1. For all multi-index α with $|\alpha| \geq 1$ and $l \geq 0$, we show

$$\lim_{(r,y)\to(0,\hat{y})} \frac{T(r,y)-1}{r^l} = 0, \quad \lim_{(r,y)\to(0,\hat{y})} \frac{\partial^\alpha T(r,y)}{r^l} = 0. \tag{13.7}$$

We observe that $t(x,y) = \partial_x g(x,y) \frac{x}{g(x,y)}$ extends smoothly to $\{(0,y)\}$ by $t(0,y) = 1$. Indeed, since we have $g(0,y) = 0$, the function $\frac{g(x,y)}{x}$ extends smoothly to $\{(0,y)\}$ as $\partial_x g(0,y)$. By $\partial_x g(0,y) \neq 0$, $\frac{g(x,y)}{x}$ is non-zero on a neighborhood of 0. Hence the extension is smooth.

Using the substitution $r = \eta^{-1}(x) = \frac{1}{\ln(-\ln(x)+e)}$, we see

$$\begin{aligned}
\lim_{(r,y)\to(0,\hat{y})} \frac{T(r,y)-1}{r^l} &= \lim_{(x,y)\to(0,\hat{y})} (t(x,y)-1)\ln(-\ln(x)+e)^l \\
&= \lim_{(x,y)\to(0,\hat{y})} \frac{t(x,y)-1}{x} \lim_{x\to 0} x\ln(-\ln(x)+e)^l \\
&= \partial_x t(0,\hat{y})\cdot 0,
\end{aligned}$$

where the last equality is by Lemma 13.10 (1). For $m+n \geq 1$, we now show $\lim_{(r,y)\to(0,\hat{y})} \frac{\partial_r^m \partial_y^n T(r,y)}{r^l} = 0$. Suppose $m \geq 1$, then the partials involve the derivatives of η. We then invoke Lemma 13.10 (3) to conclude the limit is zero. Hence we are left to find the limit of $\frac{\partial_y^n T(r,y)}{r^l}$ for $n \geq 1$. Using the same substitution $r = \eta^{-1}(x)$ as before, we have

$$\begin{aligned}
\lim_{(r,y)\to(0,\hat{y})} \frac{\partial_y^n T(r,y)}{r^l} &= \lim_{(x,y)\to(0,\hat{y})} \frac{\partial_y^n t(x,y)}{x} \lim_{x\to 0} x\ln(-\ln(x)+e)^l \\
&= \partial_x \partial_y^n t(0,\hat{y})\cdot 0.
\end{aligned}$$

This proves (13.7).

Step 2. For all $l \geq 0$, we show

$$\lim_{(r,y)\to(0,\hat{y})} \frac{U(r,y)-1}{r^l} = 0. \tag{13.8}$$

We first show the limit for $l = 0$. Using $r = \eta^{-1}(x) = \frac{1}{\ln(-\ln(x)+e)}$, we have

$$\begin{aligned}
&\lim_{(r,y)\to(0,\hat{y})} U(r,y)-1 \\
&= \lim_{(r,y)\to(0,\hat{y})} \frac{-\ln(x)+e}{-\ln(g(x,y))+e} - 1 \\
&= \lim_{(x,y)\to(0,\hat{y})} \ln\left(\frac{g(x,y)}{x}\right) \lim_{(x,y)\to(0,\hat{y})} \frac{1}{-\ln(g(x,y))+e} = \ln(\partial_x g(0,\hat{y}))\cdot 0.
\end{aligned}$$

More generally, we have

$$\begin{aligned}
&\lim_{(r,y)\to(0,\hat{y})} \frac{U(r,y)-1}{r^l} \\
&= \lim_{(x,y)\to(0,\hat{y})} \ln\left(\frac{g(x,y)}{x}\right) \frac{\ln(-\ln(x)+e)^l}{-\ln(g(x,y))+e} \\
&= \lim_{(x,y)\to(0,\hat{y})} \ln\left(\frac{g(x,y)}{x}\right) \lim_{(x,y)\to(0,\hat{y})} \frac{-\ln(x)+e}{-\ln(g(x,y))+e} \lim_{x\to 0} \frac{\ln(-\ln(x)+e)^l}{-\ln(x)+e} \\
&= \ln(\partial_x g(0,\hat{y}))\cdot 1\cdot 0,
\end{aligned}$$

where the last equality is by $\lim_{(r,y)\to(0,\hat{y})} U(r,y) = 1$ as shown above and Lemma 13.10 (2).

Step 3. For all multi-index α with $|\alpha| \geq 1$ and $l \geq 0$, we show

$$\lim_{(r,y)\to(0,\hat{y})} \frac{\partial^\alpha U(r,y)}{r^l} = 0. \tag{13.9}$$

We shall see that the following function comes about when taking derivatives of U with respect to y.

$$S(r,y) := s(\eta(r),y), \quad s(x,y) := \frac{\partial_y g(x,y)}{g(x,y)}. \tag{13.10}$$

We first observe that the function $s(x,y) = \frac{\partial_y g(x,y)}{x}\frac{x}{g(x,y)}$ extends smoothly to $\{(0,y)\}$ by $s(0,y) = \frac{\partial_x \partial_y g(0,y)}{\partial_x g(0,y)}$ because both $\frac{\partial_y g(x,y)}{x}$ and $\frac{x}{g(x,y)}$ extend smoothly. Thus $S(r,y)$ is smooth at $\{(0,y)\}$. We now express the derivative $\partial^\alpha U(r,y)$ in terms of factors we can control.

Claim 13.11. *For all multi-index α with $|\alpha| \geq 1$, the derivative $\partial^\alpha U(r,y)$ can be written as a polynomial in $U(r,y)$, $\frac{\partial^\beta T(r,y)}{r^i}$, $\frac{T(r,y)-1}{r^j}$, $\frac{U(r,y)-1}{r^k}$, and $\frac{\partial_y^l S(r,y) e^{-\frac{1}{r}}}{r^m}$ with $|\beta| \geq 1$ and $i,j,k,l,m \geq 0$; moreover, this polynomial has no summands of the form $U(r,y)^a$.*

To prove this claim, we compute

$$\partial_r U(r,y) = U(r,y)^2 \frac{T(r,y)-1}{r^2} + U(r,y)\frac{U(r,y)-1}{r^2}. \tag{13.11}$$

By iteratively taking the r derivatives, we can write $\partial_r^d U(r,y)$ as a polynomial in $U(r,y)$, $\frac{\partial_r^b T(r,y)}{r^i}$, $\frac{T(r,y)-1}{r^j}$, and $\frac{U(r,y)-1}{r^k}$, with no summands of powers of $U(r,y)$ alone. We now take the y derivatives of $\partial_r^d U(r,y)$. First of all, we have

$$\partial_y U(r,y) = U(r,y)^2 S(r,y) e^{-\frac{1}{r}}. \tag{13.12}$$

Then by iteratively taking the y derivatives of $\partial_r^d U(r,y)$, we conclude that $\partial^\alpha U(r,y)$ has the desired form as claimed.

Then by (13.7), (13.8), and $\lim_{r\to 0} \frac{e^{-\frac{1}{r}}}{r^m} = 0$, we conclude that each summand of the derivative $\partial^\alpha U(r,y)$ goes to 0. This proves (13.9).

Step 4. We show the limits of V and its derivatives in (13.5).

Since by construction $V = T \cdot U$, the proof of (13.5) is a matter of combining the limits of T, U, and their derivatives. We have $\lim_{(r,y)\to(0,\hat{y})} \frac{V(r,y)-1}{r^l} = \lim_{(r,y)\to(0,\hat{y})} T(r,y)\frac{U(r,y)-1}{r^l} + \frac{T(r,y)-1}{r^l} = 0$, and $\lim_{(r,y)\to(0,\hat{y})} \frac{\partial^\alpha V(r,y)}{r^l} = 0$ follows from the product rule. This proves (13.5).

Step 5. For all $l \geq 0$, we show

$$\lim_{(r,y)\to(0,\hat{y})} \frac{W(r,y)-1}{r^l} = 0. \tag{13.13}$$

Substituting $r = \eta^{-1}(x)$, we have

$$W(r,y) - 1 \over r^l \tag{13.14}$$

$$\frac{W(r,y)-1}{r^l} = \ln\left(\frac{-\ln(x)+e}{-\ln(g(x,y))+e}\right)\frac{1}{\ln(-\ln(g(x,y))+e)}\frac{1}{r^l}. \tag{13.14}$$

We now estimate the first factor in the above product. Since $\ln'(1) = 1$, it follows from the mean-value theorem that for (x, y) sufficient close to $(0, \hat{y})$, we have

$$\begin{aligned}\left|\ln\left(\frac{-\ln(x)+e}{-\ln(g(x,y))+e}\right)\right| &= \left|\ln\left(\frac{-\ln(x)+e}{-\ln(g(x,y))+e}\right) - \ln(1)\right| \\ < 2\left|\frac{-\ln(x)+e}{-\ln(g(x,y))+e} - 1\right| &= 2|U(r,y)-1|.\end{aligned}$$

Since by definition $\frac{1}{\ln(-\ln(g(x,y))+e)} = f(r,y)$, it follows from (13.14) that we have

$$\left|\frac{W(r,y)-1}{r^l}\right| < 2\left|\frac{U(r,y)-1}{r^l}\right| f(r,y).$$

Then by (13.8) and $\lim_{(r,y)\to(0,\hat{y})} f(r,y) = 0$, we conclude (13.13).

Step 6. For all multi-index α with $|\alpha| \geq 1$ and $l \geq 0$, we show

$$\lim_{(r,y)\to(0,\hat{y})} \frac{\partial^\alpha W(r,y)}{r^l} = 0. \tag{13.15}$$

We now express the derivative $\partial^\alpha W(r,y)$ in terms of factors we can control.

Claim 13.12. *For all multi-index α with $|\alpha| \geq 1$, the derivative $\partial^\alpha W(r,y)$ can be written as a polynomial in $V(r,y)$, $W(r,y)$, $\frac{\partial^\beta V(r,y)}{r^i}$, $\frac{V(r,y)-1}{r^j}$, $\frac{W(r,y)-1}{r^k}$, and $\frac{\partial_y^l U(r,y)\partial_y^m S(r,y) r e^{-\frac{1}{r}}}{r^n}$, with $|\beta| \geq 1$ and $i, j, k, l, m, n \geq 0$; moreover, this polynomial has no summands of the form $V(r,y)^a W(r,y)^b$.*

To prove this claim, we use (13.4) to compute

$$\partial_r W(r,y) = V(r,y)W(r,y)\frac{W(r,y)-1}{r} + W(r,y)\frac{V(r,y)-1}{r}. \tag{13.16}$$

By iteratively taking the r derivatives, we can write $\partial_r^d W(r,y)$ as a polynomial in $V(r,y)$, $W(r,y)$, $\frac{\partial_r^b V(r,y)}{r^i}$, $\frac{V(r,y)-1}{r^j}$, and $\frac{W(r,y)-1}{r^k}$, with no summands of powers of $V(r,y)$ and $W(r,y)$ alone. We now take the y derivatives of $\partial_r^d W(r,y)$. First of all, we have

$$\partial_y W(r,y) = W(r,y)^2 U(r,y) S(r,y) r e^{-\frac{1}{r}}. \tag{13.17}$$

Then by iteratively taking the y derivatives of $\partial_r^d W(r,y)$, we conclude that $\partial^\alpha W(r,y)$ has the desired form as claimed.

Then by (13.5), (13.13), and $\lim_{r\to 0} \frac{e^{-\frac{1}{r}}}{r^n} = 0$, we conclude that each summand of the derivative $\partial^\alpha W(r,y)$ goes to 0. This proves (13.15). $\square$

13.3. **Sc-smoothness Results.**

We now survey some key results on sc-smoothness (see Definition 3.6).

We first deal with the sc-smooth of disk maps away from marked points. Let $V \subset \mathbb{R}^2$ be a bounded domain with Lipshitz boundary, and $H^3(V, \mathbb{R}^N)$ the sc-Banach space with sc-structure $H^{3+m}(V, \mathbb{R}^N)$ (see Example 3.2).

The following result is analogous to Theorem 1.25 of [7].

Proposition 13.13. *Let $\hat{u} \in H^3(V, \mathbb{R}^N)$ be a map with $\hat{u}(\hat{o}) = 0$ and injective derivative $D\hat{u}(\hat{o})$ at $\hat{o} \in V$. Moreover, let H be any subspace of $\mathbb{R}^N$ with transversality* $\operatorname{im}(D\hat{u}(\hat{o})) \oplus H = \mathbb{R}^N$. *Then there is a neighborhood $\mathcal{U}_\varepsilon(\hat{u})$ of $\hat{u}$ in $H^3(V, \mathbb{R}^N)$ and a neighborhood $B(\hat{o})$ with the following properties.*

1. *For each map $u \in \mathcal{U}(\hat{u})$, there exists precisely one point $o(u)$ in $B(\hat{o})$ such that the image $u(o(u))$ lies in H.*
2. *We have transversality* $\operatorname{im}(Du(o(u))) \oplus H = \mathbb{R}^N$.

Furthermore, the map

$$\mathcal{U}_\varepsilon(\hat{u}) \to B(\hat{o}), \quad u \mapsto o(u)$$

is sc$^\infty$.

The following result is analogous to Theorem 1.26 of [7].

Proposition 13.14. *Let A be an open subset of $\mathbb{R}^k$, and $\phi_a : V \to V$ a family of maps with*

$$A \times V \to V, \quad (a, z) \mapsto \phi_a(z)$$

being smooth. Then the domain parametrization map

$$A \times H^3(V, \mathbb{R}^N) \to H^3(V, \mathbb{R}^N), \quad (a, u) \mapsto u \circ \phi_a$$

is sc$^\infty$.

The following result examines the sc-smoothness of the post-composition map.

Proposition 13.15. *Let $f : \mathbb{R}^M \to \mathbb{R}^N$ be a smooth map with all derivatives bounded. Then the post-composition map*

$$H^3(V, \mathbb{R}^M) \to H^3(V, \mathbb{R}^N), \quad u \mapsto f \circ u$$

is sc$^\infty$.

We now deal with sc-smooth of disk maps near marked points in strip coordinates. As a reminder, we denote the half-infinite intervals by

$$\mathbb{R}^+ := [0, \infty), \quad \mathbb{R}^- := (-\infty, 0].$$

Fix a strictly increasing sequence of weights $\delta_0 < \delta_1 < \cdots < 1$, and let $H^{3,\delta_0}_{\lim}(\mathbb{R}^\pm \times [0, \pi], \mathbb{C}^n; \mathbb{R}^n)$ be the weighted Sobolev space with limits with sc-structure $H^{3+m,\delta_m}_{\lim}(\mathbb{R}^\pm \times [0, \pi], \mathbb{C}^n; \mathbb{R}^n)$ (see Example 3.3).

The following result is analogous to Theorem 2.7 of [7]. (The difference in requirements $\delta < 1$ here and $\delta < 2\pi$ in [7] is explained in Remark 3.14).

Proposition 13.16. *Let A be an open subset of $\mathbb{R}^k$. Suppose a family of maps $\phi_a : \mathbb{R}^\pm \times [0,\pi] \to \mathbb{R}^\pm \times [0,\pi]$ of the form*

$$\phi_a(z) = z + \xi(a) + \zeta_a(z)$$

satisfies the following conditions.

(1) *The function $\xi : A \to \mathbb{R}^\pm$ is smooth.*
(2) *The function*

$$A \times (\mathbb{R}^\pm \times [0,\pi]) \to \mathbb{C}, \quad (a,z) \mapsto \zeta_a(z)$$

is smooth, with the map into the functional space

$$A \to H^{m,\delta}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}), \quad a \mapsto \zeta_a$$

being smooth for every $m \geq 3$ and every $\delta \in (0,1)$.

Then the domain parametrization map

$$A \times H^{3,\delta_0}_{\lim}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n) \to H^{3,\delta_0}_{\lim}(\mathbb{R}^\pm \times [0,\pi], \mathbb{C}^n; \mathbb{R}^n),$$
$$(a,u) \mapsto u \circ \phi_a$$

is sc^∞.

We now demonstrate that a class of parametrizations arising from families of strip coordinates (Definition 3.8) satisfies the conditions in Proposition 13.16.

Lemma 13.17. *Let $\hat{x}$ be a boundary marked point, and $\hat{\psi} \in \mathrm{Aut}(D)$ a disk automorphism. Moreover, let $h^\pm(x,\cdot)$ be a family of strip coordinates near $\hat{x}$, and $h'^\pm(x',\cdot)$ a family of strip coordinates near $\hat{\psi}(\hat{x})$. Then there exists $\varepsilon > 0$ and $k' > 0$ such that the family of maps*

$$\phi^\pm_{x,\psi} := h^\pm(x,\cdot)^{-1} \circ \psi^{-1} \circ h'^\pm(\psi(x),\cdot)$$

is well-defined for nearby $(x,\psi) \in \mathcal{U}_\varepsilon(\hat{x}) \times \mathcal{U}_\varepsilon(\hat{\psi})$ as maps from $[k',\infty) \times [0,\pi]$ to $\mathbb{R}^+ \times [0,\pi]$ (or from $(-\infty,-k'] \times [0,\pi]$ to $\mathbb{R}^- \times [0,\pi]$). Moreover, this family satisfies the conditions in Proposition 13.16.

Proof. By Definition 3.8, the family can be written as

$$h^\pm(x,\cdot)^{-1} \circ \psi^{-1} \circ h'^\pm(\psi(x),\cdot) = (p^\pm)^{-1} \circ (f_x)^{-1} \circ \psi^{-1} \circ f'_{\psi(x)} \circ p^\pm,$$

where $p^+(z) = -e^{-z}$, $p^-(z) = e^z$, and f_x is a family of Möbius transformations. Thus each $(f_x)^{-1} \circ \psi^{-1} \circ f'_{\psi(x)}$ is a Möbius transformation. More specifically, it is an automorphism of the extended upper half plane $\{\mathrm{Im} z \geq 0\} \cup \{\infty\}$ that fixes the origin 0. All such automorphisms can be parametrized by

$$T_{a,b}(z) = \frac{az}{bz+1},$$

for $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$. In our case, we have $T_{a,b} := (f_x)^{-1} \circ \psi^{-1} \circ f'_{\psi(x)}$ for nearby $(a,b) \in \mathcal{U}_\lambda(\hat{a},\hat{b})$. We can choose $\varepsilon > 0$ small enough such that for all $(x,\psi) \in \mathcal{U}_\varepsilon(\hat{x}) \times \mathcal{U}_\varepsilon(\hat{\psi})$, the corresponding neighborhood $\mathcal{U}_\lambda(\hat{a},\hat{b})$ is bounded away from $\{(a,b) \,|\, a = 0\}$.

Thus we have $\phi^{\pm}_{x,\psi} = \phi^{\pm}_{a,b} = (p^{\pm})^{-1} \circ T_{a,b} \circ p^{\pm}$. By a simply computation, we have

$$\phi^{+}_{a,b}(z) = z - \ln(a) + \ln(be^{-z} - 1),$$
$$\phi^{-}_{a,b}(z) = z + \ln(a) - \ln(be^{z} + 1).$$

We first examine $\phi^{+}_{a,b}$. Since a is bounded away from 0, $-\ln(a)$ is a smooth function. Moreover $b \mapsto \ln(be^{-z} - 1)$ is a smooth map. Also, since a and b are bounded, there is $k' > 0$ large enough such that $\phi^{+}_{a,b}$ maps $[k', \infty) \times [0, \pi]$ into $\mathbb{R}^{+} \times [0, \pi]$. Thus it satisfies the conditions in Proposition 13.16. The argument for $\phi^{-}_{a,b}$ is similar. $\square$

The following result examines the sc-smoothness of the post-composition map.

Proposition 13.18. *Let $f : (\mathbb{C}^m, \mathbb{R}^m) \to (\mathbb{C}^n, \mathbb{R}^n)$ be a smooth map with all derivatives bounded. Then the post-composition map*

$$H^{3,\delta_0}_{\lim}(\mathbb{R}^{\pm} \times [0, \pi], \mathbb{C}^m; \mathbb{R}^m) \to H^{3,\delta_0}_{\lim}(\mathbb{R}^{\pm} \times [0, \pi], \mathbb{C}^n; \mathbb{R}^n),$$
$$u \mapsto f \circ u$$

is sc^{∞}.

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