Asymptotic Operators in Symplectizations and the Conley-Zehnder Index

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A diploma thesis submitted to the ETH Zürich

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Abstract

We consider finite energy surfaces

$$\tilde{u}: (\dot{S}, j) \to (\mathbb{R} \times M, \tilde{J})$$

in the symplectization of a contact manifold M. These are used by H. Hofer, K. Wysocki and E. Zehnder in order to prove results about periodic orbits of the Reeb vector field. In this work, we investigate closely two details of this theory.

The first is the Conley-Zehnder index for nondegenerate arcs in the symplectic linear group $Sp(n)$. We prove in detail some topological properties of $Sp(n)$. For these, we only found sketchy or partially wrong proofs in the literature. Using these properties, we prove that the Conley-Zehnder index is uniquely defined by some natural set of axioms. In [13], a way was mentioned to define this index via the spectral flow of some asymptotic operator. In the main part of this work, we explicitly carry out this construction and prove that it meets the axioms defining the Conley-Zehnder index. We then compare this spectral flow description to an other construction of the index in [15], where some winding numbers are used for this purpose. In the generic case, we can trace back both above constructions to counting intersections with some Maslov cycle (yet another definition of the Conley-Zehnder index, introduced in [18]), and we give some idea about what actually happens for the different constructions at the intersection points.

The second detail of the theory of finite energy surfaces in symplectizations that we worked at, is the construction of a trivialization

$$\Phi: \dot{S} \times B_\varepsilon(0) \to \mathbb{R} \times M$$

of a tubular neighbourhood of the finite energy surface $\tilde{u}(\dot{S}) \subset \mathbb{R} \times M$, where $\tilde{u}$ is assumed to be an embedding. We generalize the concerning result in [12] to immersions $\tilde{u}$ and carry out in detail some small parts of the proof that were left vague in the current preprint.

Acknowledgements

I would like to thank my supervisors, Dr. Markus Kriener and Prof. Eduard Zehnder for the interesting topic and their willingness to answer all my questions.

I am also grateful to a number of people without whom writing this diploma thesis would have been much harder and less joyful: Meike Akveld made her way through all my dreadful notation when proofreading the manuscript, and I had a lot of fruitful discussions with her; Fred Hamprecht was a great help for LaTeX and life in general; Torsten Linnemann gave me his earlier results of a literature research on the spectral flow definition of the Conley-Zehnder index; Prof. Kris Wysocki explained to me several parts of his preprint and gave all kind of useful advice.

Finally, I would like to thank Prof. Helmut Hofer and Prof. Oswald Riemenschneider for their interest in my going on.

Contents

Chapter 1

Finite energy surfaces in symplectizations

In this chapter we give a short introduction to the dynamics on contact manifolds and we describe how so-called finite energy surfaces in the symplectization of a contact manifold are related to the contact dynamics on the manifold.

We consider a closed, oriented 3-manifold M. A 1-form λ on M is called a *contact form* if $\lambda \wedge \mathrm{d}\lambda$ is a volume form. This volume form induces an orientation on M, and the contact form is called *positive* if this orientation coincides with the given one of M, or *negative* if this is the reversed orientation. In the following we will only consider positive contact forms.

By a theorem of Martinet ([16] Thm.5.1), the existence of a positive contact structure is guaranteed for closed oriented 3-manifolds.

Since λ can not be identically zero in any point of the manifold, there is a 2-dimensional vectorbundle, the *contact structure* $\xi := \ker \lambda$, associated with the contact form. Again because of the nondegeneracy condition on $\lambda \wedge \mathrm{d}\lambda$, the skew symmetric form $\mathrm{d}\lambda$ has to be nondegenerate on each fibre $\xi_m = \ker \lambda_m$ and hence defines a nondegenerate 2-form on the vector bundle $\xi \to M$. In addition, $\ker \mathrm{d}\lambda \subset TM$ is 1-dimensional and hence defines a line bundle l that is transversal to ξ. A preferred section of this line bundle is the *Reeb vector field* X, defined by

$$i_X \,\mathrm{d}\lambda = 0 \qquad \text{and} \qquad \lambda(X) = 1. \tag{1.1}$$

Hence the contact form λ on M defines a natural splitting

$$TM = (l, X) \oplus (\xi, \mathrm{d}\lambda)$$

of the tangent bundle TM into a line bundle $l \to M$ with the preferred section X and a plane bundle $\xi \to M$ with the preferred nondegenerate 2-form

$d\lambda$.

The Reeb vector field has no singular points, so the natural objects to study are the periodic orbits generated by the flow of this vector field. The existence of such periodic orbits in the contact manifold is connected with the existence of so-called finite energy surfaces in the symplectization of the manifold, which we will define next.

A *symplectic vector bundle* (E, ω) over a manifold M is a real vector bundle $E \to M$, which is provided with a smooth nondegenerate section ω of $E^* \wedge E^*$. This *symplectic form* ω provides a nondegenerate skew-symmetric bilinear form ω_m on each fibre E_m, varying smoothly with $m \in M$. Note that due to the nondegeneracy condition on ω, the vector bundle E has to be of even dimension, and it can be given some almost complex structure:

An *almost complex structure* on a vector bundle $E \to M$ is a vector bundle isomorphism $J : E \to E$ satisfying $J^2 = -\mathbb{1}$. Moreover, it is said to be *compatible* with a symplectic form ω on E if

$$g_J(a, b) := \omega(a, J_m b)$$

for $a, b \in E_m$ defines a positive definite inner product on each fibre E_m. For fixed (E, ω), the space $\mathcal{J}(E, \omega)$ of such compatible almost complex structures is nonempty and contractible ([17] Proposition 2.61).

Applying these definitions to the above situation, we note that $(\xi, d\lambda)$ is a symplectic vector bundle with a distinguished class $\mathcal{J}(\xi, d\lambda)$ of almost complex structures $J : \xi \to \xi$, compatible with $d\lambda$ in the sense that

$$g_J(a, b) = d\lambda_m(a, J_m b)$$

for $a, b \in \xi_m$ defines a positive definite inner product on each fibre ξ_m. Choosing a fixed $J \in \mathcal{J}(\xi, d\lambda)$, we can now define an almost complex structure $\tilde{J}$ on the four manifold $\mathbb{R} \times M$ (i.e. on its tangent bundle) by

$$\tilde{J}(a, m)\,(\alpha, k) = (\,-\lambda_m(k)\,,\ \alpha X(m) + J_m \pi(k)\,),$$

for $(\alpha, k) \in T_{(a,m)}(\mathbb{R} \times M) \cong T_a\mathbb{R} \times T_m M$ where $\pi(k) = k - \lambda(k)X$ is the pointwise projection of TM onto ξ along X. This $\tilde{J}$ is called the *special* almost complex structure on $\mathbb{R} \times M$ associated with J. According to proposition 2.61 in [17], this almost complex structure suffices to make $T(\mathbb{R} \times M)$ into a symplectic vector bundle, i.e. there exists a symplectic form Ω on $T(\mathbb{R} \times M)$ that is compatible with $\tilde{J}$ in the above sense. We then call the symplectic vector bundle $(T(\mathbb{R} \times M), \Omega)$ the *symplectization* of the contact manifold (M, λ).

Furthermore, associated with J there is a natural Riemannian metric on $\mathbb{R} \times M$, defined by

$$\langle(\alpha, k), (\beta, h)\rangle_J = \alpha \cdot \beta + \lambda_m(k) \cdot \lambda_m(h) + g_J(\pi k, \pi h) \tag{1.2}$$

for all $(\alpha, k), (\beta, h) \in T_{(a,m)}(\mathbb{R} \times M)$, inducing the norm

$$|(\alpha, k)|_J := \langle (\alpha, k), (\alpha, k) \rangle_J^{\frac{1}{2}}.$$

Note that $\langle \cdot, \cdot \rangle_J$ is invariant under $\tilde{J}$: for all $(\alpha, k), (\beta, h) \in T_{(a,m)}(\mathbb{R} \times M)$ we have

$$\begin{aligned}
&\langle \tilde{J}(a,m)(\alpha,k), \tilde{J}(a,m)(\beta,h) \rangle_J \\
&\quad = \langle (-\lambda_m(k)\,,\, \alpha X(m) + J_m \pi k), (-\lambda_m(h)\,,\, \beta X(m) + J_m \pi h) \rangle_J \\
&\quad = (-\lambda_m(k)) \cdot (-\lambda_m(h)) + \lambda_m(\alpha X(m)) \cdot \lambda_m(\beta X(m)) + g_J(J_m \pi k, J_m \pi h) \\
&\quad = \lambda_m(k) \cdot \lambda_m(h) + \alpha \cdot \beta + \mathrm{d}\lambda(J_m \pi k, J_m^2 \pi h) \\
&\quad = \alpha \cdot \beta + \lambda_m(k) \cdot \lambda_m(h) + \mathrm{d}\lambda(\pi h, J_m \pi k) \\
&\quad = \alpha \cdot \beta + \lambda_m(k) \cdot \lambda_m(h) + g_J(\pi h, \pi k) \\
&\quad = \langle (\alpha, k), (\beta, h) \rangle_J
\end{aligned}$$

Next, let (S, j) be a compact Riemannian surface with an almost complex structure j. Then by [17] Thm.4.16, j is integrable (and hence a *complex structure*), i.e. there is an atlas of S into $\mathbb{C}$ such that j is represented in local coordinates by multiplication with i. We will call such coordinates *holomorphic.*

Furthermore, let $\Gamma \subset S$ be a finite, nonempty set of so-called *punctures*, then we call $\dot{S} := S \setminus \Gamma$ a punctured Riemannian surface.

We are now in a position to define finite energy surfaces at least in the special case of a special almost complex structure given on $\mathbb{R} \times M$.

Definition 1.1 *A* **special finite energy surface** *is a nonconstant smooth map defined on a punctured Riemannian surface $\dot{S}$,*

$$\tilde{u} = (a, u) \;:\; \dot{S} \to \mathbb{R} \times M,$$

that is **pseudoholomorphic**, *i.e. a solution of*

$$\forall z \in \dot{S}: \qquad \tilde{J}(\tilde{u}(z)) \circ T_z \tilde{u} = T_z \tilde{u} \circ j(z), \tag{1.3}$$

and has finite energy $E(\tilde{u}) < \infty$. Here $\tilde{J}$ is a given special almost complex structure on $\mathbb{R} \times M$ and the **energy** *$E(\tilde{u})$ is defined to be*

$$E(\tilde{u}) = \sup \left\{ \int_{\dot{S}} \tilde{u}^* d\lambda_\phi \mid \phi \in \Sigma \right\},$$

where for all

$$\phi \in \Sigma = \{ \phi \in \mathcal{C}^\infty(\mathbb{R}, [0,1]) \mid \phi' \geq 0 \}$$

a 1-form λ_ϕ on $\mathbb{R} \times M$ is given by

$$\lambda_\phi(a,m)\,(\alpha,k) = \phi(a) \cdot \lambda_m(k).$$

Now we consider a special finite energy surface $\tilde{u} : (\dot{S}, j) \to (\mathbb{R} \times M, \tilde{J})$ for a given special almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$.

Near every puncture $z_j \in \Gamma$ we find holomorphic coordinates on an open disk $B_r(0) \subset \mathbb{C}$,

$$h : B_r(0) \to D_j \subset S,$$

such that $h(0) = z_j$. We then make a holomorphic transformation to describe the neighbourhood of the puncture in cylindrical holomorphic coordinates

$$\begin{array}{rcl} \sigma : (s_0, \infty) \times \mathbb{R}/\mathbb{Z} & \to & D_j \setminus \{z_j\} \subset \dot{S} \\ (s, t) & \mapsto & h(\mathrm{e}^{-2\pi(s+it)}) \end{array} \tag{1.4}$$

This maps onto the open punctured disk $\dot{D}_j := D_j \setminus \{z_j\}$ around z_j such that $\lim_{s\to\infty} \sigma(s,t) = z_j$ for all $t \in \mathbb{R}/\mathbb{Z} \cong S^1$. Furthermore, σ is holomorphic in the sense that

$$T\sigma \circ J_0 = j(\sigma) \circ T\sigma, \tag{1.5}$$

where J_0 is the almost complex structure on $(s_0, \infty) \times S^1$, i.e. $J_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with respect to the basis $(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})$ of $T((s_0, \infty) \times S^1)$.

Hence in these coordinates, writing again $\tilde{u}$ for $\tilde{u} \circ \sigma$, the differential equation (1.3) is equivalent to

$$\tilde{J}(\tilde{u})\, \tilde{u}_s = \tilde{u}_t, \tag{1.6}$$

or more explicitly :

$$\begin{array}{rcl} -\lambda(u_s) & = & a_t, \\ a_s & = & \lambda(u_t), \\ J(u)\, \pi u_s & = & \pi u_t, \end{array} \tag{1.7}$$

where we write u_s for $Tu\, \frac{\partial}{\partial s}$.

In fact, by restricting $\tilde{u}$ to the cylinder $(s_0, \infty) \times S^1$ with the complex structure J_0, we again obtain a special finite energy surface (see e.g. [9] §6). For this part of the original finite energy surface we can now formulate the main theorem concerning the connection between finite energy surfaces and periodic orbits of the Reeb vector field, that was proven in [8] and extended in [10].

Theorem 1.2 ([10] Thm.1.2)
Let $\tilde{u} = (a, u) : (s_0, \infty) \times S^1 \to \mathbb{R} \times M$ be a special finite energy surface. Then the limit

$$T := \lim_{s\to\infty} \int_{S^1} u(s, \cdot)^* \lambda$$

exists and the following holds.

(i) If $T = 0$, then the corresponding curve $\tilde{u} \circ \sigma^{-1}$ on $\dot{D}$ can be extended smoothly to D, where σ is defined as in (1.4).

(ii) *If $T \neq 0$, then there exists a $|T|$-periodic orbit x of the Reeb vector field such that $\tilde{u}$ converges to this asymptotic orbit in the following sense:*

$$\lim_{s\to\infty} u(s_k,\cdot) = x(T\cdot)$$

in $\mathcal{C}^\infty(S^1)$ for some sequence $s_k \to \infty$, and for all $t \in S^1$

$$\lim_{s\to\infty} \frac{a(s,t)}{s} = T.$$

(iii) *If $T \neq 0$ and the asymptotic orbit x is nondegenerate, then $u(s,\cdot)$ converges to a parametrization of x, i.e. we have*

$$\lim_{s\to\infty} u(s,\cdot) = x(T\cdot)$$

with convergence in $\mathcal{C}^\infty(S^1)$.

In case (i), the puncture is called *removable* and, by a theorem of Gromov [5], it can be removed in such a way that the extended curve is still pseudoholomorphic. Hence we can in the following assume that all removable punctures have been removed.

In case (ii), the puncture is called *positive* or *negative* according to wether $T > 0$ or $T < 0$.

In case (iii), the puncture is moreover called *nondegenerate*. Here the periodic orbit x is called nondegenerate, if the linearization of the flow of the Reeb vector field along it has only one eigenvalue 1.

To be more precise, let ϕ_t be the flow of the Reeb vector field X satisfying $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(m) = X(\phi_t(m))$. The periodic orbit is then given by $x(t) = \phi_t(x_0)$ with $\phi_T(x_0) = x_0 = x(0)$. Since X is time-independent we conclude from $\phi_s \circ \phi_t = \phi_{s+t}$ that

$$\mathrm{d}\phi_t X = X(\phi_t).$$

Furthermore, using the defining properties (1.1) of the Reeb vector field, we obtain for all $m \in M$ and $Y \in T_mM$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_{\phi_t(m)}(\mathrm{d}\phi_t(Y))\right) = \mathcal{L}_X\lambda(\phi_t(m))\,\mathrm{d}\phi_t Y = (\mathrm{d}(i_X\lambda) + i_X\mathrm{d}\lambda)\,\mathrm{d}\phi_t Y = 0,$$

hence if $Y \in \xi_m = \ker\lambda_m$, then $d\phi_t Y \in \xi_{\phi_t(m)}$ for all $t \in \mathbb{R}$. Thus we have seen that the splitting

$$T_{x_0}M = \mathbb{R}X(x_0) \oplus \xi_{x_0}$$

is invariant under $d\phi_T$, and $d\phi_T$ is the identity on the first component, $\mathbb{R}X(x_0)$. Therefore, the requirement of x being nondegenerate is equivalent to

$$\mathrm{d}\phi_T(x_0):\ \xi_{x_0} \to \xi_{x_0}$$

having no eigenvalue equal to 1.

Moreover, we deduce using again (1.1), for all $m \in M$ and $Y, Z \in T_mM$

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{d}\lambda_{\phi_t(m)}(\mathrm{d}\phi_t(Y), \mathrm{d}\phi_t(Z))\right) &= \mathcal{L}_X \mathrm{d}\lambda(\phi_t(m))(\mathrm{d}\phi_t(Y), \mathrm{d}\phi_t(Z)) \\ &= \left(\mathrm{d}(i_X \mathrm{d}\lambda) + i_X \mathrm{d}\mathrm{d}\lambda\right)(\mathrm{d}\phi_t(Y), \mathrm{d}\phi_t(Z)) = 0,\end{aligned}$$

hence we have $\phi_t^* \mathrm{d}\lambda = \mathrm{d}\lambda(\phi_t)$, and thus

$$\mathrm{d}\phi_t(x_0) : \; (\xi_{x_0}, \mathrm{d}\lambda(x_0)) \to (\xi_{\phi_t(x_0)}, \mathrm{d}\lambda(\phi_t(m))$$

for $t \in [0, T]$ is an arc of symplectic maps, i.e. maps preserving the symplectic structure. If we have given some symplectic trivialization of $x^*\xi$ (as constructed e.g. in [12] §1 for the case of an asymptotic orbit of a finite energy surface),

$$\Psi(t) : (\xi_{x(t)}, \mathrm{d}\lambda(x(t)) \to (\mathbb{R}^2, J_0),$$

then this yields an arc

$$\begin{aligned}\Phi : \; [0,1] &\to Sp(1) \\ t &\mapsto \Psi(Tt) \circ \mathrm{d}\phi_{Tt}(x_0) \circ (\Psi(0))^{-1}\end{aligned}$$

in the symplectic group $Sp(1)$. This arc starts at $\mathbb{1}$ since $\mathrm{d}\phi_0(x_0) = \mathbb{1}_{\xi_{x_0}}$, and ends in $Sp^*(1)$, the set of symplectic matrices having no eigenvalue equal to 1, since $\sigma(\Phi(1)) = \sigma(\Psi(0) \circ \mathrm{d}\phi_T(x_0) \circ (\Psi(0))^{-1}) = \sigma(\mathrm{d}\phi_T(x_0))$.

The sets $Sp(1)$ and $Sp^*(1)$ will be discussed in chapter 2, where we then introduce the Conley-Zehnder index that is attached to nondegenerate orbits via these arcs of symplectic matrices.

For further development of the theory of pseudoholomorphic curves in [13] and [12], one needs a more general form of the almost complex structure on the symplectization $\mathbb{R} \times M$. It will not be $\mathbb{R}$-invariant anymore, but interpolate between two special almost complex structures that are associated to different contact forms on M.

Let a contact form λ on M and two smooth functions $g, h : M \to (0, \infty)$ with $h < g$ be given. We then find a smooth function $f : \mathbb{R} \times M \to (0, \infty)$ satisfying for some $0 < a_1 < a_0 < \infty$

$$f(a,u) = h(u) \quad \text{for all } a \leq -a_0, \qquad f(a,u) = g(u) \quad \text{for all } a \geq a_0,$$

$$\frac{\partial f}{\partial a}(a,u) \geq 0 \quad \text{for all } (a,u) \in \mathbb{R} \times M$$

and

$$\frac{\partial f}{\partial a}(a,u) \geq \sigma > 0 \quad \text{for all } (a,u) \in (-a_1, a_1) \times M.$$

The family of contact forms $\lambda_a = f(a,\cdot)\lambda$ interpolates between $h\lambda$ and $g\lambda$, and the closed 2-form

$$\Omega := d(f\lambda) = \tfrac{\partial f}{\partial a}\mathrm{d}a \wedge \lambda + f \cdot \mathrm{d}\lambda$$

defines a symplectic form on $(-a_1, a_1) \times M$.

We observe that the contact structure associated to λ_a is idependent of a and that because of

$$\mathrm{d}\lambda_a|_\xi = \mathrm{d}f(a,\cdot)|_\xi \wedge \lambda|_\xi + f(a,\cdot)\,\mathrm{d}\lambda|_\xi = f(a,\cdot)\,\mathrm{d}\lambda|_\xi$$

the set $\mathcal{J}(\xi, \lambda_a)$ is also independent of a. Only the Reeb vector fields X_a and projections π_a along them to ξ vary with a.

Moreover, we can choose an arc $\mathbb{R} \ni\ a \mapsto J_a\ \in \mathcal{J}(\xi, \lambda_a) \equiv \mathcal{J}(\xi, \lambda)$ of compatible almost complex structures on (ξ, λ_a) satisfying

$$J_a = J_{-a_0} \quad \text{for all } a \leq -a_0 \quad \text{and} \quad J_a = J_{a_0} \quad \text{for all } a \geq a_0.$$

A *generalized almost complex structure* on $\mathbb{R} \times M$ is now defined to be an almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ that is compatible with Ω and equals the given special almost complex structures on $(-\infty, -a_1) \times M$ and $(a_1, \infty) \times M$. That is, we have $\tilde{J} \in \mathcal{J}(\mathbb{R} \times M, \Omega)$ satisfying

$$\tilde{J}(a,m)(\alpha,k) = (\,-\lambda_a(m)(k)\,,\ \alpha X_a(m) + J_a(m)\pi_a k\,)$$

for all $a \in \mathbb{R} \setminus [-a_1, a_1]$, $m \in M$ and $(\alpha, k) \in T_{(a,m)}(\mathbb{R} \times M)$.

Note that the generalized almost complex structure is $\mathbb{R}$-invariant for $a \geq a_0$:

$$\tilde{J}(a,m)(\alpha,k) = (\,-\lambda_{a_0}(k)\,,\ \alpha X_{a_0}(m) + J_{a_0}(m)\pi_{a_0} k\,),$$

and analogously for $a \leq -a_0$.

Finally we can define the following.

Definition 1.3 *A* **generalized finite energy surface** *is a nonconstant smooth map defined on a punctured Riemannian surface $\dot{S}$,*

$$\tilde{u} = (a,u)\ :\ \dot{S} \to \mathbb{R} \times M,$$

that is a solution of

$$\forall z \in \dot{S}: \qquad \tilde{J}(\tilde{u}(z)) \circ T_z\tilde{u} = T_z\tilde{u} \circ j(z),$$

and has finite energy $\hat{E}(\tilde{u}) < \infty$. Here $\tilde{J}$ is a given generalized almost complex structure on $\mathbb{R} \times M$ and the modified energy $\hat{E}(\tilde{u})$ is defined to be

$$\hat{E}(\tilde{u}) = \sup\left\{ \int_{\dot{S}} \tilde{u}^* d\lambda_\phi \mid \phi \in \hat{\Sigma} \right\},$$

where

$$\hat{\Sigma} = \{\phi \in \mathcal{C}^\infty(\mathbb{R}, [0,1]) \mid \phi' \geq 0 \ \text{ and } \ \phi|_{[-a_1,a_1]} \equiv \tfrac{1}{2}\}.$$

As in the special case, we only consider a neighbourhood of one puncture of the finite energy surface, on which we introduce the cylindrical coordinates (1.4). The restriction of $\tilde{u}$ to this neighbourhood is again a generalized finite energy surface. It can in fact even be viewed as a special finite energy surface in a sufficiently small neighbourhood of the puncture: the following theorem has been proved in [13] for the case of the contact manifold M being the three sphere, but it does also hold in the general case.

Theorem 1.4 ([13] Thm.4.6)
Let $\tilde{u} = (a, u) : (s_0, \infty) \times S^1 \to \mathbb{R} \times M$ be a generalized finite energy surface. Then the limit

$$T := \lim_{s \to \infty} \int_{S^1} u(s, \cdot)^* \lambda$$

exists and the following holds.

(i) If $T = 0$, then the corresponding curve $\tilde{u} \circ \sigma^{-1}$ on $\dot{D}$ can be extended smoothly to D, where σ is defined as in (1.4).

(ii) If $T < 0$ or $T > 0$, then

$$a(s,t) \overset{s \to \infty}{\longrightarrow} -\infty \text{ or } \infty$$

respectively, for all $t \in S^1$.

In case (i), the puncture is — as in the special case — removable, and we do not consider this case as a genuine puncture.

In case (ii), if we choose s_0 sufficiently large, for the negative ($T < 0$) or positive ($T > 0$) case we have $a(s,t) \leq -a_0$ or $a(s,t) \geq a_0$ respectively, on all of the cylinder $(s_0, \infty) \times S^1$. Thus the considered finite energy surface lies completely within an area of $\mathbb{R} \times M$, where $\tilde{J}$ is identical to the special almost complex structure associated to $\lambda_{-a_0} = h\lambda$ or $\lambda_{a_0} = g\lambda$ respectively.

Therefore, near a puncture, the generalized finite energy surface $\tilde{u}$ can be viewed as a special finite energy surface with respect to $(M, h\lambda)$ or $(M, g\lambda)$ for a negative or positive puncture respectively. Thus all results on special finite energy surfaces near the punctures generalize to generalized finite energy surfaces.

In chapter 3 we will make one step towards a Fredholm theory for pseudoholomorphic curves (see [12]), by constructing special coordinates in the tubular neighbourhood of a generalized finite energy surface. Due to above considerations we can for this purpose always assume that near the punctures, the above identification with the case of a special finite energy surface has already been made, thus writing λ instead of $h\lambda$ or $g\lambda$.

Chapter 2

The Conley-Zehnder index

The Conley-Zehnder index is of some importance in the investigation of finite energy surfaces and their asymptotic orbits. It assigns an integer to each arc of symplectic matrices that fulfills a certain nondegeneracy condition. Such an arc is obtained, for instance, by linearizing the flow of a given Hamiltonian vector field along a nondegenerate periodic orbit (see e.g. [9] or chapter 1).

In this chapter we will only be concerned with the definition of the Conley-Zehnder index for a given symplectic arc. It can be defined axiomatically as presented in 2.1.2, but for applications one needs an explicit construction. There are several such descriptions. In 2.2 we will present in full detail the construction using the spectral flow of an asymptotic operator. In 2.3 we introduce a more geometric approach to the Conley-Zehnder index in dimension 2, and we will describe how this geometric construction relates to the spectral flow description.

2.1 Introduction

We equip $\mathbb{R}^{2n}$ with the canonical symplectic form ω which is defined by $\omega(x,y) = \langle x\,,\, Jy\rangle$ for all $x, y \in \mathbb{R}^{2n}$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product and

$$J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

is the canonical almost complex structure [1] on $\mathbb{R}^{2n}$. The objects of consideration will be arcs in the set of real symplectic matrices. The latter is defined as the set of linear maps on $\mathbb{R}^{2n}$ that preserve the symplectic structure,

$$\begin{aligned} Sp(n) &:= \{\Phi \in \mathbb{R}^{2n\times 2n} \mid \forall x, y \in \mathbb{R}^{2n} : \omega(\Phi x, \Phi y) = \omega(x,y)\} \\ &= \{\Phi \in \mathbb{R}^{2n\times 2n} \mid \Phi^T J \Phi = J\}. \end{aligned}$$

[1] In the other chapters, J is denoted by J_0, since it has to be distinguished from a number of other almost complex structures

We will show in the next section that this is actually a group, and before we define the Conley-Zehnder index in 2.1.2, we need to obtain some information about the topology of this group that is also called the *symplectic linear group.*

2.1.1 The symplectic linear group

First note that obviously $J^T = -J = J^{-1}$. Next, in the operator norm we have $\|J\| = 1$ since for $x \in \mathbb{R}^{2n}$

$$\|Jx\|^2 = \langle Jx\,,\, Jx\rangle = \langle x\,,\, J^T Jx\rangle = \|x\|^2. \tag{2.1}$$

Furthermore, the eigenvalues of J are $\pm i$. Indeed, (2.1) implies that the eigenvalues must have absolute value 1. For an eigenvector $x \in \mathbb{C}^{2n} \setminus \{0\}$ with eigenvalue λ we have

$$\overline{\lambda}\langle x, x\rangle = \langle Jx, x\rangle = \langle x, -Jx\rangle = -\lambda\langle x, x\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the Hermitian scalar product on $\mathbb{C}^{2n}$. Hence the eigenvalues also have to be purely imaginary. By lemma 1.14 in [17], the determinant of a symplectic matrix is 1. Moreover, note that the inverse of $\Phi \in Sp(n)$ is explicitly represented by

$$\Phi^{-1} = -J\Phi^T J \tag{2.2}$$

since

$$\Phi^T J\Phi = J \quad \Longleftrightarrow \quad -J\Phi^T J\Phi = \mathbb{1}.$$

Using this we deduce that for symplectic Φ also Φ^T is symplectic:

$$\begin{array}{rrcl} & \Phi^{-1} & = & -J\Phi^T J \\ \Rightarrow & (\Phi^T)^{-1} & = & -J\Phi J \\ \Rightarrow & J & = & \Phi J\Phi^T. \end{array}$$

Furthermore, products and inverses of symplectic matrices are themselves symplectic as can be seen directly from the definition. It is also useful to know that symplectic matrices have a unique polar decomposition.

Proposition 2.1 *For all $A \in Sp(n)$ there exists a unique symmetric, positive definite $P \in Sp(n)$ and a unique $U \in Sp(n) \cap O(2n)$, such that $A = PU$. Furthermore, this decomposition is continuous in A.*

Proof: Obviously for A symplectic, AA^T is also symplectic and in addition symmetric and positive definite. The latter holds since A^T is nondegenerate and thus for all $x \in \mathbb{R}^{2n} \setminus \{0\}$ we have

$$\langle x, AA^T x\rangle = \|A^T x\|^2 > 0.$$

So because of its symmetry, AA^T can be diagonalized orthonormally,

$$AA^T = S^T \operatorname{diag}(\lambda_1, \ldots, \lambda_{2n})\, S$$

with $S \in SO(2n)$ and positive eigenvalues $\lambda_1, \ldots, \lambda_{2n} \in \mathbb{R}^+$ due to AA^T being positive definite. Now define

$$P := (AA^T)^{\frac{1}{2}} = S^T \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{2n}})\, S$$

which obviously is symmetric and positive definite. Furthermore, lemma 2.19 in [17] says that any real power of a positive definite, symmetric and symplectic matrix is itself symplectic. So we also have $P \in Sp(n)$.

Next, note that $U := P^{-1}A$ is symplectic since inverses and products of symplectic matrices are again symplectic. Finally, U is also orthogonal:

$$U^T U = A^T (P^{-1})^T P^{-1} A = A^T (P^{-1})^2 A = A^T (AA^T)^{-1} A = \mathbb{1}.$$

Thus U and P have the required properties and obviously fulfill $A = PU$.

To show the uniqueness of this decomposition let $A = PU = \widetilde{P}\widetilde{U}$ be two such decompositions and obtain

$$\begin{aligned}
& P^{-1}\widetilde{P} = P^{-1}A\widetilde{U}^T = U\widetilde{U}^T \\
\Rightarrow\ & \widetilde{P}P^{-1} = (P^{-1}\widetilde{P})^T = (U\widetilde{U}^T)^T = (U\widetilde{U}^T)^{-1} = (P^{-1}\widetilde{P})^{-1} = \widetilde{P}^{-1}P \\
\Rightarrow\ & \widetilde{P}^2 = P^2.
\end{aligned}$$

But P and $\widetilde{P}$ are both symmetric and positive definite, so we can deduce that $\widetilde{P} = P$: Symmetric matrices can be diagonalized orthonormally, so there exists an orthonormal basis of eigenvectors of P. Now P^2 has the same eigenvectors as P, only with the eigenvalues squared. As the same holds for $\widetilde{P}$ it follows from $\widetilde{P}^2 = P^2$ that the eigenspaces and squared eigenvalues of P and $\widetilde{P}$ are the same. But since they are in addition both positive definite, the eigenvalues are positive and thus identical, that is $P = \widetilde{P}$. Now we also get $U\widetilde{U}^T = P^{-1}\widetilde{P} = \mathbb{1}$, hence $U = \widetilde{U}$ so the decomposition is unique.

For the continuity first consider $P = S^T \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{2n}})\, S$. First of all, AA^T is of course continuous in A. Next, the eigenvalues λ_i of AA^T are continuous since they are the solutions of $\det(AA^T - \lambda\mathbb{1}) = 0$. In addition, we obtain a continuous orthonormal basis of eigenvectors by solving for $i = 1, \ldots, 2n$ the equation $(AA^T - \lambda_i\mathbb{1})x = 0$ for x. Knowing finally that the eigenvectors constitute S we deduce that P is continuous. Now using (2.2) we find that $P^{-1} = -JP^TJ$ is continuous and thus also $U = P^{-1}A$. $\square$

So a given symplectic arc $\Phi \in \mathcal{C}([0,1], Sp(n))$ is decomposed into continuous arcs $P(t)$ and $U(t)$ of symplectic, symmetric, positive definite and symplectic, orthogonal matrices respectively. Now the P-part can be contracted to the identity.

Lemma 2.2 *The set of symplectic, symmetric and positive definite matrices is contractible.*

Proof: Let $P : S^1 \to \mathbb{R}^{2n\times 2n}$ be a continuous loop of symmetric, symplectic and positive definite matrices. Because of its symmetry it can be written as in the proof of proposition 2.1,

$$P(t) = S(t)^T \operatorname{diag}(\lambda_1(t), \ldots, \lambda_{2n}(t))S(t)$$

with continuous arcs of orthonormal matrices S and positive eigenvalues λ_i, $i = 1, \ldots, 2n$. From this representation we can construct a homotopy,

$$B(s,t) := S(t)^T \operatorname{diag}(\lambda_1(t)^{1-s}, \ldots, \lambda_{2n}(t)^{1-s})S(t),$$

between $B(0,t) = P(t)$ and $B(1,t) = \mathbb{1}$. Obviously, B is continuous in s and t, it is symmetric and positive definite everywhere and by lemma 2.19 in [17] it is also symplectic. □

So every loop Φ in $Sp(n)$ is homotopic to the loop

$$(\Phi\Phi^T)^{-\frac{1}{2}}\,\Phi : \; S^1 \to Sp(n) \cap O(2n) \cong U(n).$$

For the last isomorphism see [17], lemma 2.17. It states that

$$\begin{array}{rcl} \Theta : \; Sp(n) \cap O(2n) & \to & U(n) \\ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} & \mapsto & X + iY \end{array} \tag{2.3}$$

is a bijection and that in particular $Sp(n) \cap O(2n)$ consists of matrices of the above form. This shows that $Sp(n)$ can be retracted to $U(n)$ which is well known to be connected, hence $Sp(n)$ is connected as well. Using this retraction we can also determine the fundamental group of $Sp(n)$.

Proposition 2.3

$$\begin{array}{rcl} \rho : \; Sp(n) & \to & S^1 \\ \Phi & \mapsto & \det \Theta((\Phi\Phi^T)^{-\frac{1}{2}}\,\Phi) \end{array}$$

induces an isomorphism of the fundamental groups

$$\pi_1(Sp(n)) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Proof: We have seen above that any loop in $Sp(n)$ is homotopic to a loop in $U(n)$, that is

$$\begin{array}{rcl} Sp(n) & \to & U(n) \\ \Phi & \mapsto & \Theta((\Phi\Phi^T)^{-\frac{1}{2}}\,\Phi) \end{array}$$

induces an isomorphism $\pi_1(Sp(n)) \cong \pi_1(U(n))$. Then by proposition 2.21 in [17] we also know that $\pi_1(U(n))$ is isomorphic to $\pi_1(S^1) \cong \mathbb{Z}$ via the determinant map. □

In the following all loops will be parametrized by $S^1 \cong \mathbb{R}/\mathbb{Z}$. Furthermore, we identify $Sp(1) \subset Sp(n)$ via $(\mathbb{R}^2, J_1) \cong \mathrm{span}(e_1, e_{n+1}) \subset (\mathbb{R}^{2n}, J)$ where $(e_i)_{i=1,\dots,2n}$ is the canonical basis of $\mathbb{R}^{2n}$. Here the almost complex structure J_1 on $\mathbb{R}^2$ is identified with

$$\begin{pmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} \quad \text{where} \quad \mathrm{I} = \mathrm{diag}(1, 0, \dots, 0) \in \mathbb{R}^{n \times n}.$$

Remark 2.4 $\pi_1(Sp(n))$ *is generated by*

$$\Gamma(t) = \mathrm{e}^{2\pi t J_1} \oplus \mathbb{1}$$

with $\mathbb{1} \in \mathbb{R}^{2(n-1)\times 2(n-1)} \cong \mathrm{span}(e_2, \dots, e_n, e_{n+2}, \dots, e_{2n})$.

Proof: First note that

$$\begin{aligned} \mathrm{e}^{\alpha J} &= \sum_{k=0}^{\infty} \frac{1}{k!} \alpha^k J^k \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \alpha^{2k} (-\mathbb{1})^k + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \alpha^{2k+1} (-\mathbb{1})^k J \\ &= \cos(\alpha)\mathbb{1} + \sin(\alpha) J. \end{aligned} \tag{2.4}$$

From this representation we immediately see that $\mathrm{e}^{\alpha J}$ is symplectic for any real α, so Γ actually is a loop in $Sp(n)$.

Since $\Gamma(t)^T = \mathrm{e}^{-2\pi t J_1} \oplus \mathbb{1} = \Gamma(t)^{-1}$ we obtain

$$(\Gamma(t)\Gamma(t)^T)^{-\frac{1}{2}} \Gamma(t) = \Gamma(t) = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

with

$$\begin{aligned} X(t) &= \mathrm{diag}(\cos(2\pi t), 1, \dots, 1), \\ Y(t) &= \mathrm{diag}(\sin(2\pi t), 0, \dots, 0). \end{aligned}$$

So applying the map ρ of proposition 2.3 to Γ yields

$$\rho(\Gamma(t)) = \det(X(t) + iY(t)) = \cos(2\pi t) + i \sin(2\pi t) = \mathrm{e}^{2\pi i t}$$

which is the standard generator of $\pi_1(S^1)$. Now ρ induces an isomorphism of fundamental groups, hence Γ generates $\pi_1(Sp(n))$. □

An explicit isomorphism $\pi_1(Sp(n)) \to \mathbb{Z}$, i.e. a homotopy invariant index for loops in $Sp(n)$, is given by the Maslov index.

Theorem 2.5 *([17] Thm. 2.27)*
There exists a unique functor m, called the **Maslov index**, *which assigns an integer $m(\Phi)$ to every symplectic loop $\Phi : \mathbb{R}/\mathbb{Z} \to Sp(n)$ and satisfies the following axioms:*

(i) Two loops in $Sp(n)$ are homotopic if and only if they have the same Maslov index.

(ii) For any two loops $\Phi, \Psi : \mathbb{R}/\mathbb{Z} \to Sp(n)$

$$m(\Phi\Psi) = m(\Phi) + m(\Psi).$$

(iii) If $n = k + l$ identify $Sp(k) \oplus Sp(l)$ in the obvious way with a subgroup of $Sp(n)$. Then for any $\Phi : \mathbb{R}/\mathbb{Z} \to Sp(k), \Psi : \mathbb{R}/\mathbb{Z} \to Sp(l)$

$$m(\Phi \oplus \Psi) = m(\Phi) + m(\Psi).$$

(iv) The loop $\Phi : \mathbb{R}/\mathbb{Z} \to U(1) \subset Sp(n), t \mapsto \mathrm{e}^{2\pi i t}$ has Maslov index 1.

The Maslov index of a symplectic loop Φ is given explicitly by the winding number around 0 of $\rho(\Phi(t))$. This is equivalent to normalizing the isomorphism of proposition 2.3 by $m(\Gamma) = 1$ where Γ is the generator of $\pi_1(Sp(n))$ given by remark 2.4.

Remark 2.6 *Let $\Phi_w(t) = \mathrm{e}^{2\pi w t J} \in \Sigma(1)$ for any $w \in \mathbb{Z}$, that is $\Phi = w\Gamma$, then*

$$m(\Phi_w) = w.$$

Proof: This is deduced easily from the defining properties of the Maslov index. Indeed, for $w = 0$,

$$m(\mathbb{1}) \overset{(ii)}{=} m(\mathbb{1}) + m(\mathbb{1}) \quad \Rightarrow \quad m(\mathbb{1}) = 0$$

and for $w \in \mathbb{Z}^+$ — noting that $J_1 \cong i$ as $\mathbb{R}^2 \cong \mathbb{C}$ —

$$m(\mathrm{e}^{2\pi w t J_1}) \overset{(ii)}{=} w\, m(\mathrm{e}^{2\pi t J_1}) \overset{(iv)}{=} w$$

and hence for $w \in \mathbb{Z}^-$ we also have $m(\mathrm{e}^{2\pi w t J_1}) = w$ since

$$0 = m(\mathbb{1}) \overset{(ii)}{=} m(\mathrm{e}^{2\pi w t J_1}) + m(\mathrm{e}^{2\pi(-w)t J_1}) = m(\mathrm{e}^{2\pi w t J_1}) - w.$$

$\square$

One wishes to define an index not only for loops but also for arcs not ending at the identity. Since $Sp(n)$ is connected and we want the index to be invariant under homotopies (not necessarily with fixed endpoints), we need to restrict the set of arcs considered; otherwise all arcs would be homotopic to the constant arc and hence have the same index. This restriction is done by introducing the set of nondegenerate symplectic matrices which do not have 1 in their spectrum,

$$Sp^*(n) := \{\Phi \in Sp(n) | \det(\Phi - \mathbb{1}) \neq 0\}.$$

Proposition 2.7 *$Sp^*(n)$ has two connected components,*

$$\begin{aligned} Sp^+(n) &= \{\Phi \in Sp(n) | \det(\Phi - \mathbb{1}) > 0\}, \\ Sp^-(n) &= \{\Phi \in Sp(n) | \det(\Phi - \mathbb{1}) < 0\} \end{aligned}$$

and we have

$$\begin{array}{lcll} W^+ & := & -\mathbb{1} & \in \; Sp^+(n), \\ W^- & := & \mathrm{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1) & \in \; Sp^-(n). \end{array}$$

For the proof we need the following generalization of lemma 2.18 from [17].

Lemma 2.8 *Let $\Psi \in Sp(n)$. Then*

$$\lambda \in \sigma(\Psi) \iff \lambda^{-1} \in \sigma(\Psi)$$

and the algebraic multiplicities of λ and λ^{-1} agree. If ± 1 is an eigenvalue of Ψ then it occurs with even multiplicity. Moreover, let

$$E_\lambda = \bigcup_{k \in \mathbb{N}} \ker(\Psi - \lambda \mathbb{1})^k \subset \mathbb{C}^{2n}$$

be the generalized complex eigenspace for $\lambda \in \sigma(\Psi)$, then for $\lambda\mu \neq 1$ the eigenspaces E_λ and E_μ are ω-orthogonal, that is

$$\forall y \in E_\lambda, z \in E_\mu \; : \; \omega(y, z) = 0.$$

Proof: For the first statement note that Ψ^T and Ψ^{-1} are similar by (2.2):

$$\Psi^{-1} = J \Psi^T J^{-1},$$

hence the characteristic polynomials of Ψ and Ψ^{-1} are identical. On the other hand we have

$$(\Psi - \lambda)^k z = 0 \quad \Leftrightarrow \quad (-\lambda^{-1}\Psi^{-1})^k (\Psi - \lambda)^k z = (-\lambda^{-1} + \Psi^{-1})^k z = 0,$$

thus $\ker(\Psi - \lambda)^k = \ker(\Psi^{-1} - \lambda^{-1})^k$ for any $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}^*$, hence E_λ and $E_{\lambda^{-1}}$ are of the same dimension. So the product of all eigenvalues except for ± 1 — repeated according to their multiplicity — equals 1 and the total multiplicity of these eigenvalues is even. Since the determinant, that is the product of all eigenvalues, of a symplectic matrix is 1, it follows that if -1 is an eigenvalue, then it occurs with even multiplicity. Hence the total multiplicity of eigenvalues not equal to 1 is even and since the space is of even dimension, obviously 1 has to occur with even multiplicity as well.

The second statement is lemma 1.3 in [3]. We repeat the proof here for convenience. Let $E_\lambda^k = \ker(\Psi - \lambda)^k$, then since $E_\lambda^{2n} = E_\lambda$ it suffices to prove that $\omega(E_\lambda^k, E_\mu^l) = 0$ for all $k, l \geq 0$, which will be done by induction with respect to $m = k + l$. For $m = 0$ we have $k = l = 0$ and $E_\lambda^0 = E_\mu^0 = \{0\}$, so the statement is trivial. Now assume $\omega(E_\lambda^k, E_\mu^l) = 0$ for all $k, l \leq m$ and consider any $y \in E_\lambda^k$, $z \in E_\mu^l$ with $k+l = m+1$. We set $y^1 := (\Psi - \lambda)y \in E_\lambda^{k-1}$ and $z^1 := (\Psi - \mu)z \in E_\mu^{l-1}$, where $E_\lambda^{-1} := \{0\}$, and obtain because of Ψ being symplectic

$$\begin{aligned}
\omega(\lambda y, \mu z) &= \omega(\Psi y - y^1, \Psi z - z^1) \\
&= \omega(y, z) - \omega(\Psi y, z^1) - \omega(y^1, \Psi z) + \omega(y^1, z^1) \\
&= \omega(y, z) - \omega(\lambda y, z^1) - \omega(y^1, \mu z) - \omega(y^1, z^1)
\end{aligned}$$

The last three terms vanish by the induction hypothesis and hence

$$(\lambda\mu - 1)\, \omega(y, z) = 0$$

wich proves that $\omega(E_\lambda^k, E_\mu^l) = 0$ for $\lambda\mu \neq 1$. □

Proof of proposition 2.7: $Sp^+(n)$ and $Sp^-(n)$ are disjoint components of $Sp^*(n)$ since $\Phi \mapsto \det(\Phi - \mathbb{1})$ is continuous. One also easily checks that $W^\pm \in Sp^\pm(n)$, so it remains to be shown that any matrix in $Sp^*(n)$ can within this space be connected to either W^+ or W^-. Following [3] we first prove that any $A \in Sp^*(n)$ can be connected in $Sp^*(n)$ to a matrix with $2n$ distinct eigenvalues.

Let λ be a degenerate eigenvalue of A. Since A is symplectic (hence $\det(A) = 1$) and nondegenerate, we have $\lambda \notin \{0, 1\}$. Furthermore, $\overline{\lambda}$ is an eigenvalue of A with the same algebraic multiplicity as λ because of A being a real matrix, and from lemma 2.8 we know that λ^{-1} and $\overline{\lambda}^{-1}$ also are eigenvalues of A with again the same algebraic multiplicity as λ. We will show that A can be connected in $Sp^*(n)$ to a matrix $\tilde{A}$ such that the dimensions of E_λ, $E_{\overline{\lambda}}$, $E_{\lambda^{-1}}$ and $E_{\overline{\lambda}^{-1}}$ decrease by at least one, there are some new eigenvalues of multiplicity 1 and the rest of the spectrum of A remains unchanged. The claim then follows by repeating this construction several times for every eigenspace until there are no degeneracies left.

There are several cases to consider corresponding to whether some of $\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}$ are the same. Generally, we have $\mathbb{C}^{2n} = E \oplus F$ with

$$E = E_\lambda + E_{\overline{\lambda}} + E_{\lambda^{-1}} + E_{\overline{\lambda}^{-1}}\,, \qquad F = \bigoplus_{\substack{\mu \in \sigma(A) \\ \mu \neq \lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}}} E_\mu\,.$$

We start to construct a continuous path $A : [0,1] \to Sp^*(n)$ by defining $A(s) := B(s)A$ for some $B : [0,1] \to \mathbb{C}^{2n \times 2n}$ with $B|_F \equiv \mathbb{1}$. For E, we will in each of the subsequent cases construct an appropriate basis $(z_i)_{i\in\{1,\dots,N\}}$, $N = \dim E$ and set $Bz_i = \gamma_i z_i$ for some continuous, complex functions γ_i meeting $\gamma_i(0) = 1$, so that $A(0) = A$ and BA is continuous with respect to s.

Note that BA leaves E and F invariant, so for the nondegeneracy and the change of the spectrum we only have to consider $BA|_E$. Moreover, by the preceding lemma, E and F are ω-orthogonal, so for any $y \in F$ and $z \in E$ we have $\omega(BAy, BAz) = 0 = \omega(y, z)$. Since $BA|_F = A|_F$ is symplectic, we obviously also have $\omega(BAy, BAz) = \omega(y, z)$ for $y, z \in F$, so for $BA \in Sp^*(n)$ it remains to ensure that

(i) $\forall i \in \{1, \dots, N\} : \ \overline{Bz_i} = B\overline{z_i}$,
so B is a real matrix (for $z \in F$ the above holds trivially since for $z \in E_\mu \subset F$ also $\overline{z} \in E_{\overline{\mu}} \subset F$).

(ii) $\forall i, j \in \{1, \dots, N\} : \ \omega(z_i, z_j) = 0 \quad \text{or} \quad \gamma_i \gamma_j = 1$
since then $\omega(Bz_i, Bz_j) = \omega(z_i, z_j)$, so $B|_E$ and hence BA is symplectic,

(iii) $1 \notin \sigma(BA|_E)$.

Consequently, in the following cases we have to construct a basis (z_i) of E and choose the functions γ_i such that (i) to (iii) are fulfilled for all $s \in [0, 1]$ and when comparing A to $B(1)A$, some degeneracy of the eigenvalue λ has to be removed in such a way that some new nondegenerate eigenvalues occur.

(I) $\boldsymbol{\lambda} \in \mathbb{C} \setminus (\mathbb{R} \cup \boldsymbol{S^1})$
We choose any $e \in \mathbb{C}^{2n}$ such that $Ae = \lambda e$, then because of the preceding lemma and the nondegeneracy of ω we find an $f \in E_{\lambda^{-1}}$ such that $\omega(e, f) = 1$. These two vectors can be extended to bases $\{e, \tilde{e}_1, \dots, \tilde{e}_m\}$ and $\{f, \tilde{f}_1, \dots, \tilde{f}_m\}$ of E_λ and $E_{\lambda^{-1}}$ respectively and for the first base we can require moreover that

$$\forall k \in \{1, \dots, m\} : \ (A - \lambda)\tilde{e}_k \in \operatorname{span}(e, \tilde{e}_1, \dots, \tilde{e}_{k-1}). \tag{2.5}$$

Setting $e_k = \tilde{e}_k + \omega(f, \tilde{e}_k)e$, $f_k = \tilde{f}_k - \omega(e, \tilde{f}_k)f$ we obtain bases $\{e, e_1, \dots, e_m\}$ and $\{f, f_1, \dots, f_m\}$ of E_λ and $E_{\lambda^{-1}}$ such that (2.5) still holds for the e_k and for all $k \in \{1, \dots, m\}$ we have $\omega(e, f_k) = 0$ and $\omega(f, e_k) = 0$. Moreover, since A is real, the above construction yields bases $\{\overline{e}, \overline{e_1}, \dots, \overline{e_m}\}$ and $\{\overline{f}, \overline{f_1}, \dots, \overline{f_m}\}$ of $E_{\overline{\lambda}}$ and $E_{\overline{\lambda}^{-1}}$ that in addition meet $\omega(\overline{e}, \overline{f_k}) = 0$ and $\omega(\overline{f}, \overline{e_k}) = 0$ for all $k \in \{1, \dots, m\}$. As basis of $E = E_\lambda \oplus E_{\overline{\lambda}} \oplus E_{\lambda^{-1}} \oplus E_{\overline{\lambda}^{-1}}$ we now use $\{e, \overline{e}, f, \overline{f}, e_k, \overline{e_k}, f_k, \overline{f_k} \mid k = 1, \dots, m\}$ and define B to be the identity on all basis vectors except for

$$\begin{aligned} Be &= \gamma e, & B\overline{e} &= \overline{\gamma}\,\overline{e}, \\ Bf &= \gamma^{-1} f, & B\overline{f} &= \overline{\gamma}^{-1}\overline{f} \end{aligned}$$

for some continuous $\gamma : [0,1] \to \mathbb{C}^*$ with $\gamma(0) = 1$ and $\gamma(s)\lambda \notin \sigma(A) \cup \mathbb{R} \cup S^1$ for all $s \in (0,1]$. This obviously meets (i), and (ii) is easily checked when keeping in mind the ω-orthogonality from the previous lemma. Finally, E_λ is left invariant by $B(s)A$ for all $s \in [0,1]$ and with respect to the basis $\{e, e_1, \ldots, e_m\}$ we have

$$B(s)A|_{E_\lambda} = \begin{pmatrix} \gamma(s)\lambda & * & \cdots & * \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

From the characteristic polynomial of this matrix it is clear that $B(s)A|_E$ still has the eigenvalue λ (and hence also $\overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}$) with multiplicity m and in addition it has the nondegenerate eigenvalues $\gamma(s)\lambda$, $\overline{\gamma(s)\lambda}$, $(\gamma(s)\lambda)^{-1}$, $\overline{\gamma(s)\lambda}^{-1}$; they are distinct since $\gamma(s)\lambda \notin \mathbb{R} \cup S^1$. Since these multiplicities add up to the dimension of E, there are no other eigenvalues. Hence (iii) holds by the choice of γ and the spectrum of $B(1)A$ is of the required form.

(II) $\boldsymbol{\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}}$

First note that for $\mu \in \mathbb{R}$ and $g \in E_\mu$, that is $(A - \mu)^k g = 0$ for some $k \in \mathbb{N}$ we also have $\Re(g), \Im(g) \in E_\mu$ because of A being real.

We choose any real eigenvector e for the eigenvalue λ of A. As in the first case we find $\tilde{f} \in E_{\lambda^{-1}}$ such that $\omega(e, \tilde{f}) = 1$, then we have $f := \Re(\tilde{f}) \in E_{\lambda^{-1}}$ and $\omega(e, f) = 1$ since ω is a real linear map. There also have to be real bases of E_λ and $E_{\lambda^{-1}}$ containing e and f respectively. Starting from these and using the same construction as in (I) we obtain real bases $\{e, e_1, \ldots, e_m\}$ and $\{f, f_1, \ldots, f_m\}$ of E_λ and $E_{\lambda^{-1}}$, such that (2.5) holds for the e_k, and for all $k \in \{1, \ldots, m\}$ we have $\omega(e, f_k) = 0$ and $\omega(f, e_k) = 0$. As basis of $E = E_\lambda \oplus E_\lambda^{-1}$ we use $\{e, f, e_k, f_k \mid k = 1, \ldots, m\}$ and define B to be the identity on $\{e_k, f_k \mid k = 1, \ldots, m\}$ and on the rest of the basis

$$Be = \gamma e, \qquad Bf = \gamma^{-1} f$$

for continuous $\gamma : [0,1] \to \mathbb{R}^*$ with $\gamma(0) = 1$ and $\gamma(s)\lambda \notin \sigma(A) \cup \{-1, 1\}$ for all $s \in (0,1]$. This meets (i) since all basis vectors are real and (ii) is easily checked as well. Exactly as in (I) we deduce that $B(s)A|_E$ still has the eigenvalues λ and λ^{-1} with multiplicity m and the nondegenerate eigenvalues $\gamma(s)\lambda, (\gamma(s)\lambda)^{-1}$, which are distinct since $\gamma(s)\lambda \notin \{-1, 1\}$. So by the choice of γ we know that (iii) holds and the spectrum of $B(1)A$ is of the required form.

(III) $\boldsymbol{\lambda = -1}$

As in (II) we can choose a real eigenvector e and find some real $f \in E_{-1}$

such that $\omega(e, f) = 1$. Since ω is skewsymmetric, e and f have to be linearly independent and we can extend them to a real basis $\{e, f, g_1, \dots, g_m\}$ of E_{-1}. After changing the basis by $e_k = g_k - \omega(e, g_k)f + \omega(f, g_k)e$ to $\{e, f, e_1, \dots, e_m\}$ we have $\omega(e, e_k) = 0 = \omega(f, e_k)$ for $k \in \{1, \dots, m\}$ and moreover $(A+1)e_k \in \mathrm{span}(e, e_1, \dots, e_m)$. Indeed, A leaves E_{-1} invariant, so we have $Ae_k = -e_k + \tilde{e} + \alpha f$ with $\tilde{e} \in \mathrm{span}(e, e_1, \dots, e_m)$ and we obtain

$$\begin{aligned}
\omega(e, e_k) &= \omega(Ae, Ae_k) = \omega(-e, -e_k) + \omega(-e, \tilde{e}) + \omega(-e, \alpha f) \\
&= \omega(e, e_k) - \alpha
\end{aligned}$$

which implies $\alpha = 0$. So there is a basis of $\mathrm{span}(e_1, \dots, e_m)$, again denoted by (e_k), such that

$$\forall k \in \{1, \dots, m\}: \ (A+1)e_k \in \mathrm{span}(e, e_1, \dots, e_{k-1})$$

and the above ω-orthonormalities still hold. We define B to be the identity on $\{e_1, \dots, e_m\}$ and for the rest of the above basis of $E = E_{-1}$ we set

$$Be = \gamma e, \qquad Bf = \gamma^{-1} f$$

for $\gamma : [0,1] \to \mathbb{R}^*$ continuous, with $\gamma(0) = 1$ and $-\gamma(s) \notin \sigma(A) \cup \{-1, 1\}$ for all $s \in (0, 1]$. As before, this construction meets (i) and (ii). Furthermore, with respect to the basis $\{e, e_1, \dots, e_m\, f\}$ we have

$$B(s)A|_E = \begin{pmatrix}
-\gamma(s) & * & \cdots & \cdots & * \\
0 & -1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & -1 & * \\
0 & \cdots & \cdots & 0 & -\gamma(s)^{-1}
\end{pmatrix}.$$

Hence $B(s)A|_E$ still has the eigenvalue -1 with multiplicity m and, in addition, it has the two distinct nondegenerate eigenvalues $-\gamma(s)$ and $-\gamma(s)^{-1}$. So (iii) holds since $-\gamma(s) \neq 1$ and finally the spectrum of $B(1)A$ is of the required form.

(IV) $\boldsymbol{\lambda \in S^1 \setminus \{-1, 1\}}$

When we choose some eigenvector e for the eigenvalue λ of A, we obtain $\overline{e} \in E_{\overline{\lambda}} = E_{\lambda^{-1}}$, so $\omega(e, \overline{e})$ can either vanish or not — which are cases that have to be treated differently.

a)$\boldsymbol{\omega(e, \overline{e}) = c \neq 0}$:

In this case we can find a basis $\{e, \tilde{e}_1, \dots, \tilde{e}_m\}$ of E_λ that meets (2.5) and set $e_k = \tilde{e}_k + \omega(\overline{e}, \tilde{e}_k)c^{-1}e$ to obtain a basis $\{e, e_1, \dots, e_m\}$ such that (2.5) still holds for the e_k and for all $k \in \{1, \dots, m\}$ we have $\omega(\overline{e}, e_k) = 0$. Since A is real, this also gives us a basis $\{\overline{e}, \overline{e_1}, \ldots, \overline{e_m}\}$ of $E_{\overline{\lambda}}$ with $\omega(e, \overline{e_k}) = 0$ for all $k \in \{1, \ldots, m\}$. We use $\{e, \overline{e}, e_k, \overline{e_k} \mid k = 1, \ldots, m\}$ as basis of $E = E_\lambda \oplus E_{\overline{\lambda}}$ and define B to be the identity on all basis vectors except for

$$Be = \gamma e, \qquad B\overline{e} = \overline{\gamma}\,\overline{e}$$

for continuous $\gamma : [0,1] \to S^1$ with $\gamma(0) = 1$ and $\gamma(s)\lambda \notin \sigma(A) \cup \{-1, 1\}$ for all $s \in (0,1]$. Obviously, (i) is met and for (ii) it suffices to note that $\overline{\gamma} = \gamma^{-1}$. As in (I) one can see that $B(s)A|_E$ still has the eigenvalues λ and $\overline{\lambda}$ with multiplicity m and it has two new nondegenerate eigenvalues $\gamma(s)\lambda, \overline{\gamma(s)\lambda}$, which are distinct by the choice of γ. This was also made such that (iii) holds and the spectrum of $B(1)A$ is of the required form.

b) $\boldsymbol{\omega(e, \overline{e}) = 0}$:
In this case we find $\tilde{f} \in E_{\overline{\lambda}}$ such that $\omega(e, \tilde{f}) = 1$. From

$$\omega(\tilde{f}, \overline{\tilde{f}}) = -\omega(\overline{\tilde{f}}, \tilde{f}) = -\overline{\omega(\tilde{f}, \overline{\tilde{f}})}$$

it follows that $\alpha := \omega(\tilde{f}, \overline{\tilde{f}}) \in i\mathbb{R}$, so we have $f := \tilde{f} - \frac{1}{2}\alpha\overline{e} \in E_{\overline{\lambda}}$ with

$$\omega(f, \overline{f}) = \alpha + \tfrac{1}{2}\overline{\alpha} - \tfrac{1}{2}\alpha = 0 \quad \text{and} \quad \omega(e, f) = 1.$$

Next we can find a basis $\{e, \overline{f}, \tilde{e}_1, \ldots, \tilde{e}_m\}$ of E_λ that we then change to $\{e, \overline{f}, e_1, \ldots, e_m\}$ by

$$e_k = \tilde{e}_k + \omega(f, \tilde{e}_k)e - \omega(\overline{e}, \tilde{e}_k)\overline{f}$$

such that for all $k \in \{1, \ldots, m\}$ we have $\omega(f, e_k) = 0$ and $\omega(\overline{e}, e_k) = 0$. Moreover, for $Ae_k = \lambda e_k + \tilde{e} + \alpha\overline{f}$ with $\tilde{e} \in \operatorname{span}(e, e_1, \ldots, e_m)$ we calculate

$$\begin{aligned} \omega(\overline{e}, e_k) &= \omega(A\overline{e}, Ae_k) = \omega(\overline{\lambda}\overline{e}, \lambda e_k) + \omega(\overline{\lambda}\overline{e}, \tilde{e}) + \omega(\overline{\lambda}\overline{e}, \alpha\overline{f}) \\ &= \omega(\overline{e}, e_k) + \overline{\lambda}\alpha \end{aligned}$$

which implies $\alpha = 0$. Hence we have $(A - \lambda)e_k \in \operatorname{span}(e, e_1, \ldots, e_m)$ for all $k \in \{1, \ldots, m\}$, and thus there is a basis of $\operatorname{span}(e_1, \ldots, e_m)$, again denoted by (e_k), such that

$$\forall k \in \{1, \ldots, m\} : \ (A - \lambda)e_k \in \operatorname{span}(e, e_1, \ldots, e_{k-1}).$$

This change of basis does not affect the ω-orthonormalities constructed above. Furthermore, since A and ω are real maps, $\{f, \overline{e}, \overline{e_1}, \ldots, \overline{e_m}\}$ is a basis of $E_{\overline{\lambda}}$ such that $\omega(e, \overline{e_k}) = 0$ and $\omega(\overline{f}, \overline{e_k}) = 0$ for all $k \in \{1, \ldots, m\}$. Using $\{e, \overline{e}, f, \overline{f}, e_k, \overline{e_k} \mid k = 1, \ldots, m\}$ as basis of $E = E_\lambda \oplus E_{\overline{\lambda}}$ we define B to be the identity on $\{e_k, \overline{e_k} \mid k = 1, \ldots, m\}$ and set

$$\begin{aligned} Be &= \gamma e, & B\overline{e} &= \overline{\gamma}\,\overline{e}, \\ Bf &= \gamma^{-1} f, & B\overline{f} &= \overline{\gamma}^{-1}\overline{f} \end{aligned}$$

for some continuous $\gamma : [0,1] \to \mathbb{C}^*$ with $\gamma(0) = 1$ and $\gamma(s)\lambda \notin \sigma(A) \cup \mathbb{R} \cup S^1$ for all $s \in (0,1]$. One can check that this meets (i) and (ii). Finally, $B(s)A$ leaves E_λ invariant for all $s \in [0,1]$ and with respect to the basis $\{e, e_1, \ldots, e_m, \overline{f}\}$ it is represented by the following matrix

$$B(s)A|_{E_\lambda} = \begin{pmatrix} \gamma(s)\lambda & * & \cdots & \cdots & * \\ 0 & \lambda & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda & * \\ 0 & \cdots & \cdots & 0 & \overline{\gamma(s)}^{-1}\lambda \end{pmatrix}.$$

Obviously, $B(s)A|_E$ still has the eigenvalue λ with multiplicity m and the two new eigenvalues $\gamma(s)\lambda$ and $\overline{\gamma(s)}^{-1}\lambda = \overline{\gamma(s)\lambda}^{-1}$ are nondegenerate and distinct by the choice of γ. Moreover, (iii) holds and the spectrum of $B(1)A$ is of the required form.

So we have shown that a given matrix in $Sp^*(n)$ can be connected to a matrix with nondegenerate eigenvalues. The second step of this proof is to show that a matrix like this, again denoted by A, can be connected to a matrix that has the same eigenvalues as either W^+ or W^-. This is done by moving all or all but two eigenvalues λ to -1 for which purpose we again have to consider several cases:

1.) $\boldsymbol{\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup S^1)}$

In this case, for A there is a group of four distinct eigenvalues $\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}$ with corresponding eigenvectors $e, \overline{e}, f, \overline{f}$ that are unique up to complex multiples. By lemma 2.8 this basis of $E := E_\lambda \oplus E_{\overline{\lambda}} \oplus E_{\lambda^{-1}} \oplus E_{\overline{\lambda}^{-1}}$ meets some ω-orthogonality relations, so we can easily define a path $B : [0,1] \to \mathbb{C}^{2n\times 2n}$ that meets (i) and (ii) and hence is a path of symplectic matrices: On all eigenspaces except for E we define B to be the identity and on E we set

$$\begin{aligned} Be &= \gamma e, & B\overline{e} &= \overline{\gamma}\,\overline{e}, \\ Bf &= \gamma^{-1} f, & B\overline{f} &= \overline{\gamma}^{-1}\overline{f} \end{aligned}$$

where $\gamma : [0,1] \to \mathbb{C}^* \setminus \{\lambda^{-1}\}$ is a continuous path from $\gamma(0) = 1$ to $\gamma(1) = -\lambda^{-1}$. The path $B(s)A$ obviously lies within $Sp(n)$ and its spectrum remains unchanged except for the group $\gamma(s)\lambda, \overline{\gamma(s)\lambda}, (\gamma(s)\lambda)^{-1}, \overline{\gamma(s)\lambda}^{-1}$ that does not meet 1 and for $s = 1$ is moved to -1. So this is the required path in $Sp^*(n)$.

2.) $\boldsymbol{\lambda \in S^1 \setminus \{-1, 1\}}$

Here we have $\lambda^{-1} = \overline{\lambda}$ and thus only have to consider the pair λ, λ^{-1} of distinct eigenvalues with corresponding eigenvectors e and $\overline{e}$ that are ω-orthogonal to all other eigenspaces. A path $B : [0,1] \to Sp(n)$ can be defined by

$$Be = \gamma e, \qquad B\overline{e} = \gamma^{-1}\overline{e}$$

and the identity on all other eigenspaces of A. In this case we have to choose the continuous path γ from $\gamma(0) = 1$ to $\gamma(1) = -\lambda^{-1}$ to lie within $S^1 \setminus \{\lambda^{-1}\}$, so that $\gamma^{-1} = \overline{\gamma}$ ensures (i). This again gives us a path $B(s)A$ in $Sp^*(n)$ along which the eigenvalues $\gamma(s)\lambda, (\gamma(s)\lambda)^{-1}$ are moved to -1 and the rest of the spectrum remains unaffected.

3.) $\boldsymbol{\lambda \in \mathbb{R}^- \setminus \{-1\}}$

For the pair λ, λ^{-1} of distinct real eigenvalues we find corresponding real eigenvectors e and f that are, by lemma 2.8, ω-orthogonal to all other eigenvectors. We then define a path $B : [0,1] \to Sp(n)$ by

$$Be = \gamma e, \qquad Bf = \gamma^{-1} f$$

and on all other eigenspaces of A we let B be the identity. Since $\lambda < 0$ we can find a continuous path γ from $\gamma(0) = 1$ to $\gamma(1) = -\lambda^{-1}$ within $\mathbb{R}^+$. This path has to be real in order that (i) holds and since it does not meet λ^{-1}, it defines a path $B(s)A$ in $Sp^*(n)$. Again, the eigenvalues $\gamma(s)\lambda, (\gamma(s)\lambda)^{-1}$ are moved to -1 and the rest of the spectrum is constant.

4.) $\boldsymbol{\lambda \in \mathbb{R}^+ \setminus \{1\}}$

As in the last case we have a pair λ, λ^{-1} of distinct real eigenvalues with corresponding real eigenvectors and we define the path B as before except for γ, which in this case can not be chosen to connect the eigenvalues to -1 within $\mathbb{R} \setminus \{1\}$. Instead, we exchange λ and λ^{-1} such that $\lambda > 1$ and then find a continous path $\gamma : [0,1] \to \mathbb{R}^+ \setminus \{\lambda^{-1}\}$ from $\gamma(0) = 1$ to $\gamma(1) = 2\lambda^{-1}$. For this choice, (i) and (ii) are met, hence $B(s)A$ is a path in $Sp(n)$. Its nonconstant eigenvalues $\gamma(s)\lambda$ and $(\gamma(s)\lambda)^{-1}$ do not meet 1, hence the path even lies in $Sp^*(n)$ and moves the pair $\{\lambda, \lambda^{-1}\}$ to $\{2, \frac{1}{2}\}$.

For any two such pairs of positive real eigenvalues, this construction connects A to a matrix $\tilde{A} \in Sp^*(n)$ that has real eigenvectors e_1, e_2 with eigenvalue 2 and f_1, f_2 with eigenvalue $\frac{1}{2}$, such that between these vectors only $\omega(e_i, f_i)$ does not vanish. We can rescale these vectors such that $\omega(e_i, f_i) = 1$. Moreover, $\tilde{A}$ leaves invariant the splitting $\mathbb{C}^{2n} = E \oplus F$, where $E = \mathrm{span}(e_1, e_2, f_1, f_2)$ and F is the span of all other eigenvectors, and by construction we have $\omega(E, F) = 0$. We now set $e := e_1 + ie_2 \in E_2$ and $f := f_1 - if_2 \in E_{\frac{1}{2}}$, then

$$\omega(e, \overline{e}) = \omega(e, \overline{f}) = \omega(\overline{e}, f) = \omega(f, \overline{f}) = 0,$$

so we have a basis $\{e, \overline{e}, f, \overline{f}\}$ of E for which only $\omega(e, f)$ and $\omega(\overline{e}, \overline{f})$ do not vanish. Because of this we can define a path $B : [0,1] \to Sp(n)$ by the identity on F and

$$\begin{array}{rclcrcl} Be &=& \gamma e, && B\overline{e} &=& \overline{\gamma}\,\overline{e}, \\ Bf &=& \gamma^{-1}f, && B\overline{f} &=& \overline{\gamma}^{-1}\overline{f} \end{array}$$

for some continuous $\gamma : [0,1] \to \mathbb{C}^* \setminus \{\frac{1}{2}\}$ with $\gamma(0) = 1$ and $\gamma(1) = -\frac{1}{2}$. Along the path $B\tilde{A}$ there are only four nonconstant eigenvalues, $2\gamma(s)$, $\frac{1}{2}\gamma(s)^{-1}$, $2\overline{\gamma(s)}$ and $\frac{1}{2}\overline{\gamma(s)}^{-1}$, and they do not meet 1. Hence $\tilde{A}$ is connected within $Sp^*(n)$ to $B(1)\tilde{A}$ such that these four eigenvalues are moved to -1. This construction can be repeated until no or only one pair of eigenvalues $\{2, \frac{1}{2}\}$ is left.

If there is no pair of positive real eigenvalues left, then by the above two steps we have connected the given $A \in Sp^*(n)$ to a matrix that has $2n$ linear independent eigenvectors with the eigenvalue -1, that is to $-\mathbb{1} = W^+$.

Otherwise, A is connected to a matrix $\tilde{A}$ with the same eigenvalues and multiplicities as W^-, that is $\tilde{A} = -\mathbb{1} \oplus \left(\begin{smallmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix}\right) \in Sp^*(n)$ with respect to some splitting $\mathbb{C}^{2n} = \tilde{G} \oplus \tilde{E}$. Since all eigenvalues and $\tilde{A}$ itself are real, we can find a real basis of normalized eigenvectors $\{e, g_1, \ldots, g_{n-1}, f, g_n, \ldots, g_{2n-2}\}$ of $\tilde{A}$ with respect to which $\tilde{A} = \mathrm{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1)$. In general, this basis is not orthogonal, as the elementary example

$$\begin{pmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \in Sp^*(1)$$

shows, but we can change $\tilde{A}$ within $Sp^*(n)$ so that its eigenspaces become orthogonal. In our special case, for the orthogonality it suffices to achieve $E_{\frac{1}{2}} = JE_2$, which will be done in two steps. In the first step we change $\tilde{A}$ such that $E_{\frac{1}{2}} \perp E_2$. For the construction of the corresponding path $B : [0,1] \to Sp^*(n)$ we use the splitting $\mathbb{R}^{2n} = G \oplus \mathbb{R}e \oplus \mathbb{R}f(s)$ with $G = \mathrm{span}(g_1, \ldots, g_{2n-2})$ and $f(s) = f - s(e,f)e$ where $(\cdot,\cdot)$ is the Euclidean product on $\mathbb{R}^{2n}$: we define $B(s)$ by $B(s)|_G = -\mathbb{1}$, $B(s)e = 2e$ and $B(s)f(s) = \frac{1}{2}f(s)$. By lemma 2.8 we have $\omega(G,e) = \omega(G,f) = 0$ and thus also $\omega(G, f(s)) = 0$, hence $B(s) \in Sp^*(n)$ since it preserves ω and $1 \notin \sigma(B(s)) = \{-1, 2, \frac{1}{2}\}$. So along B the matrix $B(0) = \tilde{A}$ is changed within $Sp^*(n)$ such that $E_2 \perp E_{\frac{1}{2}}$ holds. The latter is due to $(e, f(1)) = 0$.

We now can choose $e \in E_2$ and $f \in E_{\frac{1}{2}}$ such that $\|e\| = \|f\| = 1$, $e \perp f$ and $\omega(f,e) = \alpha > 0$, then because of $\omega(G,e) = (G, Je) = 0$ and $\omega(G,f) = (G, Jf) = 0$ we have the orthogonal splitting

$$\mathbb{R}^{2n} = G \oplus^\perp \mathbb{R}Jf \oplus^\perp \mathbb{R}Je.$$

Since $\omega(z,z) = (z, Jz) = 0$ this splitting yields

$$e = g_e - \alpha Jf, \qquad f = g_f + \alpha Je \tag{2.6}$$

with $g_e, g_f \in G$. If one of g_e, g_f vanishes, then so does the other and $\alpha = 1$. This can be easily seen from (2.6) when remembering that $J^2 = -\mathbb{1}$. In this case, above orthogonal splitting becomes $\mathbb{R}^{2n} = E_{-1} \oplus^{\perp} E_2 \oplus^{\perp} E_{\frac{1}{2}}$, so the eigenspaces are orthogonal, and moreover $E_{\frac{1}{2}} = JE_2$.

If we assume $g_e, g_f \neq 0$ then because of $e \perp f$ we have

$$(g_e, g_f) = (e + \alpha Jf, f - \alpha Je) = 0$$

and hence $g_e \perp g_f$, only $Je = f$ does not necessarily hold. But again we can change $\tilde{A}$ within $Sp^*(n)$ to arrive at $JE_2 = E_{\frac{1}{2}}$. For this purpose we extend g_e, g_f to a basis $\{g_e, g_f, h_1, \ldots, h_{2n-4}\}$ of G such that $\omega(g_f, h_k) = 0$ for all $k \in \{1, \ldots, 2n-4\}$. This is possible since

$$\omega(g_f, g_e) = \omega(f - \alpha Je, e + \alpha Jf) = \omega(f, e) - \alpha^2 \omega(Je, Jf) = \alpha - \alpha^3$$

and $\alpha \neq \pm 1$ which can be seen from (2.6) because of e and f being normalized and $g_e, g_f \neq 0$. So we have $\omega(g_f, g_e) = c \neq 0$ and thus we can change a given basis $\{g_e, g_f, \tilde{h}_1, \ldots, \tilde{h}_{2n-4}\}$ of G by $h_k := \tilde{h}_k - \omega(g_f, \tilde{h}_k)c^{-1} g_e$ to meet $\omega(g_f, h_k) = 0$.

We now use the splitting

$$\mathbb{R}^{2n} = H \oplus \operatorname{span}(e, f(s), g(s), g_f)$$

with $H = \operatorname{span}(h_1, \ldots, h_{2n-4})$ and

$$f(s) = f - s g_f \ , \qquad g(s) = g_e + s \frac{\omega(g_e, g_f)}{\omega(e, f)} e$$

for the construction of a path $B : [0,1] \to Sp^*(n)$ that connects $\tilde{A}$ to a matrix with the required property: $B(s)$ can be defined by $-\mathbb{1}$ on $g_f, g(s)$ and h_k, $k = 1, \ldots, 2n-4$ and $B(s)e = 2e$, $B(s)f(s) = \frac{1}{2} f(s)$. In order that $B(s) \in Sp(n)$ we have to check that all basis vectors are either ω-orthogonal or the product of their eigenvalues is 1, since then ω is preserved on pairs of basis vectors and hence on all of $\mathbb{R}^{2n}$. By construction we have $\omega(h_k, f(s)) = \omega(h_k, e) = 0$, so $\omega(B(s)h_k, B(s)\cdot) = \omega(h_k, \cdot)$. We also have $\omega(B(s)g_f, B(s)\cdot) = \omega(g_f, \cdot)$ since $\omega(g_f, e) = \omega(g_f, f(s)) = 0$. Furthermore, $\omega(g(s), e) = \omega(g(s), f(s)) = 0$, so $\omega(B(s)g(s), B(s)\cdot) = \omega(g(s), \cdot)$. Finally we have $\omega(B(s)e, B(s)f(s)) = 2 \cdot \frac{1}{2}\, \omega(e, f(s))$, hence alltogether $B(s)$ is symplectic. It is also nondegenerate since its spectrum is $\{-1, 2, \frac{1}{2}\}$. So $B(0) = \tilde{A}$ is changed within $Sp^*(n)$ along the path B such that $E_{\frac{1}{2}}$ becomes JE_2, indeed,

$$E_{\frac{1}{2}} = \mathbb{R}f(1) = \mathbb{R}(f - g_f) = \mathbb{R}Je = JE_2.$$

Moreover, by lemma 2.8 we have $G \perp JE_2 = E_{\frac{1}{2}}$ and $G \perp JE_{\frac{1}{2}} = -E_2$, hence

$$\mathbb{R}^{2n} = G \oplus^{\perp} E_2 \oplus^{\perp} E_{\frac{1}{2}}$$

as in the case when $g_e = g_f = 0$.

In both cases we also have $JG \subset G$ since $JG \perp E_2 \oplus E_{\frac{1}{2}}$, so we can find an orthonormal basis $\{g_1, \dots, g_{n-1}, Jg_1, \dots, Jg_{n-1}\}$ of G. Finally, $\{e, g_1, \dots, g_{n-1}, Je, Jg_1, \dots, Jg_{n-1}\}$ is an orthonormal basis of $\mathbb{R}^{2n}$ with respect to which $\tilde{A} = \mathrm{diag}(2, -1, \dots, -1, \frac{1}{2}, -1, \dots, -1)$, hence with respect to the standard basis of $\mathbb{R}^{2n}$ we have

$$\tilde{A} = SDS^T$$

with

$$\begin{aligned} D &= \mathrm{diag}(2, -1, \dots, -1, \tfrac{1}{2}, -1, \dots, -1), \\ S &= (e, g_1, \dots, g_{n-1}, Je, Jg_1, \dots, Jg_{n-1}) \in O(2n). \end{aligned}$$

Moreover, one can check that $S^T JS = J$:

$$\omega(e, g_k) = \omega(e, Jg_k) = \omega(Je, g_k) = \omega(Je, Jg_k) = 0$$

holds by lemma 2.8 and because of the orthonormality of the basis we have

$$\omega(g_k, g_j) = (g_k, Jg_j) = 0, \quad \omega(Jg_k, Jg_j) = (Jg_k, -g_j) = 0$$

and

$$\omega(g_k, Jg_k) = (g_k, -g_k) = -\delta_{jk}, \quad \omega(e, Je) = (e, -e) = -1.$$

Finally, since ω is skewsymmetric, we have $\omega(e, e) = \omega(Je, Je) = 0$ and all other matrix elements are calculated by exchanging arguments in the above.

So we have $S \in Sp(n) \cap O(2n) \cong U(n)$ with the homeomorphism (2.3). Since $U(n)$ is connected, we can now find a path $T : [0,1] \to Sp(n) \cap O(2n)$ from $T(0) = S$ to $T(1) = \mathbb{1}$. This only means a change of the symplectic basis, so TDT^T is a path in $Sp^*(n)$ connecting $\tilde{A}$ to W^-. Indeed, we have $1 \notin \sigma(D) = \sigma(TDT^{-1}) = \sigma(TDT^T)$ since $T \in O(2n)$, and $TDT^T \in Sp(n)$ since

$$(TDT^T)^T J(TDT^T) = TDT^T JTDT^T = TDJDT^T = TDT^T$$

where we used that $T \in Sp(n)$ and $DJD = J$ which is easily verified. □

2.1.2 The axiomatic definition

Following [11] we define the Conley-Zehnder index.

Definition 2.9
Let $\Sigma(n) := \{\Phi \in \mathcal{C}([0,1], Sp(n)) \mid \Phi(0) = 0\,,\ \Phi(1) \in Sp^*(2n)\}$.
The **Conley-Zehnder index** *is defined to be the unique family of maps* $\mu^n : \Sigma(n) \to \mathbb{Z}, n \in \mathbb{N}$ *which are homotopy invariant (with respect to homotopy in* $\Sigma(n)$*) and have the following properties:*

(i) For any $\Phi \in \Sigma(n), \Psi \in \Sigma(m)$ identify $\Phi \oplus \Psi$ with an arc in $\Sigma(n+m)$, then

$$\mu^{n+m}(\Phi \oplus \Psi) = \mu^n(\Phi) + \mu^m(\Psi).$$

(ii) For any continuous loop g in $Sp(n)$ with $g(0) = g(1) = \mathbb{1}$ and $\Phi \in \Sigma(n)$

$$\mu^n(g\Phi) = 2\,m(g) + \mu^n(\Phi).$$

(iii) For all $\Phi \in \Sigma(n)$

$$\mu^n(\Phi^{-1}) = -\mu^n(\Phi).$$

(iv) Define $\Phi_0 \in \Sigma(1)$ by $\Phi_0(t) = \mathrm{e}^{\pi t J}$ then

$$\mu^1(\Phi_0) = 1.$$

We will also write μ_{CZ} or μ instead of μ^n when it is clear which space we refer to.

Of course one has to show that this is well-defined. The existence follows from the explicit construction in 2.2, so here we will only prove the uniqueness.

By proposition 2.7 any arc $\Phi \in \Sigma(n)$ can be homotoped in $\Sigma(n)$ to end at W^+ or W^-. So since μ_{CZ} is to be homotopy invariant, we only have to show that $\mu_{CZ}(\Phi)$ is uniquely determined for any Φ which satisfies $\Phi(1) = W^{\pm}$. In the following we will denote by $\Phi \circ \Psi$ the path that first goes along Φ to $\Phi(1) = \Psi(0)$ and then proceeds along Ψ to $\Psi(1)$. As usual, for a loop Φ and $w \in \mathbb{Z}$ we will denote by $w\Phi$ the path that runs w times through Φ (in reversed direction if $w < 0$). If nothing else is said, any arc will be parametrized by $t \in [0,1]$. Furthermore, we will denote homotopy by '$\sim$' for loops in $Sp(n)$ and by '$\overset{*}{\sim}$' for $\Sigma(n)$. Note that for homotopy with respect to $\Sigma(n)$ it suffices to have homotopy in $Sp(n)$ with fixed endpoints.

First consider the case $\Phi(1) = W^+ = -\mathbb{1}$, then $\Phi \circ \Psi$ with $\Psi(t) = \mathrm{e}^{\pi(1-t)J}$ is well-defined and a loop in $Sp(n)$. Indeed, Φ was assumed to be symplectic and by (2.4) we have $\Psi(0) = -\mathbb{1} = \Phi(1)$, $\Psi(1) = \mathbb{1} = \Phi(0)$ and also $\Psi(t) \in Sp(n)$. So because of $\pi_1(Sp(n))$ being generated by Γ as defined in remark 2.4, the above loop has to be homotopic to $w\Gamma = \mathrm{e}^{2\pi w t J_1} \oplus \mathbb{1}$ for some $w \in \mathbb{Z}$. Thus

$$\begin{aligned}
\Phi &\overset{*}{\sim} \Phi \circ \Psi \circ \mathrm{e}^{\pi t J} \\
&\overset{*}{\sim} w\Gamma \circ \mathrm{e}^{\pi t J} \\
&= \left(\mathrm{e}^{2\pi w t J_1} \circ \mathrm{e}^{\pi t J_1}\right) \oplus \left(\mathbb{1} \circ \bigoplus_{k=2}^{n} \mathrm{e}^{\pi t J_{(k)}}\right) \\
&\overset{*}{\sim} \mathrm{e}^{2\pi w t J_1}\, \mathrm{e}^{\pi t J_1} \oplus \bigoplus_{k=2}^{n} \mathrm{e}^{\pi t J_{(k)}} \qquad (2.7)
\end{aligned}$$

where we used that $J = \bigoplus_{k=1}^{n} J_{(k)}$ with $J_{(k)} := J|_{\mathrm{span}(e_k, e_{n+k})}$, hence, taking (2.4) into account, $\mathrm{e}^{\pi t J} = \bigoplus_{k=1}^{n} \mathrm{e}^{\pi t J_{(k)}}$. Of course, $J_{(k)}$ is simply J_1 for a different identification $\Sigma(1) \subset \Sigma(n)$. An explicit homotopy showing the last step for the first term is e.g.

$$\Psi(t,s) = \begin{cases} \mathrm{e}^{2\pi w(2-s)tJ_1}\,\mathrm{e}^{\pi s t J_1} & ; \quad t \in [0,(2-s)^{-1}] \\ \mathrm{e}^{\pi(2t-1)J_1} & ; \quad t \in [(2-s)^{-1},1] \end{cases}$$

for $s \in [0,1]$. Using the homotopy invariance of μ_{CZ} and (i) to (iv) of definition 2.9 we obtain

$$\begin{aligned}
\mu^n(\Phi) &= \mu^n\Big(\mathrm{e}^{2\pi w t J_1}\,\mathrm{e}^{\pi t J_1} \oplus \bigoplus_{k=2}^{n} \mathrm{e}^{\pi t J_{(k)}}\Big) \\
&\overset{(i)}{=} \mu^1\Big(\mathrm{e}^{2\pi w t J_1}\,\mathrm{e}^{\pi t J_1}\Big) + \sum_{k=2}^{n} \mu^1\Big(\mathrm{e}^{\pi t J_{(k)}}\Big) \\
&\overset{(ii)}{=} 2m\Big(\mathrm{e}^{2\pi w t J_1}\Big) + \mu^1\Big(\mathrm{e}^{\pi t J_1}\Big) + \sum_{k=2}^{n} \mu^1\Big(\mathrm{e}^{\pi t J_1}\Big) \\
&\overset{(iv)}{=} 2m\Big(\mathrm{e}^{2\pi w t J_1}\Big) + n \;=\; 2w + n,
\end{aligned}$$

where the last equality follows from remark 2.6. So $\mu_{CZ}(\Phi)$ is determined uniquely.

Similarily, this can be shown in the case $\Phi(1) = W^-$. Define the following loop in $Sp(n)$:

$$\Phi \circ \mathrm{diag}(2-t,(2-t)^{-1}) \oplus \bigoplus_{k=2}^{n} \mathrm{e}^{\pi(1-t)J_{(k)}}$$

One easily checks that $\mathrm{diag}(2-t,(2-t)^{-1}) \in Sp(1)$. As above we deduce that this loop is homotopic to $w\Gamma$ for some $w \in \mathbb{Z}$. Thus

$$\begin{aligned}
\Phi &\overset{*}{\sim} w\Gamma \circ \mathrm{diag}((1+t),(1+t)^{-1}) \oplus \bigoplus_{k=2}^{n} \mathrm{e}^{\pi t J_{(k)}} \\
&= \Big(\, \mathrm{e}^{2\pi w t J_1} \circ \mathrm{diag}(1+t,(1+t)^{-1}) \,\Big) \oplus \Big(\, \mathbb{1} \circ \bigoplus_{k=2}^{n} \mathrm{e}^{\pi t J_{(k)}} \,\Big) \\
&\overset{*}{\sim} \mathrm{e}^{2\pi w t J_1}\,\mathrm{diag}(1+t,(1+t)^{-1}) \oplus \bigoplus_{k=2}^{n} \mathrm{e}^{\pi t J_{(k)}}
\end{aligned} \tag{2.8}$$

Here we use for the last step

$$\Psi(t,s) = \begin{cases} \mathrm{e}^{2\pi(2-s)tJ_1}\,\mathrm{diag}(1+st,(1+st)^{-1}) & ; \quad t \in [0,(2-s)^{-1}] \\ \mathrm{diag}(2t,(2t)^{-1}) & ; \quad t \in [(2-s)^{-1},1] \end{cases} .$$

Then again by the defining properties of μ_{CZ} we obtain

$$\begin{aligned}
\mu^n(\Phi) &= \mu^n\Big(\mathrm{e}^{2\pi w t J_1}\operatorname{diag}(1+t,(1+t)^{-1}) \oplus \bigoplus_{k=2}^{n} \mathrm{e}^{\pi t J_{(k)}}\Big) \\
&\overset{(i)}{=} \mu^1\Big(\mathrm{e}^{2\pi w t J_1}\operatorname{diag}(1+t,(1+t)^{-1})\Big) + \sum_{k=2}^{n} \mu^1\Big(\mathrm{e}^{\pi t J_{(k)}}\Big) \\
&\overset{(ii)}{=} 2m\Big(\mathrm{e}^{2\pi w t J_1}\Big) + \mu^1\Big(\operatorname{diag}(1+t,(1+t)^{-1})\Big) + \sum_{k=2}^{n} \mu^1\Big(\mathrm{e}^{\pi t J_1}\Big) \\
&\overset{(iv)}{=} 2w + \mu^1\Big(\operatorname{diag}(1+t,(1+t)^{-1})\Big) + n - 1 \\
&= 2w + n - 1.
\end{aligned}$$

The last equality can be deduced from the yet not used property (iii). Indeed, we claim that $\mu^1\Big(\operatorname{diag}(1+t,(1+t)^{-1})\Big) = 0$ since

$$\Psi := \operatorname{diag}(1+t,(1+t)^{-1}) \overset{*}{\sim} \Psi^{-1}$$

and hence $\mu_{CZ}(\Psi^{-1}) = \mu_{CZ}(\Psi) = -\mu_{CZ}(\Psi^{-1}) = 0$. The homotopy is explicitly given by

$$\Omega(t,s) = \mathrm{e}^{-\frac{\pi}{2}sJ}\operatorname{diag}(1+t,(1+t)^{-1})\mathrm{e}^{\frac{\pi}{2}sJ}.$$

Obviously, Ω is continuous, symplectic everywhere and meets $\Omega(\cdot,0) = \Psi$ and $\Omega(0,\cdot) \equiv \mathbb{1}$. Therefore it remains to check that $\Omega(\cdot,1) = \Psi^{-1}$ and $\Omega(1,s) \in Sp^*(1)$ for all $s \in [0,1]$. Using (2.4) we calculate

$$\begin{aligned}
&\Omega(t,s) \\
&= \begin{pmatrix} \cos(-\frac{\pi}{2}s) & -\sin(-\frac{\pi}{2}s) \\ \sin(-\frac{\pi}{2}s) & \cos(-\frac{\pi}{2}s) \end{pmatrix} \begin{pmatrix} 1+t & 0 \\ 0 & (1+t)^{-1} \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi}{2}s) & -\sin(\frac{\pi}{2}s) \\ \sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix} \\
&= \begin{pmatrix} (1+t)\cos^2(\frac{\pi}{2}s) + (1+t)^{-1}\sin^2(\frac{\pi}{2}s) & (-1-t+(1+t)^{-1})\cos(\frac{\pi}{2}s)\sin(\frac{\pi}{2}s) \\ (-1-t+(1+t)^{-1})\cos(\frac{\pi}{2}s)\sin(\frac{\pi}{2}s) & (1+t)\sin^2(\frac{\pi}{2}s) + (1+t)^{-1}\cos^2(\frac{\pi}{2}s) \end{pmatrix}.
\end{aligned}$$

Putting in $s = 1$ we see that Ω actually ends at $\Psi(t)^{-1}$. Finally, from the first equality we see that $\Omega(1,s)$ is an orthonormal transformation of $\operatorname{diag}(2,\frac{1}{2})$, so it has the eigenvalues $2, \frac{1}{2}$ and thus is nondegenerate.

This finally proves that if μ_{CZ} exists, then it is uniquely determined by the properties in definition 2.9.

2.2 The spectral flow description

The Conley-Zehnder index can be constructed explicitly from the spectral flow of some asymptotic operator attached to the symplectic arc that will be introduced in this section. This operator arises from a singularity of the pseudoholomorphic curve that also gave rise to an asymptotic orbit and the symplectic arc considered (see e.g. [9] or chapter 3). However, it can also be defined for a general symplectic arc not respresented that way. For this construction some more regularity of the arc is required, so we restrict the discussion to the following set of symplectic arcs.

Definition 2.10 *Let* $\|\cdot\|$ *be the operator norm on* $\mathbb{R}^{2n\times 2n}$ *and*

$$\begin{array}{lcl} L^\infty([0,1],\mathbb{R}^{2n\times 2n}) & := & \{\Phi:[0,1]\to\mathbb{R}^{2n\times 2n} \mid \|\Phi\|_\infty<\infty\}, \\ W^{1,\infty}([0,1],\mathbb{R}^{2n\times 2n}) & := & \{\Phi:[0,1]\to\mathbb{R}^{2n\times 2n} \mid \|\Phi\|_{1,\infty}<\infty\}, \end{array}$$

with

$$\|\Phi\|_\infty := \inf\{C\in\mathbb{R} \mid \|\Phi(t)\|<C \text{ for almost all } t\in[0,1]\}, \tag{2.9}$$

$$\|\Phi\|_{1,\infty} := \|\Phi\|_\infty+\|\dot\Phi\|_\infty. \tag{2.10}$$

Then define the set of regular nondegenerate symplectic arcs

$$\Sigma^{1,\infty}(n):=\Sigma(n)\cap W^{1,\infty}([0,1],\mathbb{R}^{2n\times 2n}).$$

However, any continuous symplectic arc can be homotoped to an arc of this regularity. This is because $Sp(n)\subset\mathbb{R}^{2n\times 2n}$ is a smooth submanifold, hence $\mathcal{C}^1([0,1],Sp(n))$ is dense in $\mathcal{C}^0([0,1],Sp(n))$ with respect to the $\mathcal{C}^0$-topology (see [7] Thm.3.3). Moreover, the open condition $\Phi(1)\in Sp^*(n)$ admits $\mathcal{C}^0$-small changes of the arc.

In order to define the asymptotic operator we need the subsequent correspondence of symplectic arcs with arcs of symmetric matrices. For this purpose let $Sym(2n)$ be the set of symmetric $\mathbb{R}^{2n\times 2n}$-matrices and define

$$\mathcal{S}(n):=\mathrm{L}^\infty([0,1],Sym(2n)).$$

If not otherwise stated we will equip $\mathcal{S}(n)$ with the L^∞-metric (2.9) and $\Sigma^{1,\infty}(n)$ with the $W^{1,\infty}$-metric (2.10).

Lemma 2.11 *We have a continuous map*

$$\begin{array}{rcl} W^{1,\infty}([0,1],Sp(n)) & \to & (\mathcal{S}(n),\|\cdot\|_\infty) \\ \Phi & \mapsto & S_\Phi:=-J\dot\Phi\Phi^{-1}. \end{array}$$

Proof: Let $\Phi\in W^{1,\infty}([0,1],Sp(n))$, then $\Phi,\dot\Phi\in L^\infty([0,1],\mathbb{R}^{2n\times 2n})$. Since Φ is symplectic we can also deduce that $\Phi^{-1}\in L^\infty([0,1],\mathbb{R}^{2n\times 2n})$. Indeed,

$\Phi^{-1} = -J\Phi^T J$ by (2.2), so it is the product of three L^∞-arcs of matrices (J is a constant). Now we infer that also $S_\Phi \in L^\infty([0,1], \mathbb{R}^{2n\times 2n})$ since it is the product of L^∞-arcs and it is also symmetric for all $t \in [0,1]$, thus $S_\Phi \in \mathcal{S}(n)$. Indeed,

$$\begin{array}{ll} & \Phi^T J\Phi = J \\ \Rightarrow & \dot{\Phi}^T J\Phi + \Phi^T J\dot{\Phi} = 0 \\ \Rightarrow & (\Phi^T)^{-1}\dot{\Phi}^T J\Phi = -J\dot{\Phi} \\ \Rightarrow & S_\Phi = -J\dot{\Phi}\Phi^{-1} = (\Phi^T)^{-1}\dot{\Phi}^T J = (S_\Phi)^T. \end{array}$$

For the continuity, first remember that $\Phi^{-1} = -J\Phi^T J$ holds if $\Phi \in Sp(n)$. Moreover, $\|\Phi^T\| = \|\Phi\|$ since the adjoint of a bounded operator has the same norm as the operator (see e.g. [19] VII Thm.5). We also know that $\|J\| = 1$ and putting all this together we deduce that for all $\Phi, \Psi \in Sp(n)$

$$\|\Phi^{-1}\| = \|-J\Phi^T J\| \leq \|J\| \cdot \|\Phi^T\| \cdot \|J\| = \|\Phi\|$$

and

$$\|\Phi^{-1} - \Psi^{-1}\| = \|-J\Phi^T J + J\Psi^T J\| \leq \|-J\| \cdot \|(\Phi-\Psi)^T\| \cdot \|J\| = \|\Phi - \Psi\|.$$

Using the above we get — denoting by 'sup' the essential supremum, i.e. the minimal constant that is an upper bound almost everywhere —

$$\begin{array}{ll} \multicolumn{2}{l}{\|S_\Phi - S_\Psi\|_\infty} \\ = & \sup\limits_{t\in[0,1]} \|-J\dot{\Phi}(t)\Phi^{-1}(t) + J\dot{\Psi}(t)\Psi^{-1}(t)\| \\ \leq & \sup\limits_{t\in[0,1]} (\, \|-\dot{\Phi}(t) + \dot{\Psi}(t)\| \cdot \|\Phi^{-1}(t)\| + \|\dot{\Psi}(t)\| \cdot \|-\Phi^{-1}(t) + \Psi^{-1}(t)\| \,) \\ \leq & \|\dot{\Phi} - \dot{\Psi}\|_\infty \|\Phi\|_\infty + \|\dot{\Psi}\|_\infty \|\Phi - \Psi\|_\infty \\ \leq & \|\Phi - \Psi\|_{1,\infty}(\|\Phi\|_\infty + \|\dot{\Phi}\|_\infty + \|-\dot{\Phi} + \dot{\Psi}\|_\infty) \\ \leq & \|\Phi\|_{1,\infty}\|\Phi - \Psi\|_{1,\infty} + \|\Phi - \Psi\|_{1,\infty}^2. \end{array}$$

For given $\Phi \in W^{1,\infty}([0,1], Sp(n))$ and $\|\Phi - \Psi\|_{1,\infty} \to 0$, the last expression and hence also $\|S_\Phi - S_\Psi\|_\infty$ obviously tend to zero. □

Remark 2.12

(i) We even have a bijection

$$\begin{array}{rcl} \{\Phi \in \mathcal{C}^1([0,1], Sp(n)) \mid \Phi(0) = \mathbb{1}\} & \to & \mathcal{C}([0,1], Sym(2n)) \\ \Phi & \mapsto & S_\Phi. \end{array}$$

(ii) *Any* $S \in \mathcal{S}(n)$ *is a bounded map on* $L^2(S^1, \mathbb{R}^{2n})$*:*
For all $h \in L^2(S^1, \mathbb{R}^{2n})$

$$\|Sh\|_{L^2} \leq \|S\|_\infty \|h\|_{L^2}$$

with $\|S\|_\infty < \infty$.

Proof: Since $S \in L^\infty([0,1], \mathbb{R}^{2n\times 2n})$ we obviously have $\|S\|_\infty < \infty$. Using this we can deduce (ii): For any $h \in L^2(S^1, \mathbb{R}^{2n})$

$$\begin{aligned} \|Sh\|_{L^2} &= \left(\int_0^1 \|S(t)h(t)\|^2 \, \mathrm{d}t\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^1 \|S\|_\infty^2 \cdot \|h(t)\|^2 \, \mathrm{d}t\right)^{\frac{1}{2}} \\ &= \|S\|_\infty \|h\|_{L^2}. \end{aligned}$$

For (i) note that the preimage of $S \in \mathcal{S}(n)$ is the set of solutions of

$$\begin{cases} \dot{\Phi}(t) &= JS(t)\Phi(t) \\ \Phi(0) &= \mathbb{1} \end{cases} . \tag{2.11}$$

This is an ordinary first order differential equation with Lipschitz continuous right hand side, which has, by the Picard-Lindelöf theorem (see e.g. [6] 117.1), a unique solution in $\mathcal{C}^1([0,1], \mathbb{R}^{2n\times 2n}$ for the given initial value. From the differential equation we obtain

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}(\Phi(t)^T J\Phi(t)) &= (JS(t)\Phi(t))^T J\Phi(t) + \Phi(t)^T J(JS(t)\Phi(t)) \\ &= \Phi(t)^T S(t)\Phi(t) - \Phi(t)^T S(t)\Phi(t) \\ &= 0. \end{aligned}$$

Moreover, since $\Phi(0) = \mathbb{1}$ we know that $\Phi(0)^T J\Phi(0) = J$, hence $\Phi(t)$ is symplectic for all $t \in [0,1]$. Thus the solution of (2.11) defines the inverse of the considered map. □

So instead of constructing an index for symplectic arcs we can just as well consider arcs of symmetric matrices.

Definition 2.13 *For* $S \in \mathcal{S}(n)$ *or* $\Phi \in W^{1,\infty}([0,1], Sp(n))$ *with* $S := S_\Phi$ *we define the* **asymptotic operator**

$$L_S := -J\tfrac{\mathrm{d}}{\mathrm{d}t} - S : \quad W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n}). \tag{2.12}$$

This is well-defined since S is even bounded by remark 2.12(ii). Note that $L_S x$ for $x \in W^{1,2}(S^1, \mathbb{R}^{2n})$ does not need to have the same value in 0 and 1 since it is only defined in $L^2(S^1, \mathbb{R}^{2n})$. So there is no problem with S being not periodic.

Now, following [13], the Conley-Zehnder index can be constructed using the spectrum of the asymptotic operator.

Theorem 2.14 *There exists a unique bi-infinite sequence $(\lambda_k)_{k\in\mathbb{Z}}$ of continuous maps $\lambda_k : \mathcal{S}(n) \to \mathbb{R}$ characterized by the following requirements.*

(i) $\lambda_k(S) \leq \lambda_{k+1}(S)$ *for all* $k \in \mathbb{Z}$ *and any* $S \in \mathcal{S}(n)$.

(ii) For each $S \in \mathcal{S}(n)$ *the spectrum of* L_S *is represented by*

$$\sigma(L_S) = \{\lambda_k(S) \mid k \in \mathbb{Z}\}.$$

(iii) For each $S \in \mathcal{S}(n)$ *the multiplicity of any eigenvalue* $\tau \in \sigma(L_S)$ *is*

$$\sharp\{k \in \mathbb{Z} \mid \lambda_k(S) = \tau\}.$$

(iv) The sequence is normalized at $0 \in \mathcal{S}(n)$ *by*

$$\lambda_{-n+1}(0) = \ldots = \lambda_n(0) = 0.$$

Furthermore, the maps λ_k *have the following properties.*

a) Let $\Delta \in \mathcal{S}(n)$ *and let it satisfy* $\Delta(t) \geq \varepsilon > 0$ *for all* $t \in [0,1]$. *Then for all* $S \in \mathcal{S}(n)$ *and* $k \in \mathbb{Z}$ *we have*

$$\lambda_k(S + \Delta) < \lambda_k(S).$$

b) The maps

$$\begin{array}{rcl} W^{1,\infty}([0,1], Sp(n)) & \to & \mathbb{R} \\ \Phi & \mapsto & \lambda_k(S_\Phi), \end{array}$$

also denoted by λ_k, *are continuous for all* $k \in \mathbb{Z}$.

From the maps in b) we can read off the Conley-Zehnder index: For any $\Phi \in W^{1,\infty}([0,1], Sp(n))$ *define*

$$\mu_{spec}(\Phi) := \max\{k \in \mathbb{Z} \mid \lambda_k(\Phi) < 0\},$$

then for all $\Phi \in \Sigma^{1,\infty}(n)$ *we have*

$$\mu_{CZ}(\Phi) = \mu_{spec}(\Phi).$$

At the moment we can only remark that b) follows easily when assuming the existence of the continuous maps $\lambda_k : (\mathcal{S}(n), \|\cdot\|_\infty) \to \mathbb{R}$. Simply note that $\Phi \mapsto S_\Phi$ is continuous by lemma 2.11.

The maps λ_k can also be seen as continuous sections in the spectrum bundle

$$\mathcal{B} := \bigcup_{S\in\mathcal{S}(n)} \sigma(L_S) \;\to\; \mathcal{S}(n).$$

The theorem then states that the spectrum bundle can be represented by a sequence of continuous real functions on $\mathcal{S}(n)$ and that the sheets of the bundle can be globally numbered in ascending order and with some normalization. Only to the branching points of this discrete bundle are several numbers assigned according to the multiplicity of this eigenvalue.

The proof of theorem 2.14 requires detailed investigation of the spectrum of the asymptotic operator. This will be performed in 2.2.2 using parts of Kato's perturbation theory which will be presented in the next section. We finally prove the spectrum bundle part of theorem 2.14 in section 2.2.3 and show in 2.2.4 that the Conley-Zehnder index can indeed be read off from the sections λ_k of the spectral bundle.

2.2.1 Kato's perturbation theory

In this section we present the part of Kato's perturbation theory [14] that will later be applied to the asymptotic operator.

We are only interested in operators with compact resolvent and these are characterized as follows.

Theorem 2.15 ([14] III Thm. 6.29.)
Let T be a closed operator on a Banach space X such that the resolvent $R(\zeta)$ exists and is compact for some $\zeta \in \mathbb{C}$. Then the spectrum of T consists entirely of isolated eigenvalues with finite multiplicities, and $R(\zeta)$ is compact for every ζ in the resolvent set.

Such an operator T is called an *operator with compact resolvent* and a spectrum as the above is called a *discrete spectrum.*

We also need some concepts of holomorphic families of operators. If an object is defined on a real interval only, we will from now on also call it holomorphic if it is analytic, i.e. it can be locally represented by its Taylor expansion. In this case the object can be extended to a complex neighbourhood of the real interval on which it is holomorphic.

Definition 2.16

(i) Let $T(x)$ be a family of bounded operators on a Hilbert space H defined on some domain $U \in \mathbb{C}$. It is called **holomorphic** *if it is complex differentiable in the norm on its domain, that is for any $z_0 \in U$ there is a $T_1 \in \mathcal{L}(H)$ such that for all sufficiently small $z \in \mathbb{C}$*

$$T(z_0 + z) = T(z_0) + zT_1 + \mathcal{O}(z^2).$$

(ii) A **holomorphic family of type (A)** *on a Hilbert space H is a family $T(x)$ of closed operators on H defined for x in a domain $U \subset \mathbb{C}$ such that*

a) The domain of $T(x)$ is independent of x, that is $D(T(x)) = D$,

b) $T(x)\phi$ is holomorphic on U for every $\phi \in D$.

(iii) A **selfadjoint holomorphic family of type (A)** *is a holomorphic family of type (A) such that $T(x)$ is selfadjoint for all real $x \in U$.*

(iv) A family $T(x)$ of operators or vectors in a Hilbert space that is defined on some real interval is called **piecewise holomorphic** *if there is only a finite number of points in the interval at which T is not holomorphic. Furthermore, we require that at these points T is still continuous and that its derivatives from the left and right hand side exist though they need not be the same.*

In terms of the above we can formulate the theorem that will constitute the central part in the proof of theorem 2.14.

Theorem 2.17 ([14] VII Thm. 3.9.)
Let $T(x)$ be a selfadjoint holomorphic family of type (A) on a Hilbert space H defined for x in a neighbourhood of an interval $I_0 \subset \mathbb{R}$. Furthermore, let $T(x)$ have compact resolvent for $x \in I_0$. There are then two sequences of holomorphic maps, $\nu_n : I_0 \to \mathbb{R}$ and $\psi_n : I_0 \to H$ such that for any $x \in I_0$ the sequence $(\nu_n(x))_{n\in\mathbb{N}}$ represents all the repeated eigenvalues of $T(x)$ and $(\psi_n(x))_{n\in\mathbb{N}}$ forms a complete orthonormal family of the corresponding eigenvectors.

Now consider a family $T(x) = T_0 + P(x)$ of operators defined in a neighbourhood of 0, that meets the assumptions of the previous theorem and that has a bounded holomorphic perturbation $P(x)$ with $P(0) = 0$. So we have $P(x) = xT_1 + \mathcal{O}(x^2)$ for sufficiently small x with T_1 a bounded operator. We then obtain by the above theorem holomorphic functions of eigenvalues and eigenvectors. This justifies the following ansatz for the investigation of the quantitative behaviour of the spectrum under perturbation:

$$\begin{aligned} \eta(x) &= \eta_0 + x\eta_1 + \mathcal{O}(x^2) \qquad \text{for the eigenvalue and} \\ \phi(x) &= \phi_0 + x\phi_1 + \mathcal{O}(x^2) \qquad \text{for the eigenvector.} \end{aligned}$$

Inserting this into the eigenvalue equation

$$(T_0 + xT_1 + \mathcal{O}(x^2))(\phi_0 + x\phi_1 + \mathcal{O}(x^2)) = (\eta_0 + x\eta_1 + \mathcal{O}(x^2))(\phi_0 + x\phi_1 + \mathcal{O}(x^2))$$

yields up to first order in x

$$T_0\phi_0 = \eta_0\phi_0, \tag{2.13}$$

$$T_0\phi_1 + T_1\phi_0 = \eta_0\phi_1 + \eta_1\phi_0. \tag{2.14}$$

From (2.13) we infer that η_0 is an eigenvalue of T_0 — which is obvious anyway. Now let Π_0 be the projection onto the eigenspace for the eigenvalue η_0 of T_0, then

$$\Pi_0 T_0 = \eta_0 \Pi_0. \tag{2.15}$$

Indeed, we know from theorem 2.17 that there exists a complete family of eigenvectors of T_0 and (2.15) obviously holds when applied to any of these eigenvectors. So applying Π_0 to (2.14) we get

$$\Pi_0 T_1 \phi_0 = \eta_1 \phi_0 \tag{2.16}$$

and hence, due to ϕ_0 being normalized and the fact that projections are selfadjoint,

$$\eta_1 = (\phi_0\,,\, \Pi_0 T_1 \phi_0)_H = (\phi_0\,,\, T_1 \phi_0)_H, \tag{2.17}$$

where $(\cdot,\cdot)_H$ is the scalar product on the Hilbert space H. Note that $\eta_1 = \frac{\mathrm{d}\eta}{\mathrm{d}x}(0)$ and $\phi_0 = \phi(0)$. We also write $T_1 = \frac{\mathrm{d}P}{\mathrm{d}x}(0)$.

From this expression for the first order perturbation of the eigenvalues we get the following theorem.

Theorem 2.18 *Let $L(x) = T + A(x)$ be a family of operators as in theorem 2.17 with A bounded holomorphic on $D = D(L(x))$. Then with the notation of that theorem the following holds for all $n \in \mathbb{N}$ and $x_0 \le x_1 \in I_0$:*

$$\frac{\mathrm{d}\nu_n}{\mathrm{d}x}(x_0) = \left(\psi_n(x_0)\,,\, \frac{\mathrm{d}A}{\mathrm{d}x}(x_0)\,\psi_n(x_0)\right)_H \tag{2.18}$$

and the growth rate of the eigenvalues is uniformly bounded,

$$\begin{aligned} |\nu_n(x_1) - \nu_n(x_0)| &\le \sup_{x\in[x_0,x_1]} \left\| \frac{\mathrm{d}A}{\mathrm{d}x}(x) \right\| \cdot |x_1 - x_0| \\ &\le \left\| \frac{\mathrm{d}A}{\mathrm{d}x} \right\|_{sup} |x_1 - x_0| \end{aligned} \tag{2.19}$$

where $\|\cdot\|$ denotes the operator norm on H and

$$\left\| \frac{\mathrm{d}A}{\mathrm{d}x} \right\|_{sup} := \sup_{x\in I_0} \left\| \frac{\mathrm{d}A}{\mathrm{d}x}(x) \right\| < \infty.$$

Proof: We will consider $L(x)$ to be a perturbation of $L(x_0)$, that is we write $L(x_0 + x) = T_0 + P(x)$ with $T_0 = T + A(x_0)$ and the perturbation $P(x) = A(x_0 + x) - A(x_0)$. Then (2.18) is simply the result (2.17) of the calculation above for the eigenvalue $\eta = \nu_n$ and integration of (2.18) gives (2.19). Indeed, keeping in mind that $\psi_n(x)$ is normalized for all $x \in I_0$,

$$|\nu_n(x_1) - \nu_n(x_0)| = \left| \int_{x_0}^{x_1} \left(\psi_n(x)\,,\, \frac{\mathrm{d}A}{\mathrm{d}x}(x)\,\psi_n(x)\right)_H \mathrm{d}x \right|$$

$$
\begin{aligned}
&\leq \int_{x_0}^{x_1} \|\psi_n(x)\| \cdot \left\| \frac{\mathrm{d}A}{\mathrm{d}x}(x)\, \psi_n(x) \right\| \, \mathrm{d}x \\
&\leq \int_{x_0}^{x_1} \left\| \frac{\mathrm{d}A}{\mathrm{d}x}(x) \right\| \, \mathrm{d}x \\
&\leq \sup_{x \in [x_0, x_1]} \left\| \frac{\mathrm{d}A}{\mathrm{d}x}(x) \right\| \cdot |x_1 - x_0| \\
&\leq \left\| \frac{\mathrm{d}A}{\mathrm{d}x} \right\|_{sup} |x_1 - x_0|
\end{aligned}
$$

where we wrote $\|\cdot\|$ for the norm as well as the operator norm on H. Finally, $\|\frac{\mathrm{d}A}{\mathrm{d}x}\|_{sup} < \infty$ holds since A is bounded holomorphic and thus has bounded and continuous derivatives. $\square$

Of course, the holomorphic functions ν_n from theorem 2.17 representing the eigenvalues are not necessarily in any order as demanded in theorem 2.14(i). Their graphs may even cross at so-called crossing points. So we introduce functions κ_n that result from the ν_n by putting them in ascending order at each point. The graphs of the κ_n then follow branches of the ν_n and at crossing points may jump over to another branch of such a holomorphic function. We will sharpen this concept by a subsequent definition. But first we have to think about the numbering of (κ_n). If, for example, the spectrum has an upper bound but is unbounded below, then it is impossible to number the ascending (κ_n) by $\mathbb{N}$, but we can still number it by $-\mathbb{N}$. Assume — as it is the case for the asymptotic operator — that the Hilbert space is of infinite dimension. Then because of its discreteness (by theorem 2.15) the spectrum of each $L(x)$ consists of a countable infinite number of isolated eigenvalues that are unbounded at at least one side. Furthermore, the type of unboundedness of $\sigma(L(x))$ is the same for all x in a finite interval if — as for the asymptotic operator — the perturbation is bounded. This follows from the uniform bound (2.19) on the growth rate of the eigenvalues. So assuming the Hilbert space to be infinite dimensional and the family of operators to have bounded perturbation as in theorem 2.18 we have the following cases:

- If the spectrum is unbounded from above and below we can number (κ_n) by $I := \mathbb{Z}$.
- If the spectrum is bounded from above but without lower bound we can number (κ_n) by $I := -\mathbb{N}$.
- If the spectrum is bounded from below but without upper bound we can number (κ_n) by $I := \mathbb{N}$.

Definition 2.19 *In the situation of theorem 2.18 define $(\kappa_n)_{n\in I}$, I as indicated above, to be the family of piecewise holomorphic functions $\kappa_n : I_0 \to \mathbb{R}$ that satisfies the following conditions:*

(i) $\kappa_n(x) \leq \kappa_{n+1}(x)$ *for all* $n \in I$ *and* $x \in I_0$*,*

(ii) for any $x \in I_0$ *there is a bijection* $\pi : I \to \mathbb{N}$ *such that* $\kappa_n(x) = \nu_{\pi(n)}(x)$ *for all* $n \in I$*.*

In the case of $I = \mathbb{Z}$, of course, $(\kappa_k)_{k\in\mathbb{Z}}$ is only defined up to a shift in the numbering. In applications one can fix this shift by normalization at some $x_0 \in I_0$.

To make sure that these functions are indeed piecewise holomorphic, we pick any $\kappa := \kappa_n, n \in \mathbb{N}$ and an $x_0 \in I_0$ where κ jumps to another holomorphic branch. We will show that there is no other crossing point along κ in some neighbourhood of x_0. Then it follows that in any compact interval of x there are only finitely many crossing points along κ in which it may cease to be holomorphic.

Note that by theorem 2.15 all eigenvalues of $T(x_0)$ are isolated and of finite multiplicity. Hence we know that there is a neighbourhood of $\kappa(x_0)$ in which there is no other eigenvalue of $T(x_0)$. Consequently there is a finite set $X \subset \mathbb{N}$ with $\nu_n(x_0) = \kappa(x_0)$ for $n \in X$ and $|\nu_n(x_0) - \kappa(x_0)| \geq \varepsilon > 0$ for all $n \in \mathbb{N} \setminus X$. First consider the holomorphic functions that do not cross κ in x_0. These stay away from κ in some neighbourhood of x_0: They are separated from κ by at least ε in x_0 and their growth rate is uniformly bounded by (2.19). Hence there is an open interval around x_0 on which $|\nu_n - \kappa(x_0)| > \frac{1}{2}\varepsilon$ for all $n \in \mathbb{N} \setminus X$. But κ is obviously continuous, so on some other open interval around x_0 we have $|\kappa(x) - \kappa(x_0)| < \frac{1}{2}\varepsilon$. Hence on the intersection of both intervals there is no crossing between κ and any ν_n, $n \in \mathbb{N} \setminus X$. So in this neighbourhood κ can only jump to a ν_n with $n \in X$. But any two of these holomorphic functions are either identical or do not cross again in some neighbourhood of x_0. As X is finite we infer that there is a neighbourhood of x_0 in which none of the functions going through $\kappa(x_0)$ at $x = x_0$ crosses any other of them again. Thus κ cannot jump a second time in some neighbourhood of x_0.

Now because of the κ_n being piecewise holomorphic, we can transfer theorem 2.18 to this other representation of the spectrum of $L(x)$.

Remark 2.20 *In the situation of theorem 2.18 the bound for the growth rate is also valid for the* κ_n*: For all* $n \in \mathbb{N}$ *and* $x_0 \leq x_1 \in I_0$

$$|\kappa_n(x_1) - \kappa_n(x_0)| \leq \sup_{x\in[x_0,x_1]} \left\| \frac{\mathrm{d}A}{\mathrm{d}x}(x) \right\| \cdot |x_1 - x_0| \leq \left\| \frac{\mathrm{d}A}{\mathrm{d}x} \right\|_{sup} |x_1 - x_0| \quad (2.20)$$

and we also have

$$\frac{\mathrm{d}\kappa_n}{\mathrm{d}x}(x_0) = \left(\phi\,,\, \frac{\mathrm{d}A}{\mathrm{d}x}(x_0)\,\phi \right)_H \quad (2.21)$$

where ϕ *is the eigenvector* $\psi_{\pi(n)}(x_0)$ *from theorem 2.17 corresponding to* $\nu_{\pi(n)}(x_0) = \kappa_n(x_0)$*. At crossing points* $x_0 \in I_0$ *where* κ_n *jumps from one* *holomorphic branch to another, (2.21) has to be interpreted as the left or right hand side derivative. In that case for ϕ we have to put in $\psi_{\pi(n)}(x_0)$ corresponding to $\nu_{\pi(n)}(x) = \kappa_n(x)$ for $x \leq x_0$ or $x \geq x_0$, respectively.*

Remark 2.21 *Consider a family of operators that meets all assumptions of theorem 2.17 except for that it is only piecewise holomorphic of type (A). Then we still get continuous but only piecewise holomorphic functions $\nu_n : I_0 \to \mathbb{R}$ that represent the spectrum of the family of operators and we can define $(\kappa_n)_{n\in I}$ exactly as in definition 2.19. Remark 2.20 also can be generalized to this case: The bound for the growth rate still holds, one only needs to define $\|\cdot\|_{sup}$ to be the essential supremum — neglecting undefined values at a finite number of points. For the first derivative of the eigenvalue, one has to restrict the formula to derivatives from one side at the nonholomorphic points.*

Proof: Theorem 2.17 can be applied to each of the intervals where the family is holomorphic, giving holomorphic functions for the eigenvalues that can be glued together continuously. For the definition of $(\kappa_n)_{n\in I}$ one only has to repeat the argument after definition 2.19 for each of these intervals. □

2.2.2 Perturbation theory for the asymptotic operator

Let $S \in \mathcal{S}(n)$ be given and consider the asymptotic operator

$$L_S := -J\tfrac{\mathrm{d}}{\mathrm{d}t} - S : \quad W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n}).$$

We will treat L_S as perturbation of the following selfadjoint operator.

Proposition 2.22 $T := -J\frac{\mathrm{d}}{\mathrm{d}t}$ *with domain* $D(T) = W^{1,2}(S^1, \mathbb{R}^{2n})$ *is a selfadjoint operator on the Hilbert space* $L^2(S^1, \mathbb{R}^{2n})$.

Proof: Following [2] we first remark that T is closed. To see this let $(x_n)_{n\in\mathbb{N}} \subset D(T)$ with $x_n \to x$ and $Tx_n \to y$ converging in $L^2(S^1, \mathbb{R}^{2n})$. This obviously implies $\dot{x}_n \to Jy$. Now calculate $\dot{x}$ as a distribution: For all test functions ϕ in the Schwarz space we get with $(\cdot,\cdot)$ the usual pairing

$$(\dot{x}, \phi) = -(x, \dot{\phi}) = \lim_{n\to\infty} -(x_n, \dot{\phi}) = \lim_{n\to\infty} (\dot{x}_n, \phi) = (Jy, \phi).$$

Thus $\dot{x} = Jy \in L^2(S^1, \mathbb{R}^{2n})$ and it follows that $x \in W^{1,2}(S^1, \mathbb{R}^{2n}) = D(T)$ and $Tx = -J\dot{x} = y$ which was to be shown. Furthermore, T is densely defined since $W^{1,2}(S^1, \mathbb{R}^{2n}) \subset L^2(S^1, \mathbb{R}^{2n})$ is dense. And it is also symmetric. Indeed, for $x, y \in D(T)$ we get by partial integration (note $\partial S^1 = \emptyset$)

$$(x, Ty)_{L^2} = \int_{S^1} \langle\, x(t)\,,\, -J\dot{y}(t)\,\rangle\ \mathrm{d}t$$

$$
\begin{aligned}
&= \int_{S^1} Jx(t) \cdot \dot{y}(t)\,\mathrm{d}t \\
&= \int_{S^1} -J\dot{x}(t) \cdot y(t)\,\mathrm{d}t \quad = \ (Tx, y)_{L^2}.
\end{aligned}
$$

Now a densely defined, symmetric operator T is extended by its adjoint (see e.g. [4] XII 1.), $T^* \supset T$, that is $D(T) \subset D(T^*)$ and $T^*|_{D(T)} = T$. So it remains to show that $D(T^*) \subset D(T)$. In order to see this we will first establish

$$
\mathrm{im}(T) = \ker(T)^{\perp}. \tag{2.22}
$$

As J is nondegenerate, the kernel of $-J\frac{\mathrm{d}}{\mathrm{d}t}$ is the set of constant functions on S^1. So for $\zeta \in \ker(T)^{\perp}$ we necessarlily have $(\zeta, 1)_{L^2} = \int_{S^1} \zeta = 0$ and thus we can define

$$
z(t) := J \int_0^t \zeta(\tau) d\tau \quad \in \mathcal{C}(S^1, \mathbb{R}^{2n}) \subset L^2(S^1, \mathbb{R}^{2n}).
$$

Obviously $\dot{z} = J\zeta \in L^2(S^1, \mathbb{R}^{2n})$, thus $z \in W^{1,2}(S^1, \mathbb{R}^{2n}) = D(T)$ and $Tz = -J\dot{z} = \zeta$, hence $\zeta \in \mathrm{im}(T)$. So we have shown $\ker(T)^{\perp} \subset \mathrm{im}(T)$.

On the other hand, remember from functional calculus (e.g. [4] XII 1.) that $\ker(T^*) = \mathrm{im}(T)^{\perp}$, thus

$$
\mathrm{im}(T) \subset \overline{\mathrm{im}(T)} = \mathrm{im}(T)^{\perp\perp} = \ker(T^*)^{\perp} \subset \ker(T)^{\perp}
$$

where the last inclusion is by $T^* \supset T$. This proves (2.22).

Now choose any $u \in D(T^*)$, then $(T^*u, v) = (u, Tv) = 0$ holds for all $v \in \ker(T)$, hence $T^*u \in \ker(T)^{\perp}$. Because of (2.22) this implies that $T^*u = Tw$ for some $w \in D(T)$ and hence for all $v \in D(T)$

$$
(u, Tv) = (T^*u, v) = (Tw, v) = (w, Tv).
$$

Using again (2.22) we deduce from this

$$
u - w \in \mathrm{im}(T)^{\perp} = \ker(T)^{\perp\perp} = \overline{\ker(T)}.
$$

Finally, the fact that T is closed comes into play: it implies that $\ker(T)$ is closed and hence $u \in w + \ker(T) \subset D(T)$. $\square$

As we are going to use the Sobolev embedding theorem several times we state here the part of it that we will need.

Theorem 2.23 *(Sobolev embedding)*

$$
W^{1,2}((0,1), \mathbb{R}^{2n}) \subset \mathcal{C}([0,1], \mathbb{R}^{2n}) \tag{2.23}
$$

and also

$$
W^{1,2}(S^1, \mathbb{R}^{2n}) \subset \mathcal{C}(S^1, \mathbb{R}^{2n}) \tag{2.24}
$$

are compact embeddings.

Proof: (2.23) is the standard form of the theorem and is proven e.g. in [1]. For (2.24) choose any $f \in W^{1,2}(S^1, \mathbb{R}^{2n})$, break up S^1 at some point $A \in S^1$ and note that $S^1 \setminus \{A\}$ is diffeomorphic to $(0,1)$. Therefore we get $f|_{S^1 \setminus \{A\}} \in W^{1,2}((0,1), \mathbb{R}^{2n})$ and deduce by (2.23) that f is continuous on $S^1 \setminus \{A\}$ after redefenition on a zero set. Now break up S^1 at some other point $B \in S^1 \setminus \{A\}$ and consider the redefined f. By the same argument as above we get that f is continuous on $S^1 \setminus \{B\}$ where alteration may only be necessary in A. So altogether f is continuous on S^1 after redefinition on a zero set. But in $W^{1,2}(S^1, \mathbb{R}^{2n})$ we only consider maps up to changes on zero sets anyway.

To see the compactness of the embedding note that $W^{1,2}(S^1, \mathbb{R}^{2n})$ and $\mathcal{C}(S^1, \mathbb{R}^{2n})$ are homeomorphic to subsets of $W^{1,2}([0,1], \mathbb{R}^{2n})$ and $\mathcal{C}([0,1], \mathbb{R}^{2n})$ respectively, then the claim follows from the standard form of the theorem. $\square$

L_S is a bounded perturbation of T since $-S : L^2(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ is bounded by remark 2.12(ii). But in order to apply the theory of the previous section we need a family of operators. So choose a path $P : [0,1] \to \mathcal{S}(n)$ from $P(0) = 0$ to $P(1) = S$ in $\mathcal{S}(n)$ that is piecewise holomorphic (with respect to the L^∞-metric) and then define

$$L(x) := -J\tfrac{\mathrm{d}}{\mathrm{d}t} - P(x): \quad W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n}). \tag{2.25}$$

One such path is for example $P(x) = xS$.

In the following we will have to use two different norms on the space of piecewise holomorphic paths in $\mathcal{S}(n)$. So in order to avoid confusion with the norms on $\Sigma^{1,\infty}(n)$ and $\mathcal{S}(n)$ defined in (2.9) and (2.10) we introduce some further notation. For a holomorphic path $P : [0,1] \to \mathcal{S}(n)$ we define

$$\begin{aligned} \|P\|_{\mathcal{C}^0} &:= \sup_{x \in [0,1]} \|P(x)\|_\infty, \\ \|P\|_{\mathcal{C}^1} &:= \|P\|_{\mathcal{C}^0} + \left\|\tfrac{\mathrm{d}P}{\mathrm{d}x}\right\|_{\mathcal{C}^0}. \end{aligned}$$

When P is only piecewise holomorphic the first norm is still well-defined, and for the second norm we consider at the nonholomorphic points the one side derivatives only. When we apply Kato's perturbation theory, $\|\cdot\|_\infty$ will take the part of the operator norm $\|\cdot\|$ on H and we will replace $\|\frac{\mathrm{d}\cdot}{\mathrm{d}x}\|_{sup}$ in theorem 2.18 by $\|\frac{\mathrm{d}\cdot}{\mathrm{d}x}\|_{\mathcal{C}^0}$ or $\|\cdot\|_{\mathcal{C}^1}$.

Remark 2.24 *Any piecewise holomorphic path $P : [0,1] \to \mathcal{S}(n)$ and also its first derivative is bounded uniformly as operator on $L^2(S^1, \mathbb{R}^{2n})$: For all $x \in [0,1]$ and $h \in L^2(S^1, \mathbb{R}^{2n})$ we have*

$$\|P(x)\, h\|_{L^2} \leq \|P\|_{\mathcal{C}^0} \|h\|_{L^2} \tag{2.26}$$

and — only considering derivatives from one side at the nonholomorphic points —

$$\left\| \frac{\mathrm{d}P}{\mathrm{d}x}(x)\, h \right\|_{L^2} \leq \left\| \frac{\mathrm{d}P}{\mathrm{d}x} \right\|_{\mathcal{C}^0} \|h\|_{L^2} \tag{2.27}$$

with $\|P\|_{\mathcal{C}^0} < \infty$ *and* $\|\frac{\mathrm{d}P}{\mathrm{d}x}\|_{\mathcal{C}^0} \leq \|P\|_{\mathcal{C}^1} < \infty$.

Proof: Remark 2.12(ii) says that each $P(x)$ and $\frac{\mathrm{d}P}{\mathrm{d}x}(x)$ is bounded by its L^∞-norm as a map on $L^2(S^1, \mathbb{R}^{2n})$. Furthermore, P is piecewise holomorphic in the L^∞-metric on $\mathcal{S}(n)$, so $\|P(x)\|_\infty$ is continuous in $x \in [0,1]$ and hence $\|P\|_{\mathcal{C}^0}$ is finite. And we also know that $\frac{\mathrm{d}P}{\mathrm{d}x}$ is a piecewise continuous arc in $\mathcal{S}(n)$ and has finite one side limits at the noncontinuous points, so $\|\frac{\mathrm{d}P}{\mathrm{d}x}(x)\|_\infty$ is bounded as well, hence $\|\frac{\mathrm{d}P}{\mathrm{d}x}\|_{\mathcal{C}^0} \leq \|P\|_{\mathcal{C}^0} + \|\frac{\mathrm{d}P}{\mathrm{d}x}\|_{\mathcal{C}^0} = \|P\|_{\mathcal{C}^1} < \infty$. □

Now we have to check that $L(x)$ is in fact of the type that we considered in the previous section.

Proposition 2.25 $L(x)$ *is a piecewise holomorphic family of type (A) for* $x \in [0,1]$.

Proof: $D(x) = W^{1,2}(S^1, \mathbb{R}^{2n})$ meets (iia) of definition 2.16 and it is easy to see that L is also piecewise holomorphic in the sense of (iib) and (iv). Let $h \in W^{1,2}(S^1, \mathbb{R}^{2n})$, then obviously $-J\dot{h} \in L^2(S^1, \mathbb{R}^{2n})$ and this is a constant when considering $L(x)h$ with respect to x. The remaining term $-P(x)h$ is piecewise holomorphic in x since P is bounded (see remark 2.24) and piecewise holomorphic. It remains to be shown that $L(x)$ is closed for all $x \in U$. This will follow from the next lemma since each $P(x)$ is bounded (by remark 2.12(ii)) and $-J\frac{\mathrm{d}}{\mathrm{d}t}$ is closed as seen in proposition 2.22. □

Lemma 2.26 *Let T be a closed operator on a Hilbert space H. If A is a bounded operator defined on $D(T)$, then $T + A$ with $D(T + A) = D(T)$ is also closed.*

Proof: Let $(x_n)_{n\in\mathbb{N}} \subset D(T)$ with $x_n \to x$ and $(T + A)x_n \to y$ converging in H. Then $Ax_n \to Ax$ since A is bounded and hence $Tx_n \to y - Ax$. From this we deduce by the closedness of T that $x \in D(T) = D(T + A)$ and $Tx = y - Ax$, thus $(T + A)x = y$ which was to be shown. □

The selfadjointness of the $L(x)$ follows from a similar theorem of Kato.

Theorem 2.27 ([14] V Thm. 4.3.)
Let T be a selfadjoint operator on a Hilbert space. If A is a symmetric and bounded operator defined on $D(T)$ then $T + A$ with $D(T + A) = D(T)$ is selfadjoint.

Proposition 2.28 $L(x)$ *is a selfadjoint operator with compact resolvent for all* $x \in [0,1]$.

Proof: We write $L(x) = T + A$ with $T = -J\frac{\mathrm{d}}{\mathrm{d}t}$, $D(T) = W^{1,2}(S^1, \mathbb{R}^{2n})$ and $A = -P(x)$. Then T is selfadjoint by proposition 2.22 and A is bounded on $D(T)$ as seen in remark 2.12(ii), and it is also symmetric on $D(T)$ since $P(x)$ is a path of real symmetric matrices for real x. Now we can apply the previous theorem and deduce that $L(x) = T + A$ is selfadjoint. This also tells us that the resolvent $(L(x) - i)^{-1}$ exists and is an L^2-bounded map $L^2(S^1, \mathbb{R}^{2n}) \to W^{1,2}(S^1, \mathbb{R}^{2n})$. This is because selfadjoint operators have a real spectrum and thus i is in the resolvent set. From the special form of $L(x)$ we can even deduce that $(L(x) - i)^{-1}$ is bounded with respect to the $W^{1,2}$-norm on $W^{1,2}(S^1, \mathbb{R}^{2n})$. To show this, we choose any $h \in L^2(S^1, \mathbb{R}^{2n})$ and set $g = (L(x) - i)^{-1}h$, then using $-J\dot{g} = h + (P(x) + i)g$ we obtain

$$\begin{aligned}
&\|(L(x) - i)^{-1}h\|_{W^{1,2}} \\
&\quad = \ \|\dot{g}\|_{L^2} + \|g\|_{L^2} \\
&\quad \leq \ \|h\|_{L^2} + \|(P(x) + i)g\|_{L^2} + \|g\|_{L^2} \\
&\quad \leq \ \|h\|_{L^2} + \|P(x)\|_\infty \|g\|_{L^2} + 2\|g\|_{L^2} \\
&\quad \leq \ (\ 1 + \|P(x)\|_\infty \|(L(x) - i)^{-1}\| + 2\|(L(x) - i)^{-1}\|\ )\ \|h\|_{L^2}.
\end{aligned}$$

Here $\|(L(x) - i)^{-1}\|$ is the finite L^2-operator norm. Now apply the Sobolev embedding theorem 2.23 to see that $(L(x)-i)^{-1} : L^2(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ is even compact. As we have seen in proposition 2.25, $L(x)$ is also closed, so according to theorem 2.15 we know that $L(x)$ is an operator with compact resolvent for every $x \in [0, 1]$. □

Now we are in the position to apply Kato's theorems 2.15 and 2.17 to get a detailed description of the spectrum of the $L(x)$.

Corollary 2.29 *The spectrum of $L(x)$ is discrete for all $x \in [0, 1]$, that is $\sigma(L(x))$ consists of isolated eigenvalues with finite multiplicities. Moreover, all eigenvalues are real and, if $\Phi(x)$ is continuous, their multiplicities are less than or equal to $2n$.*

Proof: We can apply theorem 2.15 because of proposition 2.28 and as $L(x)$ is selfadjoint, its spectrum is real. For continuous $P(x) =: S$ we fix any $x \in [0, 1]$, so that the equation for an eigenvector $h \in W^{1,2}(S^1, \mathbb{R}^{2n})$ corresponding to a fixed eigenvalue λ of $L(x)$ is equivalent to

$$\dot{h}(t) = J\,(S(t) + \lambda)\,h(t).$$

This is an ordinary differential equation of first order and obviously the right hand side is Lipschitz continuous in $t \in [0, 1]$ and h. Hence by the Picard-Lindelöf theorem ([6] 117.1) there is a unique solution for any initial value $h(0) \in \mathbb{R}^{2n}$. For the eigenspace, there is the additional condition $h(0) = h(1)$, so it has to be of dimension less than or equal to $2n$. □

Corollary 2.30 *There exists a bi-infinite sequence* $(\kappa_k)_{k\in\mathbb{Z}}$ *of continuous, piecewise holomorphic functions* $\kappa_k : [0,1] \to \mathbb{R}$ *satisfying the following:*

(i) $\kappa_k(x) \le \kappa_{k+1}(x)$ *for all* $k \in \mathbb{Z}$ *and any* $x \in [0,1]$.

(ii) For each $x \in [0,1]$ *the spectrum of* $L(x)$ *is given by*

$$\sigma(L(x)) = \{\kappa_k(x) \mid k \in \mathbb{Z}\}.$$

(iii) For each $x \in [0,1]$ *the multiplicity of any eigenvalue* $\tau \in \sigma(L(x))$ *is*

$$\sharp\{k \in \mathbb{Z} \ \mid \kappa_k(x) = \tau\}.$$

Proof: Because of the propositions 2.25 and 2.28 we can apply theorem 2.17 to get piecewise holomorphic functions ν_n representing the spectrum of $L(x)$ (see also remark 2.21). Then we define $(\kappa_n)_{n\in\mathbb{Z}}$ according to definition 2.19. The sequence is numbered by $\mathbb{Z}$ since P is a bounded perturbation and because of the spectrum of $L(0)$ being unbounded from above and below, which will be seen in the next lemma. □

Lemma 2.31 *The spectrum of* $L(0) = -J\frac{\mathrm{d}}{\mathrm{d}t}$ *on* $W^{1,2}(S^1, \mathbb{R}^{2n}) \subset L^2(S^1, \mathbb{R}^{2n})$ *is* $\sigma(L(0)) = 2\pi\mathbb{Z}$ *and each of these eigenvalues has multiplicity* $2n$.

Proof: By corollary 2.29 the spectrum consists of isolated eigenvalues. The eigenvector equation for $h \in W^{1,2}(S^1, \mathbb{R}^{2n})$ is

$$-J\dot{h}(t) = \lambda h(t)$$

implying

$$h(t) = \mathrm{e}^{\lambda t J} h(0).$$

For this solution to be in $W^{1,2}(S^1, \mathbb{R}^{2n}) \subset \mathcal{C}(S^1, \mathbb{R}^{2n})$ (by theorem 2.23) there is the additional condition $h(1) = h(0)$, that is

$$\mathrm{e}^{\lambda J} h(0) = \cos(\lambda) h(0) + \sin(\lambda) J h(0) = h(0)$$

where (2.4) has been used. Since J has no real eigenvalues, the above can only hold for $\sin(\lambda) = 0$, that is $\lambda \in \pi\mathbb{Z}$. For $\lambda = (2k+1)\pi$ we would get $-h(0) = h(0)$, hence $h(0) = 0$ and thus $h \equiv 0$ — which is no eigenvector. But for each $\lambda \in 2\pi\mathbb{Z}$ we even have $\mathrm{e}^{\lambda J} = \mathbb{1}$. Thus the space of solutions, that is the eigenspace, is $2n$-dimensional just as the space of possible initial values $h(0) \in \mathbb{R}^{2n}$. □

We can also apply theorem 2.18 to families of asymptotic operators. This gives us a bound on the growth rate of the eigenvalues along such families which will be useful in the construction of sections of the spectrum bundle.

Corollary 2.32 *Let $B : [0,1] \to \mathcal{S}(n)$ be any piecewise holomorphic path and $L(x) = -J\frac{\mathrm{d}}{\mathrm{d}t} - B(x) : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ the corresponding family of asymptotic operators. Then considering the ordered sequence $(\kappa_k)_{k\in\mathbb{Z}}$ representing $\sigma(L(\cdot))$ as defined in 2.19, we have for all $k \in \mathbb{Z}$ and $x_0 \le x_1 \in [0,1]$*

$$|\kappa_k(x_0) - \kappa_k(x_1)| \le \sup_{x\in[x_0,x_1]} \left\| \frac{\mathrm{d}B}{\mathrm{d}x}(x) \right\|_\infty |x_0 - x_1| \le \left\| \frac{\mathrm{d}B}{\mathrm{d}x} \right\|_{\mathcal{C}^0} |x_0 - x_1|.$$

And we also have

$$\frac{\mathrm{d}\kappa_k}{\mathrm{d}x}(x_0) = \left(\phi \, , \, -\frac{\mathrm{d}B}{\mathrm{d}x}(x_0)\, \phi \right)_{L^2}$$

which at nonholomorphic points has to be interpreted as the left or right hand side derivative. For ϕ we have to put in $\psi_{\pi(n)}(x_0)$ from theorem 2.17 corresponding to $\nu_{\pi(n)}(x) = \kappa_n(x)$ for $x \le x_0$ or $x \ge x_0$ respectively.

Proof: $L(x)$ is a selfadjoint, piecewise holomorphic family with compact resolvent as in proposition 2.25 and 2.28 — we did not use $B(0) = 0$ to prove this. Moreover, B is a bounded perturbation as seen in remark 2.24. So we can apply remark 2.21 to $T = -J\frac{\mathrm{d}}{\mathrm{d}t}$ and $A(x) = -B(x)$. □

2.2.3 The spectrum bundle

In this section we will prove theorem 2.14, only leaving property a) and the connection with the Conley-Zehnder index for the next section. So we have to construct continuous maps $\lambda_k : (\mathcal{S}(n), \|\cdot\|_\infty) \to \mathbb{R}$ in ascending order with $k \in \mathbb{Z}$, that represent the spectra of the asymptotic operators L_S and are normalized by

$$\lambda_{-n+1}(0) = \ldots = \lambda_n(0) = 0. \tag{2.28}$$

This will be done in a way so that it becomes clear that the maps are uniquely determined by the above requirements.

Let an arbitrary $S \in \mathcal{S}(n)$ be given and choose any piecewise holomorphic path $P : [0,1] \to \mathcal{S}(n)$ from 0 to S. Since the arc is continuous, we know that the $\lambda_k(P(x))$ have to be continuous in x. They also have to be in ascending order and represent the spectrum of $L_{P(x)} = L(x)$ (compare (2.12) to (2.25) with the arc P). Thus $(\lambda_k(P(x)))_{k\in\mathbb{Z}}$ has to be identical to the sequence $(\kappa_k(x))_{k\in\mathbb{Z}}$ of corollary 2.30 up to a shift in the numbering that is the same for all x. The latter is because of the continuity of both sequences and the fact that due to finite degeneracy they do not consist of only one value. The numbering of $(\lambda_k)_{k\in\mathbb{Z}}$ is fixed by the normalization (2.28). We also shift the numbering of $(\kappa_k)_{k\in\mathbb{Z}}$ such that

$$\kappa_{-n+1}(0) = \ldots = \kappa_n(0) = 0. \tag{2.29}$$

This shift is uniquely determined since there are exactly $2n$ zeros in $(\kappa_k)_{k\in\mathbb{Z}}$ representing the eigenvalue 0 of $L(0)$ that has multiplicity $2n$ by lemma 2.31. So from each path P we get a uniquely determined *normalized sequence* $(\kappa_k)_{k\in\mathbb{Z}}$ representing $\sigma(L_{P(\cdot)})$ and fulfilling (2.29). Now by above considerations, the maps λ_k are necessarily defined as follows.

Definition 2.33 *For given $S \in \mathcal{S}(n)$ choose any piecewise holomorphic path $P : [0,1] \to \mathcal{S}(n)$ starting at 0 and ending at S. Let $(\kappa_k)_{k\in\mathbb{Z}}$ be the normalized sequence of functions representing $\sigma(L_{P(\cdot)})$. Then for all $k \in \mathbb{Z}$*

$$\lambda_k(S) := \kappa_k(1).$$

The main task will be to show that this definition does not depend upon the choice of the path P. But taking this for granted first note that by construction these maps are ordered, normalized by (2.28) and represent the spectra of the asymptotic operators L_S including multiplicities.

Moreover, with the above definition, one can easily compute differences $\lambda_k(S+\Delta)-\lambda_k(S)$ for some $S, S+\Delta \in \mathcal{S}(n)$. Consider a piecewise holomorphic path from 0 to S defined on $[0,1]$. When reparametrizing it holomorphically, the functions representing the spectrum of the corresponding operator family are simply reparametrized the same way. We reparametrize the above path to $[0,\frac{1}{2}]$ and on $[\frac{1}{2},1]$ continue it from S to $S+\Delta$. This gives a normalized sequence $(\tilde{\kappa}_k)_{k\in\mathbb{Z}}$ such that $\lambda_k(S) = \tilde{\kappa}_k(\frac{1}{2})$ and $\lambda_k(S+\Delta) = \tilde{\kappa}_k(1)$. So in order to investigate the difference between $\lambda_k(S)$ and $\lambda_k(S+\Delta)$ we only need to consider any path from S to $S+\Delta$, again reparametrized to $[0,1]$. There also is an ordered sequence $(\kappa_k)_{k\in\mathbb{Z}}$ for this path and the difference in question is equal to $\tilde{\kappa}_k(1) - \tilde{\kappa}_k(\frac{1}{2}) = \kappa_j(1) - \kappa_j(0)$ for $j = j(k) \in \mathbb{Z}$ such that the eigenfunctions corresponding to $\kappa_j(0)$ and $\tilde{\kappa}_k(\frac{1}{2})$ are the same. Summarizing we have the following.

Remark 2.34 *Let $P : [0,1] \to \mathcal{S}(n)$ be any piecewiese holomorphic path from S to $S+\Delta$ and $(\kappa_k)_{k\in\mathbb{Z}}$ the ordered sequence representing $\sigma(L_{P(\cdot)})$. Normalize this sequence such that the same eigenfunctions correspond to both $\lambda_k(S)$ and $\kappa_k(0)$, then for all $k \in \mathbb{Z}$*

$$\lambda_k(S+\Delta) - \lambda_k(S) = \kappa_k(1) - \kappa_k(0).$$

Now using this remark we can also deduce that the λ_k are continuous with respect to the L^∞-metric on $\mathcal{S}(n)$: Choose any $S_0 \in \mathcal{S}(n)$ then

$$U_\varepsilon := \{S \in \mathcal{S}(n) \mid \|S - S_0\|_\infty < \varepsilon\} = \{S_0 + \Delta \mid \Delta \in \mathcal{S}(n), \|\Delta\|_\infty < \varepsilon\}$$

is a neighbourhood of S_0 for all $\varepsilon > 0$. For $S = S_0+\Delta \in U_\varepsilon$ let $(\kappa_k)_{k\in\mathbb{Z}}$ be the normalized sequence of eigenvalue functions along the path $P(x) = S_0 + x\Delta$ as in remark 2.34. This and corollary 2.32 imply that for all $k \in \mathbb{Z}$

$$|\lambda_k(S) - \lambda_k(S_0)| = |\kappa_k(1) - \kappa_k(0)| \le \left\| \frac{\mathrm{d}P}{\mathrm{d}x} \right\|_{\mathcal{C}^0} = \|\Delta\|_\infty < \varepsilon.$$

So we even have uniform continuity of $(\lambda_k)_{k\in\mathbb{N}}$, that is for any $S_0 \in \mathcal{S}(n)$ and $\varepsilon > 0$ there is a neighbourhood U_ε of S_0 such that $|\lambda_k(S) - \lambda_k(S_0)| < \varepsilon$ for all $S \in U_\varepsilon$ and $k \in \mathbb{Z}$.

In order to see that the λ_k are actually well-defined by definition 2.33, first note that the fundamental group of $\mathcal{S}(n)$ is trivial. In fact, any loop $P : S^1 \to \mathcal{S}(n)$ can easily be contracted to the constant loop at $0 \in \mathcal{S}(n)$ by the homotopy sP, $s \in [0,1]$. So any two paths in $\mathcal{S}(n)$ connecting 0 to S can be homotoped into one another with fixed endpoints. Of course, this can also be done continuously with respect to the $\mathcal{C}^1$-metric on the space of piecewise holomorphic paths in $\mathcal{S}(n)$. We also know that $\lambda_k(S)$ can only assume the discrete values given by $\sigma(L_S)$. So to make sure that it does not depend on the choice of the path P it remains to prove the following.

Theorem 2.35 *Let $S \in \mathcal{S}(n)$ and $P_0 : [0,1] \to \mathcal{S}(n)$ be any path as used in definition 2.33 for $\lambda_k(S)$. Then for all $k \in \mathbb{Z}$, $\lambda_k(S)$ is continuous with respect to $\mathcal{C}^1$-small variations of P_0 preserving the analyticity and end points of the path.*

Proof: Choose any $\varepsilon > 0$ and denote the normalized sequence of eigenvalue functions corresponding to P_0 by $(\kappa_k^0)_{k\in\mathbb{Z}}$. Then we have to show that there is a $\delta > 0$ such that for all piecewise holomorphic paths $P : [0,1] \to \mathcal{S}(n)$ that meet

$$P(0) = 0\,, \quad P(1) = S \quad \text{and} \quad \|P - P_0\|_{\mathcal{C}^1} < \delta \tag{2.30}$$

we have $|\kappa_k(1) - \kappa_k^0(1)| \leq \varepsilon$ for all $k \in \mathbb{Z}$ where $(\kappa_k)_{k\in\mathbb{Z}}$ is the normalized sequence attached to P.

For any P as above and fixed $y \in [0,1]$ consider the following path in $\mathcal{S}(n)$ which is obviously holomorphic:

$$B(x) = P_0(y) + x(P(y) - P_0(y))\,, \quad x \in [0,1]. \tag{2.31}$$

Denote the ordered sequence of functions representing the spectrum of $L_{B(x)}$ by $(\tilde{\kappa}_k)_{k\in\mathbb{Z}}$, then corollary 2.32 yields for all $k \in \mathbb{Z}$

$$|\tilde{\kappa}_k(0) - \tilde{\kappa}_k(1)| \leq \|P(y) - P_0(y)\|_\infty. \tag{2.32}$$

This gives an upper bound for the difference between the eigenvalues $\tilde{\kappa}_k(1)$ of $L_{P(y)}$ and the ones $\tilde{\kappa}_k(0)$ of $L_{P_0(y)}$. But it is not obvious that for a given $j \in \mathbb{Z}$ this bound is also valid between κ_j and κ_j^0. In order to show this, one has to make sure that by tracing the eigenvalues along the above path B, actually $\kappa_j^0(y)$ goes to $\kappa_j(y)$ and not to another branch of (κ_k).

We know that both $(\kappa_k^0(y))_{k\in\mathbb{Z}}$ and $(\tilde{\kappa}_k(0))_{k\in\mathbb{Z}}$ represent $\sigma(L_{P_0(y)})$, so because of their ascending order, the sequences have to be equal up to a shift in the numbering. The same holds for $(\kappa_k(y))$ and $(\tilde{\kappa}_k(1))$ and hence for every $y \in [0,1]$ there is a shift $s(y) \in \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ (2.32) becomes

$$|\kappa_{k+s}(y) - \kappa_k^0(y)| \leq \|P(y) - P_0(y)\|_\infty. \tag{2.33}$$

Moreover, we know from the normalization (2.29) that $\kappa_k(0) = \kappa_k^0(0)$ and hence $s(0) = 0$, so in order to prove the claim we have to make sure that for some sufficiently small δ this shift can not change along $y \in [0,1]$.

As a first step towards this we deduce from the discreteness of the spectrum that for every $j \in \mathbb{Z}$ and $y \in [0,1]$ there is a gap around $\kappa_j^0(y)$ in $\sigma(L_{P_0(y)})$, i.e. there is a maximal $\gamma(y) > 0$ such that

$$\sigma(P_0(y)) \cap (\ \kappa_j^0(y) - \gamma(y)\,,\ \kappa_j^0(y) + \gamma(y)\ ) = \ \{\kappa_j^0(y)\}. \tag{2.34}$$

Now let $0 < \delta < \frac{1}{4}\gamma(y)$ and P be a path with (2.30). Then $\sigma(L_{P(y)})$ must have a gap corresponding to the gap for $L_{P_0(y)}$ because of the bound (2.33) on the distance of eigenvalues. That is, there are no eigenvalues of $L_{P(y)}$ in

$$(\ \kappa_j^0(y) - \gamma(y) + \delta\,,\ \kappa_j^0(y) - \delta\ ) \cup (\ \kappa_j^0(y) + \delta\,,\ \kappa_j^0(y) + \gamma(y) - \delta\ )\ . \tag{2.35}$$

If we knew now that $|\kappa_j(y) - \kappa_j^0(y)| < \gamma(y) - \delta$, then it would follow from this gap that even $|\kappa_j(y) - \kappa_j^0(y)| < \delta$. This is the main effect that we will use to make sure that κ_j stays close to κ_j^0 all along $[0,1]$. The rest of the proof would be easy if the gap width γ was bounded away from zero. But from the continuity of the κ_k^0 we can only deduce that γ is continuous on $[0,1]$ except for the crossing points. At these points, $\gamma(y)$ is positive but from both sides tends to zero since several branches of eigenvalue functions meet in $\kappa_j^0(y)$.

Next note that $A := \|P_0\|_{\mathcal{C}^1} + 1 < \infty$ by remark 2.24. Choose $\delta < 1$, then for P as in (2.30) we have $\|P\|_{\mathcal{C}^1} < A$ and corollary 2.32 implies that for any $x, y \in [0,1]$ and $k \in \mathbb{Z}$

$$|\kappa_k(x) - \kappa_k(y)| < A\,|x-y| \tag{2.36}$$

as well as

$$|\kappa_k^0(x) - \kappa_k^0(y)| < A\,|x-y|. \tag{2.37}$$

Now we choose a fixed $j \in \mathbb{Z}$ for the rest of the proof. Since all κ_k^0 are piecewise holomorphic and have a uniform bound (2.37) on the growth rate, there only are a finite number of crossing points along κ_j^0, at which some other κ_k^0 actually crosses this graph, i.e. is not identical to it on a neighbourhood of the point. On compact intervals $[a,b] \subset [0,1]$ not containing any crossing point we can use the finite gap width $\Gamma := \min_{y\in[a,b]} \gamma(y) > 0$ to make sure that the shift s stays constant.

There might be several functions κ_k^0 that on $[a,b]$ are identical to κ_j^0 and differ from it on the other side of some crossing point or not. Because of the order of (κ_k^0) these functions have to be consecutively numbered by $\{j-l, \dots, j+m\} =: N$ for some $l, m \in \mathbb{N}_0$. By (2.34) all other κ_k^0, $k \notin N$ are on $[a,b]$ separated from κ_j^0 by at least Γ. Choose some $0 < \delta < \min\{1, \frac{1}{4}\Gamma\}$, let P be a path with (2.30), $i \in N$, and set $s := s(a)$. For any $y \in [a,b]$

assume $|\kappa_{i+s}(y) - \kappa_i^0(y)| \le \delta$. Combining this with (2.36) and (2.37) we deduce for all $x \in (\, y - \frac{\delta}{A}\, ,\, y + \frac{\delta}{A}\,)$

$$
\begin{aligned}
&|\kappa_{i+s}(x) - \kappa_i^0(x)| \\
&\quad \le \quad |\kappa_{i+s}(x) - \kappa_{i+s}(y)| + |\kappa_{i+s}(y) - \kappa_i^0(y)| + |\kappa_i^0(y) - \kappa_i^0(x)| \\
&\quad < \quad 2A\,|x-y| + \delta \; < \; 3\delta \; < \; \Gamma - \delta .
\end{aligned}
$$

The gap (2.35) of $\sigma(L_{P(y)})$ implies that even

$$|\kappa_{i+s}(x) - \kappa_i^0(x)| \le \delta \tag{2.38}$$

for $x \in (\, y - \frac{\delta}{A}\, ,\, y + \frac{\delta}{A}\,) \cap [a,b]$.

From (2.33) we obtain $|\kappa_{i+s}(a) - \kappa_i^0(a)| \le \delta$, therefore we know that $a \in I := \{x \in [a,b] \,\big|\, |\kappa_{i+s}(x) - \kappa_i^0(x)| \le \delta\}$, so I is not empty. By the above, I is open with respect to $[a,b]$, and it is also closed because of the continuity of κ_{i+s} and κ_i^0. Hence we have $I = [a,b]$, that is the inequality (2.38) extends to the whole interval $[a,b]$. On the other hand, for any $x \in [a,b]$, $i \in N$ and $k \in \mathbb{Z} \setminus N$ we have

$$|\kappa_{i+s}(x) - \kappa_k^0(x)| \ge |\kappa_i^0(x) - \kappa_k^0(x)| - |\kappa_{i+s}(x) - \kappa_i^0(x)| \ge \Gamma - \delta > 3\delta,$$

hence in (2.38) we can not change the shift s — for any other shift, some of the κ_{i+s}, $i \in N$ would have to be closer than δ to some κ_k^0, $k \notin N$. Thus we have shown that $s(x) = s(a)$ for all $x \in [a,b]$, that is the shift s is constant along closed intervals not containing a crossing point of κ_j^0.

To complete the proof it remains to find an interval around every crossing point of κ_j^0 on which the shift s is also constant. So we have to consider a crossing point y where several graphs of (κ_k^0) meet κ_j^0. These will be κ_k^0 for $k \in N := \{j-l, \ldots, j+m\}$ with some $l, m \in \mathbb{N}_0$ since the sequence is ordered. From the gap of width $\gamma(y)$ in the spectrum of $L_{P_0(y)}$ and the maximal growth rate (2.37) we can deduce some restrictions on the spectrum of $L_{P_0(x)}$ for $x \in [\, y - \frac{\gamma(y)}{8A}\, ,\, y + \frac{\gamma(y)}{8A}\,]$: For the graphs that meet in $\kappa_j^0(y)$ we have

$$\forall k \in N:\; \kappa_k^0(x) \in (\, \kappa_j^0(y) - \tfrac{1}{8}\gamma(y)\, ,\, \kappa_j^0(y) + \tfrac{1}{8}\gamma(y)\,) \tag{2.39}$$

and for all other graphs of (κ_k^0)

$$\forall k \in \mathbb{Z} \setminus N:\; \kappa_k^0(x) \in \mathbb{R} \setminus (\, \kappa_j^0(y) - \tfrac{7}{8}\gamma(y)\, ,\, \kappa_j^0(y) + \tfrac{7}{8}\gamma(y)\,)\,. \tag{2.40}$$

Now let $a = \frac{\gamma(y)}{8A}$, choose some $0 < \delta < \min\{1, \frac{1}{8}\gamma(y)\}$ and assume P to fulfill (2.30). Moreover, we set $s := s(y-a)$ such that from (2.33) we have $|\kappa_{k+s}(y-a) - \kappa_k^0(y-a)| < \delta$ for all $k \in \mathbb{Z}$. Therefore, using (2.39), we obtain for all $i \in N$

$$\kappa_{i+s}(y-a) \in (\, \kappa_j^0(y) - \tfrac{1}{8}\gamma(y) - \delta\, ,\, \kappa_j^0(y) + \tfrac{1}{8}\gamma(y) + \delta\,)$$

From this, the maximal growth rate (2.36) and $aA < \frac{1}{8}\gamma(y)$ we deduce for all $i \in N$ and $x \in [y-a, y+a]$

$$\begin{aligned} \kappa_{i+s}(x) &\in \left(\kappa_j^0(y) - \tfrac{1}{8}\gamma(y) - \delta - 2aA \,,\, \kappa_j^0(y) + \tfrac{1}{8}\gamma(y) + \delta + 2aA \right) \\ &\subset \left(\kappa_j^0(y) - \tfrac{1}{2}\gamma(y) \,,\, \kappa_j^0(y) + \tfrac{1}{2}\gamma(y) \right) . \end{aligned} \tag{2.41}$$

Now consider any $x \in [y-a, y+a]$: in view of (2.40) and (2.41) we have for all $i \in N$ and $k \in \mathbb{Z} \setminus N$

$$|\kappa_{i+s}(x) - \kappa_k^0(x)| \geq |\kappa_j^0(y) - \kappa_k^0(x)| - |\kappa_{i+s}(x) - \kappa_j^0(y)| \geq \tfrac{7}{8}\gamma(y) - \tfrac{1}{2}\gamma(y) > \delta.$$

Hence

$$\forall k \in \mathbb{Z}: \; |\kappa_{k+s(x)}(x) - \kappa_k^0(x)| < \delta \tag{2.42}$$

can only hold if for any $k \in \mathbb{Z} \setminus N$ we have $k + s(x) - s \notin N$, that is if and only if $s(x) = s$. Since $s(x)$ is defined by (2.42) we have shown that on $[y-a, y+a]$ the shift s is constant.

We have seen that for sufficiently small $\delta > 0$ the shift s is constant on some small closed intervals around crossing points of κ_j^0 (with the crossing point in the interior) and on closed intervals not containing crossing points. Since there are only finitely many crossing points for a fixed κ_j^0, the interval $[0,1]$ is covered by finitely many intervals on which for sufficiently small $\delta > 0$ the shift s is constant. Therefore there is a $\delta > 0$ such that on all of these intervals s is constant and hence on all of $[0,1]$ we have $s \equiv s(0) = 0$. Now if we also choose $\delta < \varepsilon$, then (2.33) yields for $y = 1$ and all $k \in \mathbb{Z}$

$$|\kappa_k(1) - \kappa_k^0(1)| \leq \|P(1) - P_0(1)\|_\infty \leq \|P - P_0\| < \varepsilon,$$

which was to be shown. □

2.2.4 Proof of $\mu_{CZ} = \mu_{spec}$

In the previous section we established the unique existence of globally numbered sections λ_k of the spectrum bundle that have the properties (i) to (iv) of theorem 2.14. Now we will prove the remaining property a) of these sections and we will finish the proof of theorem 2.14 by checking that the index constructed from the spectral bundle sections,

$$\mu_{spec}(\Phi) = \max\{k \in \mathbb{Z} \mid \lambda_k(\Phi) < 0\} \quad \text{for all} \quad \Phi \in \Sigma^{1,\infty}(n), \tag{2.43}$$

meets the conditions of the axiomatic definition 2.9 of the Conley-Zehnder index.

When using the explicit definition of the λ_k and the subsequent remark 2.34, the proof of theorem 2.14 a) reduces to the following lemma.

Lemma 2.36 *Let $\Delta \in \mathcal{S}(n)$ and let it satisfy $\Delta(t) \geq \varepsilon > 0$ for all $t \in [0,1]$. For any $S \in \mathcal{S}(n)$ let $(\kappa_k)_{k\in\mathbb{Z}}$ be the ordered sequence representing the spectrum of $L_{S+x\Delta}$. Then we have for all $k \in \mathbb{Z}$*

$$\kappa_k(1) < \kappa_k(0).$$

Proof: $S+x\Delta$ obviously is a piecewise holomorphic path in $\mathcal{S}(n)$, so we can apply corollary 2.32 and obtain for any $k \in \mathbb{Z}$ and some unknown normalized family $\phi_x \in L^2(S^1, \mathbb{R}^{2n})$ of eigenfunctions of $L_{S+x\Delta}$

$$\begin{aligned}
\kappa_k(1) - \kappa_k(0) &= \int_0^1 \left(\phi_x\,,\, -\frac{\mathrm{d}(S+x\Delta)}{\mathrm{d}x}\phi_x\right)_{L^2} \mathrm{d}x \\
&= \int_0^1 \int_0^1 -\langle \phi_x(t)\,,\, \Delta(t)\,\phi_x(t)\rangle \;\mathrm{d}t\,\mathrm{d}x \\
&\leq -\int_0^1 \int_0^1 \varepsilon |\phi_x(t)|^2 \;\mathrm{d}t\,\mathrm{d}x \\
&\leq -\varepsilon \int_0^1 \|\phi_x\|_{L^2}^2 \mathrm{d}x \\
&= -\varepsilon \;<\; 0.
\end{aligned}$$

□

In order to show the homotopy invariance of μ_{spec} we need to understand the situation in which eigenvalue functions of L_Φ for some symplectic arc Φ cross 0. Here and in the following we denote by L_Φ the asymptotic operator L_{S_Φ} corresponding to the symplectic arc Φ.

Proposition 2.37 *For any $\Phi \in W^{1,\infty}([0,1], Sp(n))$ with $\Phi(0) = \mathbb{1}$ we have $\ker(L_\Phi) = \{0\}$, i.e. $0 \notin \sigma(L_\Phi)$ holds if and only if $\Phi(1) \in Sp^*(n)$, that is $\Phi \in \Sigma^{1,\infty}(n)$.*

Proof: Substituting $h = \Phi g$ we have the equivalence

$$\begin{aligned}
L_\Phi h = 0 \quad &\Longleftrightarrow \quad -J\dot{h}(t) = -J\dot{\Phi}(t)\Phi(t)^{-1}h(t) \\
&\Longleftrightarrow \quad -J\dot{\Phi}(t)g(t) - J\Phi(t)\dot{g}(t) = -J\dot{\Phi}(t)g(t) \\
&\Longleftrightarrow \quad \dot{g}(t) = 0
\end{aligned}$$

where the last equivalence follows from the nondegeneracy of the symplectic $\Phi(t)$. Therefore the set of solutions of $L_\Phi h = 0$ is

$$\{\Phi h_0 \mid h_0 \in \mathbb{R}^{2n}\} \subset W^{1,\infty}([0,1], \mathbb{R}^{2n}) \subset W^{1,2}([0,1], \mathbb{R}^{2n}).$$

In order that $\Phi h_0 \in W^{1,2}(S^1, \mathbb{R}^{2n})$ we also need $h_0 = \Phi(0)h_0 = \Phi(1)h_0$ (due to the Sobolev embedding 2.23). Hence remembering that $\Phi(0) = \mathbb{1}$ we obtain

$$\ker(L_\Phi) = \{\Phi h_0 \mid h_0 \in \mathbb{R}^{2n} \text{ such that } \Phi(1)h_0 = h_0\}.$$

Also note that $\Phi h_0 \not\equiv 0$ for $h_0 \neq 0$ since $\Phi(0) = \mathbb{1}$, so the kernel of L_Φ is trivial if and only if $\Phi(1)$ has no eigenvalue 1, that is if Φ is nondegenerate. □

Now consider a homotopy $\Psi : [0,1] \to \Sigma^{1,\infty}(n)$. Because of the above lemma and the nondegeneracy of all $\Psi(s)$ we have $0 \notin \sigma(L_{\Psi(s)})$ for all $s \in [0,1]$. Moreover, we know that for all $k \in \mathbb{Z}$ the function λ_k representing a part of this spectrum is continuous with respect to the $W^{1,\infty}$-metric on $\Sigma^{1,\infty}(n)$. But Ψ is a homotopy on that metric space, hence for any $k \in \mathbb{Z}$, $\lambda_k(\Psi(s))$ is also continuous in s and thus cannot cross 0. Therefore we deduce that $\lambda_k(\Psi(1)) < 0$ holds if and only if $\lambda_k(\Psi(0)) < 0$ and hence

$$\begin{aligned}\mu_{spec}(\Psi(1)) &= \min\{k \in \mathbb{Z} \mid \lambda_k(\Psi(1)) < 0\} \\ &= \min\{k \in \mathbb{Z} \mid \lambda_k(\Psi(0)) < 0\} \quad = \mu_{spec}(\Psi(0)).\end{aligned}$$

This proves that μ_{spec} is invariant with repect to homotopies in $\Sigma^{1,\infty}(n)$.

Before proceding with checking the conditions for the Conley-Zehnder index let us introduce an equivalent definition for μ_{spec} in terms of the spectral flow through 0.

Proposition 2.38 *For $\Phi, \Psi \in W^{1,\infty}([0,1], Sp(n))$ with $\Phi(0), \Psi(0) = \mathbb{1}$ let P be any piecewise holomorphic path in $\mathcal{S}(n)$ from $P(0) = S_\Phi$ to $P(1) = S_\Psi$ and let $(\nu_n)_{n\in\mathbb{N}}$ be the sequence of functions representing $\sigma(L_{P(\cdot)})$ as given by theorem 2.17. Then*

$$\begin{aligned}\mu_{spec}(\Psi) - \mu_{spec}(\Phi) \; = \; & \sharp\{n \in \mathbb{N} \mid \nu_n(0) \geq 0 > \nu_n(1)\} \\ & - \; \sharp\{n \in \mathbb{N} \mid \nu_n(1) \geq 0 > \nu_n(0)\}.\end{aligned}$$

This number can also be obtained from the flow of (ν_n) through zero: For every $n \in \mathbb{N}$, we count each point in $[0,1]$ at which ν_n goes from $[0,\infty)$ to $(-\infty, 0)$ as $+1$, and each point in $[0,1]$ at which ν_n goes from $(-\infty,0)$ to $[0,\infty)$ is counted as -1. In particular, for a path P as above with $P(0) = 0$ (i.e. $\Phi \equiv \mathbb{1}$) one obtains

$$\begin{aligned}\mu_{spec}(\Psi) \; = -n \; + & \; \sharp\{n \in \mathbb{N} \mid \nu_n(0) \geq 0 > \nu_n(1)\} \\ - & \; \sharp\{n \in \mathbb{N} \mid \nu_n(1) \geq 0 > \nu_n(0)\}.\end{aligned}$$

Proof: For the definition of $\mu_{spec}(\Psi)$, some piecewise holomorphic path $\tilde{P} : [0,1] \to \mathcal{S}(n)$ from 0 to S_Ψ is needed. We choose a path going from 0 to $\tilde{P}(\frac{1}{2}) = \Phi$ and then along P reparametrized to $[\frac{1}{2}, 1]$. Let $(\tilde{\kappa}_k)_{k\in\mathbb{Z}}$ be the ordered, normalized sequence representing $L_{\tilde{P}(\cdot)}$ and normalize the ordered sequence $(\kappa_k)_{k\in\mathbb{Z}}$ attached to P such that $\tilde{\kappa}_k(\frac{1}{2}) = \kappa_k(0)$ for all $k \in \mathbb{Z}$. Then we can express μ_{spec} in terms of (κ_k):

$$\begin{aligned}\mu_{spec}(\Psi) - \mu_{spec}(\Phi) \; = \; & \sharp\{k \in \mathbb{Z} \mid \kappa_k(1) < 0 \leq \kappa_k(0)\} \\ & - \; \sharp\{k \in \mathbb{Z} \mid \kappa_k(0) < 0 \leq \kappa_k(1)\}. \qquad (2.44)\end{aligned}$$

In order to see this, first assume $\mu_{spec}(\Psi) \geq \mu_{spec}(\Phi)$, then we know that

$$\begin{aligned} \tilde{\kappa}_k(1), \tilde{\kappa}_k(\tfrac{1}{2}) < 0 \quad &\text{for} \quad k \leq \mu_{spec}(\Phi), \\ \tilde{\kappa}_k(1) < 0 \leq \tilde{\kappa}_k(\tfrac{1}{2}) \quad &\text{for} \quad k \in \{\mu_{spec}(\Phi)+1, \ldots, \mu_{spec}(\Psi)\}, \\ \tilde{\kappa}_k(1), \tilde{\kappa}_k(\tfrac{1}{2}) \geq 0 \quad &\text{for} \quad k > \mu_{spec}(\Psi). \end{aligned}$$

Hence obviously

$$\mu_{spec}(\Psi) - \mu_{spec}(\Phi) = \sharp\{k \in \mathbb{Z} \mid \tilde{\kappa}_k(1) < 0 \leq \tilde{\kappa}_k(\tfrac{1}{2})\}$$

and the other set in (2.44) is empty. If $\mu_{spec}(\Psi) \leq \mu_{spec}(\Phi)$ we get in complete analogy

$$\mu_{spec}(\Phi) - \mu_{spec}(\Psi) = \sharp\{k \in \mathbb{Z} \mid \tilde{\kappa}_k(\tfrac{1}{2}) < 0 \leq \tilde{\kappa}_k(1)\}$$

and

$$\{k \in \mathbb{Z} \mid \tilde{\kappa}_k(1) < 0 \leq \tilde{\kappa}_k(\tfrac{1}{2})\} = \emptyset.$$

Moreover, $\tilde{P}|_{[\frac{1}{2},1]}$ only is a reparametrization of P, so from the normalization of (κ_k) we deduce $\kappa_k(1) = \tilde{\kappa}_k(1)$ for all $k \in \mathbb{Z}$, which proves (2.44).

Furthermore, only a finite number out of (κ_k) can cross zero at all on $[0,1]$. This is because of the uniform bound on the growth rate of κ_k from corollary 2.32 and the fact that in any bounded interval there are only finitely many points of $\sigma(L_0) = \{\kappa_k(0) \mid k \in \mathbb{Z}\}$ (see lemma 2.31). In addition, every κ_k is composed of a finite number of holomorphic branches that equal zero either identically or at only a finite number of points. So the finitely many κ_k meeting 0 at all can cross it only finitely often. Hence when counting the flow of (κ_k) through zero as described above, the sum is well-defined and from (2.44) it is clear that we get $\mu_{spec}(\Psi) - \mu_{spec}(\Phi)$ this way.

Now just as for (κ_k), the flow of (ν_n) through zero is also well-defined and it obviously adds up to

$$\sharp\{n \in \mathbb{N} \mid \nu_n(0) \geq 0 > \nu_n(1)\} - \sharp\{n \in \mathbb{N} \mid \nu_n(1) \geq 0 > \nu_n(0)\}.$$

But (κ_k) results from (ν_n) by glueing together the holomorphic branches differently and this does not change the flow through zero. This finally finishes the proof of the assertions with respect to $\mu_{spec}(\Psi) - \mu_{spec}(\Phi)$. For the last statement of the proposition it suffices to prove $\mu_{spec}(\mathbb{1}) = -n$. To see this, simply recall from (2.28) that for $S_{\mathbb{1}} = 0$ we have the normalization $\lambda_{-n+1}(0) = \ldots = \lambda_n(0) = 0$ and hence $\lambda_{-n} = -2\pi$ because of our knowledge of $\sigma(L_0)$ from lemma 2.31. □

Using these alternative definitions of μ_{spec} and the homotopy invariance we can show now that μ_{spec} also meets the properties (i) to (iv) of definition 2.9 and hence $\mu_{CZ}(\Phi) = \mu_{spec}(\Phi)$ for all $\Phi \in \Sigma^{1,\infty}(n)$.

(i) *For any* $\Phi \in \Sigma^{1,\infty}(n), \Psi \in \Sigma^{1,\infty}(m)$ *identify* $\Phi \oplus \Psi$ *with an arc in* $\Sigma^{1,\infty}(n+m)$*, then*

$$\mu_{spec}(\Phi \oplus \Psi) = \mu_{spec}(\Phi) + \mu_{spec}(\Psi).$$

Proof: The 'obvious way' to identify $\Sigma^{1,\infty}(n) \oplus \Sigma^{1,\infty}(m)$ with $\Sigma^{1,\infty}(n+m)$ is by changing basis in the latter such that

$$J = \begin{pmatrix} J_n & 0 \\ 0 & J_m \end{pmatrix}$$

where $J_n = \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right) \in \mathbb{R}^{2n \times 2n}$ and 0 stands for $0 \in \mathbb{R}^{2n \times 2m}$ or $0 \in \mathbb{R}^{2m \times 2n}$ respectively. Then we set

$$\Omega := \Phi \oplus \Psi = \begin{pmatrix} \Phi & 0 \\ 0 & \Psi \end{pmatrix}.$$

This yields

$$\begin{aligned} S_\Omega &= -J\dot{\Omega}\Omega^{-1} \\ &= \begin{pmatrix} -J_n & 0 \\ 0 & -J_m \end{pmatrix} \begin{pmatrix} \dot{\Phi} & 0 \\ 0 & \dot{\Psi} \end{pmatrix} \begin{pmatrix} \Phi^{-1} & 0 \\ 0 & \Psi^{-1} \end{pmatrix} \\ &= \begin{pmatrix} -J_n\dot{\Phi}\Phi^{-1} & 0 \\ 0 & -J_m\dot{\Psi}\Psi^{-1} \end{pmatrix} \\ &= S_\Phi \oplus S_\Psi, \end{aligned}$$

so for the calculation of $\lambda_k(S_\Omega)$ we can use the path $P \oplus B$ with

$$P(x) = \begin{cases} 2xS_\Phi & ; x \in [0, \frac{1}{2}] \\ S_\Phi & ; x \in [\frac{1}{2}, 1] \end{cases} \quad , \quad B(x) = \begin{cases} 0 & ; x \in [0, \frac{1}{2}] \\ (2x-1)S_\Psi & ; x \in [\frac{1}{2}, 1] \end{cases} .$$

The corresponding family of asymptotic operators

$$L(x) = -J\tfrac{\mathrm{d}}{\mathrm{d}t} - (P(x) \oplus B(x))$$

obviously splits with $h = h^n \oplus h^m \in W^{1,2}([0,1], \mathbb{R}^{2n}) \oplus W^{1,2}([0,1], \mathbb{R}^{2m})$:

$$L(x)h = \left(-J\tfrac{\mathrm{d}}{\mathrm{d}t} - P(x)\right) h^n \oplus \left(-J\tfrac{\mathrm{d}}{\mathrm{d}t} - B(x)\right) h^m.$$

Thus we get $\sigma(L(x)) = \sigma(L_{P(x)}) \cup \sigma(L_{B(x)})$ for all $x \in [0,1]$, where also the degeneracies are added up. Let $(\nu_n^P)_{n \in \mathbb{N}}$ be the sequence from theorem 2.17 representing $\sigma(L_{P(\cdot)})$ and $(\nu_n^B)_{n \in \mathbb{N}}$ the analoguous sequence for $\sigma(L_{B(\cdot)})$, then the union of these two sequences represents $\sigma(L(\cdot))$ and thus from proposition 2.38 we deduce

$$\begin{aligned} \mu^{n+m}(\Phi \oplus \Psi) &= -(n+m) + \sharp\{n \in \mathbb{N} \mid \nu_n^P(0) \geq 0 > \nu_n^P(1)\} \\ &\quad + \sharp\{n \in \mathbb{N} \mid \nu_n^B(0) \geq 0 > \nu_n^B(1)\} \\ &\quad - \sharp\{n \in \mathbb{N} \mid \nu_n^P(1) \geq 0 > \nu_n^P(0)\} \\ &\quad - \sharp\{n \in \mathbb{N} \mid \nu_n^B(1) \geq 0 > \nu_n^B(0)\} \\ &= \mu^n(\Phi) + \mu^m(\Psi). \end{aligned}$$

$\square$

(ii) *For any $W^{1,\infty}$-loop g in $Sp(n)$ with $g(0) = g(1) = \mathbb{1}$ and $\Phi \in \Sigma(n)$*

$$\mu_{spec}(g\Phi) = 2m(g) + \mu_{spec}(\Phi).$$

Proof: In the following, $A \oplus B$ always means A acting on span(e_1, e_{n+1}) and B acting on span$(e_2, \ldots, e_{n-1}, e_{n+1}, \ldots, e_{2n})$. We deduce by remark 2.4 that $g \sim w\Gamma = \mathrm{e}^{2\pi wtJ_1} \oplus \mathbb{1}$ for some $w \in \mathbb{Z}$. Of course, this homotopy can be realized $W^{1,\infty}$-regular and with the endpoints fixed at $\mathbb{1}$. Moreover, as in the proof of the uniqueness of μ_{CZ} in 2.1.2, Φ can be homotoped in $\Sigma(n)$ to $\Phi^{(1)} \oplus \Phi^{(n-1)}$ (see (2.7) and (2.8)) and this can also be done with $W^{1,\infty}$-regularity. By multiplying these homotopies it follows that $g\Phi$ is homotopic to $\mathrm{e}^{2\pi wtJ_1}\Phi^{(1)} \oplus \Phi^{(n-1)}$ in $\Sigma^{1,\infty}(n)$. So because of (i) and the homotopy invariance of μ_{spec} we have

$$\begin{aligned} \mu_{spec}(g\Phi) &= \mu_{spec}(\mathrm{e}^{2\pi wtJ_1}\Phi^{(1)}) + \mu_{spec}(\Phi^{(n-1)}) \\ &= \mu_{spec}(\mathrm{e}^{2\pi wtJ_1}\Phi^{(1)}) + \mu_{spec}(\Phi) - \mu_{spec}(\Phi^{(1)}) \end{aligned} \tag{2.45}$$

On the other hand, using the properties of the Maslov index from theorem 2.5 and remark 2.6, we obtain

$$m(g) \overset{(i)}{=} m(\mathrm{e}^{2\pi wtJ_1} \oplus \mathbb{1}) \overset{(iii)}{=} m(\mathrm{e}^{2\pi wtJ_1}) + m(\mathbb{1}) = w.$$

After this and (2.45) it remains to be shown that

$$\mu_{spec}(\mathrm{e}^{2\pi wtJ_1}\Phi^{(1)}) - \mu_{spec}(\Phi^{(1)}) = 2w.$$

So — dropping the index 1 — we have to consider the symplectic arc $\mathrm{e}^{2\pi wtJ}\Phi$. The corresponding symmetric arc is

$$\begin{aligned} S &= -J(2\pi wJ\mathrm{e}^{2\pi wtJ}\Phi + \mathrm{e}^{2\pi wtJ}\dot{\Phi})\Phi^{-1}\mathrm{e}^{-2\pi wtJ} \\ &= -J2\pi wJ - J\mathrm{e}^{2\pi wtJ}\dot{\Phi}\Phi^{-1}\mathrm{e}^{-2\pi wtJ} \\ &= 2\pi w\mathbb{1} + \mathrm{e}^{2\pi wtJ}S_\Phi\mathrm{e}^{-2\pi wtJ}. \end{aligned}$$

For the family of asymptotic operators $L(x)$ we use the path $P(x) = xS$, then $\sigma(L(x))$ is given by the solutions of the eigenvalue equation

$$-J\dot{h} - x2\pi w\, h - x\,\mathrm{e}^{2\pi wtJ}S_\Phi\mathrm{e}^{-2\pi wtJ}h = \lambda(x)h.$$

Substituting $h = \mathrm{e}^{2\pi wtJ}g$ we get the following equivalence:

$$\begin{aligned} & L(x)h = \lambda(x)h \\ \Leftrightarrow\;& \mathrm{e}^{2\pi wtJ}\,(\,-J2\pi wJg - J\dot{g} - x2\pi w\, g - xS_\Phi g\,) = \lambda(x)\mathrm{e}^{2\pi wtJ}g \\ \Leftrightarrow\;& -J\dot{g} - xS_\Phi g = (\lambda(x) - (1-x)2\pi w)g \end{aligned}$$

The left hand side in the last equation is the family of asymptotic operators L_{xS_Φ} from which $\mu_{spec}(\Phi)$ can be calculated. Let $(\kappa_k^\Phi)_{k\in\mathbb{Z}}$ be the normalized, ordered sequence representing $\sigma(L_{xS_\Phi})$ and compare it to the sequence $(\kappa_k)_{k\in\mathbb{Z}}$ that represents $\sigma(L(x))$. The above equivalence of eigenvalue equations implies that $\sigma(L(x))$ results from $\sigma(L_{xS_\Phi})$ by a shift of $(1-x)2\pi w$, hence there is an $s \in \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ and $x \in [0,1]$

$$\kappa_k(x) = \kappa_{k+s}^\Phi(x) + (1-x)2\pi w. \tag{2.46}$$

We know the eigenvalues of L_0 from lemma 2.31 and from the normalization at $x = 0$ we obtain

$$\kappa_k(0) = 2\pi[\tfrac{k}{2}] = \kappa_k^\Phi(0)$$

for all $k \in \mathbb{Z}$, where $[r]$ denotes the maximal integer less or equal to $r \in \mathbb{R}$. Hence by (2.46) we have

$$2\pi[\tfrac{k}{2}] = 2\pi[\tfrac{k+s}{2}] + 2\pi w = 2\pi[\tfrac{k+s+2w}{2}].$$

Since this holds for all $k \in \mathbb{Z}$, the shift has to be $s = -2w$. So finally (2.43) with definition 2.33 yields

$$\begin{aligned}
\mu_{spec}(\mathrm{e}^{2\pi w t J}\Phi) &= \max\{k \in \mathbb{Z} \mid \kappa_k(1) = \kappa_{k-2w}^\Phi(1) < 0\} \\
&= \max\{k \in \mathbb{Z} \mid \kappa_k^\Phi(1) < 0\} + 2w \\
&= \mu_{spec}(\Phi) + 2w.
\end{aligned}$$

□

(iv) *Define* $\Phi_0 \in \Sigma^{1,\infty}(1)$ *by* $\Phi_0(t) = \mathrm{e}^{\pi t J}$ *then*

$$\mu_{spec}(\Phi_0) = 1.$$

Proof: The symmetric path attached to Φ_0 evaluates to

$$S_{\Phi_0} = -J\pi J\Phi_0(t)\Phi_0(t)^{-1} = \pi\mathbb{1}.$$

For the calculation of $\mu_{spec}(\Phi_0)$ we use the family $L(x) = -J\frac{\mathrm{d}}{\mathrm{d}t} - x\pi$ of asymptotic operators, for which obviously $\sigma(L(x))$ is identical to $\sigma(L_0)$ shifted by $-x\pi$. From lemma 2.31 we know that $\sigma(L_0) = 2\pi\mathbb{Z}$ with each eigenvalue of multiplicity 2. The normalized, ordered sequence representing $\sigma(L(x))$ is thus given by

$$\kappa_k(x) = 2\pi\left[\tfrac{k}{2}\right] - x\pi \quad ; k \in \mathbb{Z}.$$

By (2.43) this yields $\mu_{spec}(\Phi_0) = 1$ since we have $\kappa_k(1) = (2\left[\frac{k}{2}\right] - 1)\pi < 0$ if and only if $k \leq 1$. □

(iii) *For all* $\Phi \in \Sigma^{1,\infty}(n)$
$$\mu_{spec}(\Phi^{-1}) = -\mu_{spec}(\Phi).$$

Proof: As in the proof of the uniqueness of μ_{CZ} in 2.1.2, Φ can be homotoped in $\Sigma(n)$ to either
$$\Phi^+ = \mathrm{e}^{2\pi w t J_1}\,\mathrm{e}^{\pi t J_1} \oplus \bigoplus_{k=2}^{n} \mathrm{e}^{\pi t J_{(k)}}$$
by (2.7) or, by (2.8), to
$$\Phi^- = \mathrm{e}^{2\pi w t J_1}\,\mathrm{diag}(1+t,(1+t)^{-1}) \oplus \bigoplus_{k=2}^{n} \mathrm{e}^{\pi t J_{(k)}}$$
for some $w \in \mathbb{Z}$. This homotopy, of course, can also be realized with $W^{1,\infty}$-regularity, that is in $\Sigma^{1,\infty}(n)$. Furthermore, from (2.2) we know that $\Phi^{-1} = J^{-1}\Phi^T J$, so if $\Psi(s,t)$ is a homotopy in $\Sigma^{1,\infty}(n)$ connecting Φ to one of the above arcs, then $J^{-1}\Psi(s,t)J$ homotopes Φ^{-1} to the inverse of that arc. And since $\sigma(J^{-1}\Psi(s,1)J) = \sigma(\Psi(s,1)) \not\ni 1$, this also is a homotopy in $\Sigma^{1,\infty}(n)$. Because of this we only have to prove the claim for the arcs $\Phi^{\pm}$.

Let us first use the above proven properties of μ_{spec} to calculate the index as far as possible:
$$\begin{aligned}
\mu_{spec}(\Phi^+) &\overset{(i)}{=} \mu_{spec}(\mathrm{e}^{2\pi w t J_1}\,\mathrm{e}^{\pi t J_1})) + \sum_{k=2}^{n} \mu_{spec}(\mathrm{e}^{\pi t J_1}) \\
&\overset{(ii)}{=} 2m(\mathrm{e}^{2\pi w t J_1}) + \mu_{spec}(\mathrm{e}^{\pi t J_1}) + \sum_{k=2}^{n} \mu_{spec}(\mathrm{e}^{\pi t J_1}) \\
&\overset{(iv)}{=} 2w + n,
\end{aligned}$$
where in the last step we also used remark 2.6. Using (2.4) we see that
$$\begin{aligned}
\Phi^+(t)^{-1} &= \mathrm{e}^{-\pi t J_1}\,\mathrm{e}^{-2\pi w t J_1}) \oplus \sum_{k=2}^{n} \mathrm{e}^{-\pi t J_1} \\
&= \mathrm{e}^{-2\pi w t J_1}\,\mathrm{e}^{-\pi t J_1}) \oplus \sum_{k=2}^{n} \mathrm{e}^{-\pi t J_1},
\end{aligned}$$
and thus
$$\begin{aligned}
\mu_{spec}((\Phi^+)^{-1}) &\overset{(i)}{=} \mu_{spec}(\mathrm{e}^{-2\pi w t J_1}\,\mathrm{e}^{-\pi t J_1}) + \sum_{k=2}^{n} \mu_{spec}(\mathrm{e}^{-\pi t J_1}) \\
&\overset{(ii)}{=} 2m(\mathrm{e}^{-2\pi w t J_1}) + \mu_{spec}(\mathrm{e}^{-\pi t J_1}) + \sum_{k=2}^{n} \mu_{spec}(\mathrm{e}^{-\pi t J_1}) \\
&= -2w + n\,\mu_{spec}(\mathrm{e}^{-\pi t J_1}).
\end{aligned}$$

For Φ^- we obtain

$$\begin{aligned}
\mu_{spec}(\Phi^-) &\overset{(i)}{=} \mu_{spec}(\mathrm{e}^{2\pi wtJ_1}\operatorname{diag}(1+t,(1+t)^{-1})) + \sum_{k=2}^{n}\mu_{spec}(\mathrm{e}^{\pi tJ_1}) \\
&\overset{(ii)}{=} 2m(\mathrm{e}^{2\pi wtJ_1}) + \mu_{spec}(\operatorname{diag}(1+t,(1+t)^{-1})) + \sum_{k=2}^{n}\mu_{spec}(\mathrm{e}^{\pi tJ_1}) \\
&\overset{(iv)}{=} 2w + \mu_{spec}(\operatorname{diag}(1+t,(1+t)^{-1})) + n - 1
\end{aligned}$$

and

$$\begin{aligned}
\Phi^-(t)^{-1} &= \operatorname{diag}((1+t)^{-1}, 1+t)\,\mathrm{e}^{-2\pi wtJ_1} \oplus \sum_{k=2}^{n}\mu_{spec}(\mathrm{e}^{-\pi tJ_1}) \\
&= \mathrm{e}^{-2\pi wtJ_1}\Psi(t) \oplus \sum_{k=2}^{n}\mu_{spec}(\mathrm{e}^{-\pi tJ_1})
\end{aligned}$$

where

$$\Psi(t) = \mathrm{e}^{2\pi wtJ_1}\operatorname{diag}((1+t)^{-1}, 1+t)\,\mathrm{e}^{-2\pi wtJ_1},$$

thus

$$\begin{aligned}
\mu_{spec}((\Phi^-)^{-1}) &\overset{(i)}{=} \mu_{spec}(\mathrm{e}^{-2\pi wtJ_1}\,\Psi) + \sum_{k=2}^{n}\mu_{spec}(\mathrm{e}^{-\pi tJ_1}) \\
&\overset{(ii)}{=} 2m(\mathrm{e}^{-2\pi wtJ_1}) + \mu_{spec}(\Psi) + \sum_{k=2}^{n}\mu_{spec}(\mathrm{e}^{-\pi tJ_1}) \\
&= 2w + \mu_{spec}(\Psi) + (n-1)\mu_{spec}(\mathrm{e}^{-\pi tJ_1}).
\end{aligned}$$

It only remains to prove that

$$\mu_{spec}(\mathrm{e}^{-\pi tJ_1}) = -1 \tag{2.47}$$

and

$$\mu_{spec}(\operatorname{diag}(1+t,(1+t)^{-1})) = -\mu_{spec}(\Psi). \tag{2.48}$$

(2.47) follows in complete analogy to (iv): The corresponding family of asymptotic operators is $L(x) = -J\frac{\mathrm{d}}{\mathrm{d}t} + x\pi$, so $\sigma(L(x)) = \sigma(L_0) + x\pi$ and thus, because of the normalization

$$\kappa_k(x) = 2\pi\left[\tfrac{k}{2}\right] + x\pi.$$

So we have $\kappa_k(1) = (2\left[\frac{k}{2}\right]+1)\pi < 0$ if and only if $k \leq -1$ and hence (2.43) yields $\mu_{spec}(\mathrm{e}^{-\pi tJ_1}) = -1$.

For (2.48) first note that Ψ can be homotoped by

$$\mathrm{e}^{-2\pi wsJ_1}\,\Psi\,\mathrm{e}^{2\pi wsJ_1} \quad, s \in [0,1]$$

to $\mathrm{diag}((1+t)^{-1}, 1+t)) =: \Phi(t)$. This indeed is a homotopy in $\Sigma^{1,\infty}(n)$ because of $\sigma(\mathrm{e}^{-2\pi w s J_1}\Psi(1)\mathrm{e}^{-2\pi w s J_1}) = \sigma(\Psi(1)) \not\ni 1$. Since μ_{spec} is a homotopy invariant, it remains to show that $\mu_{spec}(\Phi) = -\mu_{spec}(\Phi^{-1})$. We set

$$T := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and then calculate

$$\begin{aligned} S_\Phi &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -(1+t)^{-2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+t & 0 \\ 0 & (1+t)^{-1} \end{pmatrix} \\ &= (1+t)^{-1}T. \end{aligned}$$

and

$$\begin{aligned} S_{\Phi^{-1}} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -(1+t)^{-2} \end{pmatrix} \begin{pmatrix} (1+t)^{-1} & 0 \\ 0 & 1+t \end{pmatrix} \\ &= -(1+t)^{-1}T \end{aligned}$$

We use the paths xS_Φ and $xS_{\Phi^{-1}}$ to define families $L^\Phi(x)$ and $L^{\Phi^{-1}}(x)$ of asymptotic operators. Substituting $h = Tg$ for the eigenfunction and noting that $TJT = -J$, we have the following equivalence of eigenvalue equations:

$$\begin{aligned} & L^{\Phi^{-1}}(x)h = \lambda(x)h \\ \Leftrightarrow\ & T(-JT\dot{g} + x(1+t)^{-1}T^2 g) = T\lambda(x)Tg \\ \Leftrightarrow\ & J\dot{g} + x(1+t)^{-1}Tg = \lambda(x)g \\ \Leftrightarrow\ & L^\Phi(x)g = -\lambda(x)g. \end{aligned}$$

This implies $\sigma(L^{\Phi^{-1}}(x)) = -\sigma(L^\Phi(x))$, hence for the normalized, ordered sequences $(\kappa_k^\Phi)_{k\in\mathbb{Z}}$ and $(\kappa_k^{\Phi^{-1}})_{k\in\mathbb{Z}}$ that represent these spectra, we have

$$\kappa_k^{\Phi^{-1}}(x) = -\kappa_{-k+s}^\Phi(x)$$

for all $k \in \mathbb{Z}$, $x \in [0,1]$ and some shift $s \in \mathbb{Z}$. The numbering of (κ_k^Φ) is reversed for this equality because of the ascending order of both sequences. From the normalization

$$\kappa_0^{\Phi^{-1}}(0) = \kappa_1^{\Phi^{-1}}(0) = 0 = \kappa_0^\Phi(0) = \kappa_1^\Phi(0)$$

we conclude $s = 1$ and thus

$$\begin{aligned} \mu_{spec}(\Phi^{-1}) &= \max\{k \in \mathbb{Z} \mid \kappa_k^{\Phi^{-1}}(1) < 0\} \\ &= \max\{k \in \mathbb{Z} \mid \kappa_{-k+1}^\Phi(1) > 0\} \\ &= 1 - \min\{j \in \mathbb{Z} \mid \kappa_j^\Phi(1) > 0\} \\ &= -\max\{j \in \mathbb{Z} \mid \kappa_j^\Phi(1) \le 0\}. \end{aligned}$$

Finally we remark that $\Phi(1)$ is nondegenerate, so in view of proposition 2.37 we have $\kappa_j^\Phi \neq 0$ for all $j \in \mathbb{Z}$ and hence

$$\mu_{spec}(\Phi^{-1}) = -\max\{j \in \mathbb{Z} \mid \kappa_j^\Phi(1) < 0\} = -\mu_{spec}(\Phi).$$

□

2.3 The winding interval description for $\Sigma(1)$

There is a particularly simple, geometric construction of the Conley-Zehnder index for symplectic arcs in $\Sigma(1)$, that was introduced in [15]. For this purpose we identify $\mathbb{R}^2 \cong \mathbb{C}$ in the canonical way, so J corresponds to multiplication by i. The construction then uses the fact that for a continuous arc Φ in $Sp(1)$ and any $v \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ a winding number $\Delta(\Phi v) \in \mathbb{R}$ can be defined. Indeed, $t \mapsto \Phi(t)v$ is a continuous path in $\mathbb{C}^*$, since $v \neq 0$ and $\Phi(t)$ is nondegenerate for all $t \in [0,1]$. So we can use the standard definition of a winding number for continuous paths:

$$\Delta(\Phi v) := \tfrac{1}{2\pi}(\phi(1) - \phi(0))$$

where $\phi : [0,1] \to \mathbb{R}$ is a continuous argument of Φv, i.e. for all $t \in [0,1]$

$$\Phi(t)v = |\Phi(t)v| \, \mathrm{e}^{i\phi(t)}.$$

As a first step we get a new representation of the Maslov index from the above winding number.

Proposition 2.39 ([15] Proposition 48)
The Maslov index of a loop Φ in $Sp(1)$ is given by

$$m(\Phi) = \Delta(\Phi v)$$

for any $v \in \mathbb{C}^*$.

This does make sense since for $\Phi(0) = \Phi(1)$ the winding interval

$$I(\Phi) := \{\Delta(\Phi v) \mid v \in \mathbb{C}^*\}$$

consists of only a single integer. When considering symplectic arcs Φ, we can also make use of the interval $I(\Phi)$ which will, in general, consist of more than only one point. Note that $\Delta(\Phi v)$ only depends on the real line defined by v, that is for any $r \in \mathbb{R}^*$ we have $\Delta(\Phi rv) = \Delta(\Phi v)$. Moreover, $\Delta(\Phi v)$ is continuous with respect to v. Hence the map

$$\begin{aligned} \pi_\Phi : S^1 \cong \mathbb{R}/\pi\mathbb{Z} &\to I(\Phi) \\ \alpha &\mapsto \Delta(\Phi \mathrm{e}^{i\alpha}) \end{aligned} \tag{2.49}$$

is continuous and surjective. From this it is obvious that $I(\Phi)$ actually is an interval. Furthermore, it has the following crucial property.

Lemma 2.40 ([15] Lemma 49)
For every continuous arc Φ in $Sp(1)$ we define the winding interval to be

$$I(\Phi) := \{\Delta(\Phi v) \mid v \in \mathbb{C}^*\}.$$

Then we have $|I(\Phi)| \leq \frac{1}{2}$ for the length of this interval.

As shown in [15] this can be used for the following explicit construction of the Conley-Zehnder index.

Theorem 2.41 *For any continuous arc Φ in $Sp(1)$ define*

$$\mu_{wind}(\Phi) := \begin{cases} 2w+1 & ; \mathrm{I}(\Phi) \subset (w, w+1) \quad , w \in \mathbb{Z} \\ 2w & ; \mathrm{I}(\Phi) \ni w \quad , w \in \mathbb{Z} \end{cases}, \tag{2.50}$$

then for all $\Phi \in \Sigma(1)$ we have

$$\mu_{CZ}(\Phi) = \mu_{wind}(\Phi).$$

Of course, from checking the axioms of the Conley-Zehnder index for both μ_{wind} and μ_{spec} it is clear that $\mu_{wind}(\Phi) = \mu_{spec}(\Phi)$ for all $\Phi \in \Sigma^{1,\infty}(1)$. But we want to understand how it comes about that these two very different constructions end up at the same result. For this purpose we restrict ourselfes to piecewise holomorphic $W^{2,\infty}$- arcs $\Phi \in \Sigma(1)$ and consider $\Phi_x(t) = \Phi(xt)$ for $x \in [0,1]$. Obviously, all Φ_x are $W^{1,\infty}$-arcs in $Sp(1)$, though they need not be nondegenerate. But we do not require nondegeneracy for the definition of $\mu_{wind}(\Phi_x)$ and $\mu_{spec}(\Phi_x)$ in theorem 2.41 and 2.14 respectively. Furthermore, note that $S_{\Phi_0} = 0$ and due to the required regularity

$$S_{\Phi_x} = -J\dot{\Phi}_x(\Phi_x)^{-1} \overset{(2.2)}{=} Jx\dot{\Phi}(x\cdot)J\Phi^T(x\cdot)J$$

is a piecewise holomorphic path in $\mathcal{S}(1)$ with respect to x. So for the definition of $\mu_{spec}(\Phi_x)$, we can use the family of asymptotic operators

$$L^{(x)}(z) = L(xz) := -J\tfrac{\mathrm{d}}{\mathrm{d}t} - S_{\Phi_{xz}}, \quad z \in [0,1]. \tag{2.51}$$

Let $(\kappa_k^{(x)})_{k\in\mathbb{Z}}$ be the normalized, ordered sequence representing $\sigma(L^{(x)}(\cdot))$, then obviously $\kappa_k^{(x)}(z) = \kappa_k^{(1)}(xz) =: \kappa_k(xz)$ holds for all $x, z \in [0,1]$ and $k \in \mathbb{Z}$. Hence we have for all $x \in [0,1]$

$$\mu_{spec}(\Phi_x) = \max\{k \in \mathbb{Z} \mid \kappa_k^{(x)}(1) < 0\} = \max\{k \in \mathbb{Z} \mid \kappa_k(x) < 0\}. \tag{2.52}$$

Now we can directly compare the development of $\mu_{wind}(\Phi_x)$ and $\mu_{spec}(\Phi_x)$ as x goes from 0 to 1 and we claim that for all $x \in [0,1]$ where $\Phi(x) \in Sp^*(1)$ we have

$$\mu_{spec}(\Phi_x) = \mu_{wind}(\Phi_x),$$

that is

$$\max\{k \in \mathbb{Z} \mid \kappa_k(x) < 0\} = \begin{cases} 2w+1 & ; I(\Phi_x) \subset (w, w+1) \quad , w \in \mathbb{Z} \\ 2w & ; I(\Phi_x) \ni w \quad , w \in \mathbb{Z} \end{cases}.$$

Unfortunately, the perturbation theory for $\mu_{spec}(\Phi_x)$ becomes very complicated in higher orders, so we will only be able to prove the claim for the generic case, where only the first order is needed. Before going into more detail, we mention another equivalent definition of the Conley-Zehnder index for $\Sigma(1)$. It can also be defined as intersection number of the symplectic arc with the *Maslov cycle*

$$\mathcal{C}^+ := \{\Psi \in Sp(1) \mid \sigma(\Psi) = \{1\}\} = Sp(1) \setminus Sp^*(1).$$

For the last equality recall that by lemma 2.8 the eigenvalue 1 of any symplectic matrix has even (algebraic) multiplicity. This construction is explained in full detail in [18]. Here we will reduce both the spectral flow description and the winding interval description to counting intersections with $\mathcal{C}^+$. The first step towards this is the following lemma.

Lemma 2.42 *Let Φ and Φ_x be as above and let $\Phi(x) \in Sp^*(1)$ for all x in some interval, then on this interval, $\mu_{wind}(\Phi_x)$ as well as $\mu_{spec}(\Phi_x)$ are constant.*

Proof: From the invariance of both μ_{wind} and μ_{spec} under homotopies in $\Sigma(1)$ this follows obviously. But we will give another, more intuitive proof.

From (2.52) one sees that $\mu_{spec}(\Phi_x)$ only changes when some $\kappa_k(x)$ becomes zero, therefore it has to be constant along intervals in which we have $0 \notin \sigma(-J\frac{\mathrm{d}}{\mathrm{d}t} - S_{\Phi_x})$. But by proposition 2.37 this is equivalent to $\Phi_x(1) = \Phi(x) \in \mathcal{C}^+$.

From the definition (2.50) it is clear that $\mu_{wind}(\Phi_x)$ can only change when one of the boundaries of $I(\Phi_x)$ becomes an integer. For this to be true, firstly there must be a $v \in \mathbb{C}^*$ with $\Delta(\Phi_x v) \in \mathbb{Z}$, hence $\Phi_x(1)v = \Phi(x)v = \lambda v$ for some positive eigenvalue λ of $\Phi(x)$. But this only means that some integer is contained in $I(\Phi_x)$, so secondly the integer actually has to be the boundary of the interval. We claim that this can not be the case if the above eigenvalue $\lambda \neq 1$. To prove this, we assume $\lambda \neq 1$, then λ^{-1} also is an eigenvalue of $\Phi_x(1)$ by lemma 2.8. Thus there are $\alpha \neq \beta \in [0, \pi)$ such that

$$\Phi_x(1)\mathrm{e}^{i\alpha} = \lambda \mathrm{e}^{i\alpha} \quad \text{and} \quad \Phi_x(1)\mathrm{e}^{i\beta} = \lambda^{-1}\mathrm{e}^{i\beta}.$$

Now consider $\Delta_\pm := \Delta(\Phi_x(\mathrm{e}^{i\alpha} \pm \varepsilon \mathrm{e}^{i\beta}))$ for small $\varepsilon > 0$: From

$$\Phi_x(1)(\mathrm{e}^{i\alpha} \pm \varepsilon \mathrm{e}^{i\beta}) = \mathrm{e}^{2\pi i \Delta_\pm}(\mathrm{e}^{i\alpha} \pm \varepsilon \mathrm{e}^{i\beta})$$

we deduce

$$\begin{aligned} \mathrm{e}^{2\pi i \Delta_\pm} &= (\mathrm{e}^{i\alpha} \pm \varepsilon \mathrm{e}^{i\beta})^{-1}(\lambda \mathrm{e}^{i\alpha} \pm \varepsilon \lambda^{-1} \mathrm{e}^{i\beta}) \\ &= |\mathrm{e}^{i\alpha} \pm \varepsilon \mathrm{e}^{i\beta}|^{-2}(\lambda + \varepsilon^2 \lambda^{-1} \pm \varepsilon(\lambda \mathrm{e}^{i(\alpha-\beta)} + \lambda^{-1}\mathrm{e}^{-i(\alpha-\beta)})) \end{aligned}$$

and hence

$$\sin(2\pi\Delta_\pm) = \pm\varepsilon|\mathrm{e}^{i\alpha} \pm \varepsilon \mathrm{e}^{i\beta}|^{-2}(\lambda - \lambda^{-1})\sin(\alpha - \beta).$$

Since $\alpha - \beta \notin \pi\mathbb{Z}$ and $\lambda \neq 1$ the last expression does not vanish and this finally implies

$$\sin(2\pi\Delta_+) > 0 > \sin(2\pi\Delta_-) \quad \text{or} \quad \sin(2\pi\Delta_+) < 0 < \sin(2\pi\Delta_-).$$

Because of lemma 2.40, $\Phi_x \mathrm{e}^{i\alpha}$ and $\Phi_x \mathrm{e}^{i\beta}$ must have the same integer winding number k and from the above we get for sufficiently small ε

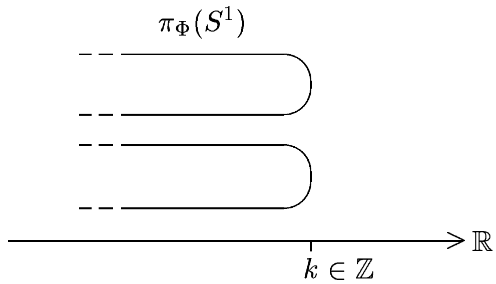
$$I(\Phi_x) \ni \Delta_+ > k \quad \text{and} \quad I(\Phi_x) \ni \Delta_- < k$$

or

$$I(\Phi_x) \ni \Delta_+ < k \quad \text{and} \quad I(\Phi_x) \ni \Delta_- > k,$$

so obviously k is not a boundary of $I(\Phi_x)$. $\square$

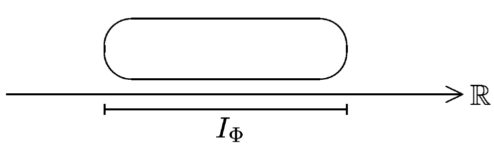
What we proved above is that for a symplectic arc Φ such that $P(1)$ has a positive eigenvalue $\lambda \neq 1$, a situation as pictured below does not occur.



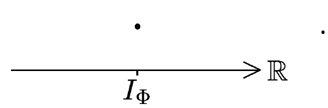
In this picture, as in all further pictures, S^1 is mapped to $\mathbb{R}^2$ with the x-component given by π_Φ from (2.49). Moreover, we can even prove for every $r \in I(\Phi)$ that $\pi_\Phi^{-1}(r)$ consists of either one or two points or is the entire S^1: Assume that we have $0 \leq \alpha_1 < \alpha_2 < \alpha_3 < \pi$ with $\pi_\Phi(\alpha_j) = \Delta(\Phi \mathrm{e}^{i\alpha_j}) = r$, then

$$\mathrm{e}^{-2\pi i r}\Phi(1)\mathrm{e}^{i\alpha_j} = \|\Phi(1)\mathrm{e}^{i\alpha_j}\|\mathrm{e}^{i\alpha_j}.$$

So there would be three eigenvectors $\mathrm{e}^{i\alpha_j}$ of $\mathrm{e}^{-2\pi i r}\Phi(1)$ with real eigenvalues, each of them defining a different real line. But since we are in dimension 2, this implies that the map considered is a multiple of the identity, and because of $\Phi(1)$ being symplectic (that is $\det(\Phi(1)) = 1$) and $\det(\mathrm{e}^{-2\pi r J_1}) = 1$ this multiple has to be 1. Thus we would get $\Phi(1) = \mathrm{e}^{2\pi i r}$ and hence $\pi_\Phi(S^1) = \{r\}$. Our picture thus has to look like either



or



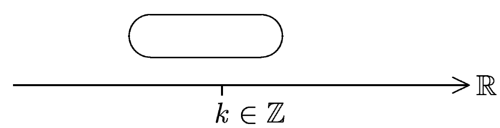
.

Note that

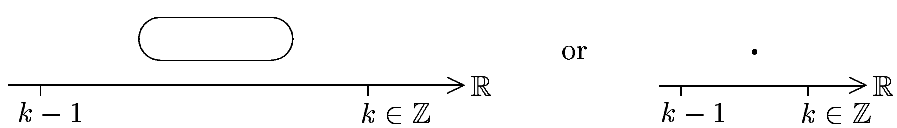
$$\begin{aligned}
Sp^+ &:= \{\Phi \in Sp(1) | \det(\Phi - \mathbb{1}) > 0\} \\
&= \{\Phi \in Sp(1) | \Phi \text{ has complex or negative, real eigenvalues}\}, \\
Sp^- &:= \{\Phi \in Sp(1) | \det(\Phi - \mathbb{1}) < 0\} \\
&= \{\Phi \in Sp(1) | \Phi \text{ has positive, real eigenvalues}\}.
\end{aligned}$$

This is since by lemma 2.8 any symplectic matrix in $Sp(1)$ can be written as $\Phi = T^{-1}\mathrm{diag}(\lambda, \lambda^{-1})T$ with $\lambda \in \mathbb{R} \cap S^1$ or as $\Phi = T^{-1}\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)T$ for $\lambda = \pm 1$ and with some $T \in Gl(2, \mathbb{R})$. Thus we have $\det(\Phi - \mathbb{1}) = (\lambda - 1)(\lambda^{-1} - 1)$ from which the above is easily seen. Knowing this, we can identify the different components of $Sp(1)$ that $\Phi(x)$ can lie in with pictures of $I(\Phi_x)$. We remark that as seen in the proof of lemma 2.42, there is an integer in $I(\Phi_x)$ if and only if $\Phi(x) \in Sp^-$. For $\Phi(x) \in Sp^+$, one can have $I(\Phi)$ consisting of just one point, i.e. the angle $\sphericalangle(h, \Phi(x)h)$ being constant w.r.t. $h \in \mathbb{R}^2$ — for $\Phi(x) = \mathrm{e}^{2\pi w J_1}$ with $w \in (0, 1)$ —, but for $\Phi(x)$ having positive eigenvalues, this can only be the case for $\Phi(x) = \mathbb{1}$. Finally, for $\Phi(x) \in \mathcal{C}^+$, the eigenvalue 1 of $\Phi(x)$ can be of multiplicity 1 or 2. In the first case, since there is only one eigenvector h with a positive eigenvalue, i.e. $\sphericalangle(h, \Phi(x)h) = 0$, the winding interval has to have a boundary at the integer $\Delta(\Phi_x h)$. In the second case, $\Phi(x) = \mathbb{1}$ and hence the winding interval consists of only one integer. Therefore, the possible pictures are the following.

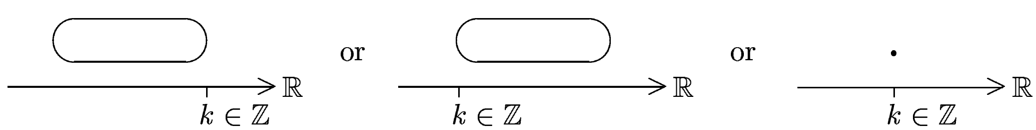
$\Phi(x) \in Sp^-$:



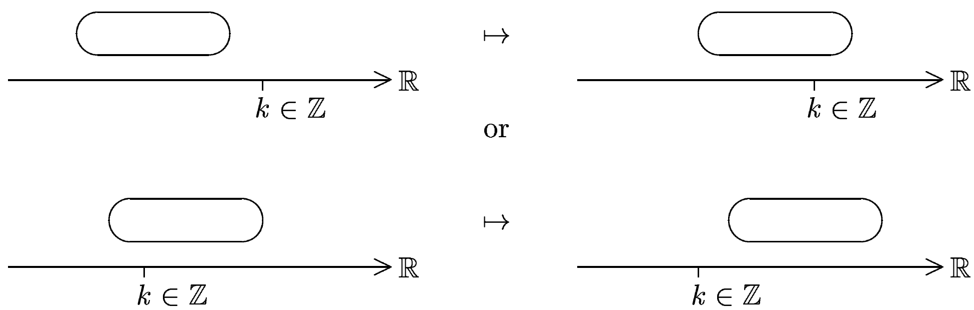
$\Phi(x) \in Sp^+$:



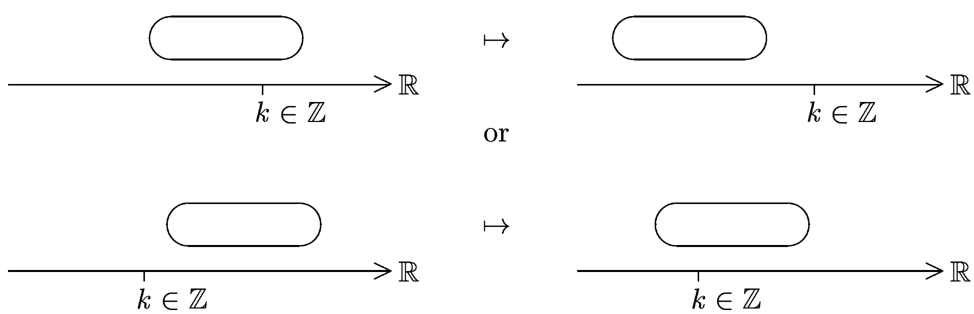
$\Phi(x) \in \mathcal{C}^+$:



In order to read off the change of $\mu_{wind}(\Phi_x)$ when $\Phi(x)$ goes through $\mathcal{C}^+$ at $x = x_0$, we have to somehow determine the direction in which the boundary of $I(\Phi)$ crosses the integer. For the first two nondegenerate cases of $\Phi(x_0) \in \mathcal{C}^+ \setminus \{\mathbb{1}\}$ there is — up to real multiples — only one eigenvector h_0 of $\Phi(x_0)$ and the change of its winding number determines the change of $I(\Phi_x)$ that we are interested in. Note that multiplication with $i \cong J$ rotates h_0 by $\frac{1}{2}\pi$, so — with $(\cdot,\cdot)$ the Euclidean product on $\mathbb{R}^2$ — we have $(Jh_0, \Phi(x_0)h_0) = 0$. If $(Jh_0, \Phi(x)h_0)$ increases or decreases with x near x_0, then so does the winding number $\Delta(\Phi_x h_0)$ since e.g. in the first case, h_0 is turned from $\Phi(x_0)h_0 = h_0$ towards Jh_0, that is in positive direction. Then if the winding number increases, $\mu_{wind}(\Phi_x)$ increases by 1 corresponding to one of the following changes of the picture



Analogously, for decreasing winding number, we have one of the following pictures and $\mu_{wind}(\Phi_x)$ decreases by 1.



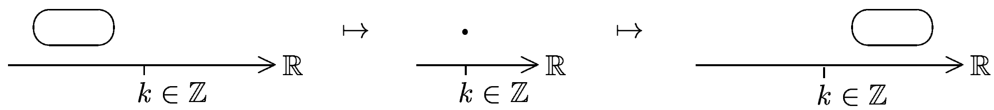
In the degenerate case $\Phi(x_0) = \mathbb{1}$ we can determine the change of $\mu_{wind}(\Phi_x)$ from the behaviour of the ends of $I(\Phi_x)$. Close to x_0, these ends are the winding numbers of h_M and h_m where $\frac{\mathrm{d}}{\mathrm{d}x}\big|_{x=x_0}(Jh, \Phi h)$ assumes its maximum M and minimum m with respect to $h \in \exp(i[0,\pi))$. This is because at $x = x_0$ we have $\sphericalangle(h, \Phi h) = 0$ for all h and the change of this angle and hence the winding number of Φh is described by

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=x_0}(Jh, \Phi h) \\ &\quad= \cos\sphericalangle(Jh, \Phi(x_0)h)\,\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=x_0}\|\Phi h\| + \|\Phi(x_0)h\|\,\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=x_0}\cos\sphericalangle(Jh, \Phi h) \\ &\quad= \frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=x_0}\cos\sphericalangle(Jh, \Phi h) \end{aligned}$$

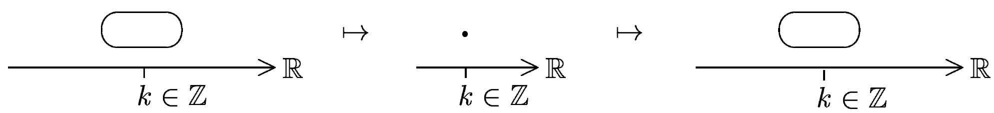
$= \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_0} \sin \sphericalangle(h, \Phi h).$

For $M, m \neq 0$, the change of $\mu_{wind}(\Phi_x)$ is then $+2$, 0 or -2 according to the following three cases.

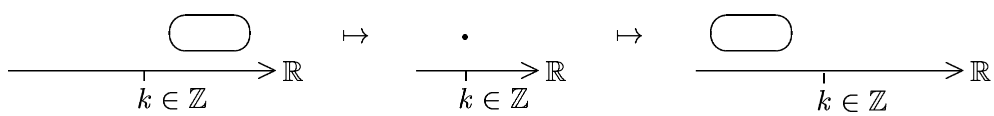
$M, m > 0$:



$M > 0 > m$:



$M, m < 0$:



Now we restrict ourselves to the generic case of Φ intersecting $\mathcal{C}^+$ transversally. Moreover, we require, that at any intersection point x_0 with $\Phi(x_0) \neq \mathbb{1}$ we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_0} (Jh_0, \Phi h_0) \neq 0$$

for the eigenvector h_0 of $\Phi(x_0)$. Finally, at intersection points $\Phi(x_0) = \mathbb{1}$ we assume that

$$\max_{h \in S^1} \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_0} (Jh, \Phi h) \neq 0, \ \min_{h \in S^1} \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_0} (Jh, \Phi h) \ \neq \ 0.$$

For $\mu_{wind}(\Phi_x)$ the intersections are then counted as follows.

- $\Phi(x_0) \neq \mathbb{1}$:
  $\mu_{wind}(\Phi_x)$ increases or decreases by 1 according to whether
  $$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_0} (Jh_0, \Phi h_0) \ > 0 \ \text{ or } \ < 0.$$
- $\Phi(x_0) = \mathbb{1}$:
  $\mu_{wind}(\Phi_x)$ increases or decreases by 2 according to whether $M, m > 0$ or $M, m < 0$ and it does not change for $M > 0 > m$.

Let us now consider $\mu_{spec}(\Phi_x)$ at a point $x_0 \in [0,1]$ where we have $\Phi(x_0) \in \mathcal{C}^+ \setminus \{\mathbb{1}\}$. From the proof of proposition 2.37 we know that for $h_0 \in \mathbb{R}^2$ with $\Phi(x_0)h_0 = h_0$ we have an eigenfunction $\Phi_{x_0}h_0$ with eigenvalue 0 of the asymptotic operator $L(x_0)$ given by (2.51). Moreover, up to real multiples this is the only eigenfunction for the eigenvalue 0. Hence there is a unique $k \in \mathbb{Z}$ such that $\kappa_k(x_0) = 0$ is the eigenvalue corresponding to the eigenfunction $t \mapsto \Phi(x_0 t)h_0$. Now from κ_k we can read off the change of $\mu_{spec}(\Phi)$: it decreases by 1 if κ_k increases near x_0, and increases by 1 if κ_k decreases near x_0. Therefore the first order perturbation of the eigenvalue can be used to decide how to count the intersection, providing it does not vanish. In order to calculate this perturbation, we need to expand S_{Φ_x} in terms of x. Since we assumed Φ to be piecewise holomorphic, we have

$$\begin{aligned}\Phi_{x+\varepsilon}(t) &= \Phi(xt+\varepsilon t)\\ &= \Phi(xt) + \varepsilon t\dot{\Phi}(xt) + \mathcal{O}(\varepsilon^2)\end{aligned}$$

and thus by (2.2)

$$\begin{aligned}\Phi_{x+\varepsilon}(t)^{-1} &= -J\Phi(xt+\varepsilon t)^T J\\ &= -J\Phi(xt)^T J - \varepsilon tJ\dot{\Phi}(xt)^T J + \mathcal{O}(\varepsilon^2)\\ &= \Phi(xt)^{-1} - \varepsilon tJ\dot{\Phi}(xt)^T J + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Using this we get the following expansion:

$$\begin{aligned}&S_{\Phi_{x+\varepsilon}}(t)\\ &= -J(\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi(xt) + \varepsilon\tfrac{\mathrm{d}}{\mathrm{d}t}(t\dot{\Phi}(xt)) + \mathcal{O}(\varepsilon^2))\ (\Phi(xt)^{-1} - \varepsilon tJ\dot{\Phi}(xt)^T J + \mathcal{O}(\varepsilon^2))\\ &= S_{\Phi_x}(t) - \varepsilon J\tfrac{\mathrm{d}}{\mathrm{d}t}(t\dot{\Phi}(xt))\Phi(xt)^{-1} + \varepsilon tJ\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi(xt)J\dot{\Phi}(xt)^T J + \mathcal{O}(\varepsilon^2)\\ &= S_{\Phi_x}(t) - \varepsilon J\tfrac{\mathrm{d}}{\mathrm{d}t}(t\dot{\Phi}(xt))\Phi(xt)^{-1} + \varepsilon txJ\dot{\Phi}(xt)J\dot{\Phi}(xt)^T J + \mathcal{O}(\varepsilon^2).\end{aligned}$$

By corollary 2.32 we can now calculate the first order perturbation of the eigenvalue $\kappa_k(x_0)$: Using that $(y, Jy) = 0$ for any $y \in \mathbb{R}^{2n}$ we obtain

$$\begin{aligned}\frac{\mathrm{d}\kappa_k}{\mathrm{d}x}(x_0) &= \int_0^1 \Big(\Phi(x_0 t)h_0\,,\, -\frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)\,\Phi(x_0 t)h_0 \Big)\ \mathrm{d}t \qquad (2.53)\\ &= \int_0^1 \Big(\Phi(x_0 t)h_0\,,\, J\tfrac{\mathrm{d}}{\mathrm{d}t}(t\dot{\Phi}(x_0 t))h_0 \Big)\\ &\qquad - \Big(\Phi(x_0 t)h_0\,,\, tx_0 J\dot{\Phi}(x_0 t)J\dot{\Phi}(x_0 t)^T J\Phi(x_0 t)h_0 \Big)\ \mathrm{d}t\\ &= \int_0^1 \tfrac{\mathrm{d}}{\mathrm{d}t}\Big(t\Big(\Phi(x_0 t)h_0\,,\, J\dot{\Phi}(x_0 t)h_0 \Big)\Big)\\ &\qquad -tx_0\Big(\dot{\Phi}(x_0 t)h_0\,,\, J\dot{\Phi}(x_0 t)h_0 \Big)\\ &\qquad +tx_0\Big(\dot{\Phi}(x_0 t)^T J\Phi(x_0 t)h_0\,,\, J\dot{\Phi}(x_0 t)^T J\Phi(x_0 t)h_0 \Big)\ \mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \left[\, t \left(\Phi(x_0 t) h_0 \,,\, J\dot{\Phi}(x_0 t) h_0 \right) \right]_{t=0}^{t=1} \\
&= \left(\Phi(x_0) h_0 \,,\, J\dot{\Phi}(x_0) h_0 \right) \\
&= -\left(J h_0, \dot{\Phi}(x_0) h_0 \right) \\
&= -\left.\tfrac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_0} (\, J h_0, \Phi h_0 \,)\,.
\end{aligned}
$$

From this we read off that in the case $\Phi(x_0) \neq \mathbb{1}$, we have the same intersection formula for $\mu_{spec}(\Phi_x)$ as for $\mu_{wind}(\Phi_x)$.

For $\Phi(x_0) = \mathbb{1}$ we are in the case of degenerate perturbation theory: the eigenspace for the eigenvalue 0 of $L(x_0)$ is

$$
E_0 = \Pi_0 W^{1,2}(S^1, \mathbb{R}^{2n}) = \{ \Phi_{x_0} h \mid h \in \mathbb{R}^2 \}
$$

and is of dimension 2, so we have to determine the change of two eigenvalues κ_k. For the first order perturbation, corollary 2.32 still holds, that is

$$
\frac{\mathrm{d}\kappa_k}{\mathrm{d}x}(x_0) = \left(\psi \,,\, -\frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)\, \psi \right)_{L^2},
$$

but we have to determine the normalized $\psi \in \Pi_0 W^{1,2}(S^1, \mathbb{R}^{2n})$ from (2.16), that is

$$
-\Pi_0 \frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)\psi = \frac{\mathrm{d}\kappa_k}{\mathrm{d}x}(x_0)\psi.
$$

Hence we have to diagonalize $-\Pi_0 \frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)$ on E_0, then the first order perturbation is given by the eigenvalues. For this we first remark that this operator is a selfadjoint operator on a 2-dimensional space (namely E_0), thus its spectrum consists of exactly two eigenvalues (or one eigenvalue with multiplicity 2). Moreover it is known from e.g. [19] XI.8 Thm.2, that

$$
\begin{aligned}
\max_{\substack{\psi \in E_0 \\ \|\psi\|=1}} \left(\psi \,,\, -\Pi_0 \frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)\, \psi \right)_{L^2} &\in \sigma\left(-\Pi_0 \tfrac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0) \right), \\
\min_{\substack{\psi \in E_0 \\ \|\psi\|=1}} \left(\psi \,,\, -\Pi_0 \frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)\, \psi \right)_{L^2} &\in \sigma\left(-\Pi_0 \tfrac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0) \right).
\end{aligned}
$$

Next we note that

$$
\begin{aligned}
&\max_{\substack{\psi \in E_0 \\ \|\psi\|=1}} \left(\psi \,,\, -\Pi_0 \frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)\, \psi \right)_{L^2} \\
&\quad = \max_{h \in S^1} \|\Phi_{x_0} h\|_{L^2}^{-2} \left(\Phi_{x_0} h \,,\, -\frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)\, \Phi_{x_0} h \right)_{L^2} \\
&\quad \overset{(2.53)}{=} \max_{h \in S^1} -\|\Phi_{x_0} h\|_{L^2}^{-2} \left.\tfrac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_0} (\, J h, \Phi h \,)
\end{aligned}
$$

has the same sign as

$$-m \;=\; \max_{h\in S^1}\; -\; \tfrac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=x_0}\,(\,Jh,\Phi h\,)\;\neq\;0.$$

Analogously,

$$\min_{\substack{\psi\in E_0\\ \|\psi\|=1}}\left(\psi\,,\,-\Pi_0\frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)\,\psi\right)_{L^2} = \min_{h\in S^1}\; -\|\Phi_{x_0}h\|_{L^2}^{-2}\,\tfrac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=x_0}\,(\,Jh,\Phi h\,)$$

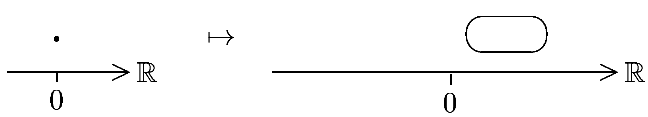
has the same sign as

$$-M \;=\; \min_{h\in S^1}\; -\; \tfrac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=x_0}\,(\,Jh,\Phi h\,)\;\neq\;0.$$

Hence we have $\sigma\left(-\Pi_0\frac{\mathrm{d}S_{\Phi_x}}{\mathrm{d}x}(x_0)\right)=\{-\tilde{M},-\tilde{m}\}$, where $\tilde{M},\tilde{m}\in\mathbb{R}^*$ have the same sign as M and m respectively. Now the first order perturbation is $\frac{\mathrm{d}\kappa_k}{\mathrm{d}x}(x_0) = -\tilde{M}$ or $-\tilde{m}$ respectively for the two eigenvalues of $L(x)$ crossing 0 at x_0. This shows that again we have the same intersection formula for $\mu_{spec}(\Phi_x)$ and $\mu_{wind}(\Phi_x)$, since e.g. for $M,m>0$ we also have $\tilde{M},\tilde{m}>0$, therefore the two eigenvalues go from $[0,\infty)$ to $(-\infty,0)$ and hence $\mu_{spec}(\Phi_x)$ increases by 2.

So we have seen that for $\mu_{wind}(\Phi_x)$ and $\mu_{spec}(\Phi_x)$ all intersections with $\mathcal{C}^+$ are counted equally. Finally we know that $\mu_{wind}(\Phi_0)=\mu_{wind}(\mathbb{1})=0$ and from proposition 2.38 we have $\mu_{spec}(\Phi_0)=-1$. This looks like $\mu_{wind}(\Phi_x)$ and $\mu_{spec}(\Phi_x)$ will always differ by 1, but their intersection with $\mathcal{C}^+$ at $x=0$ has to be considered separately since Φ only goes from $\mathcal{C}^+$ into Sp^* and does not actually cross $\mathcal{C}^+$. Also, we only claim that $\mu_{wind}(\Phi_x)=\mu_{spec}(\Phi_x)$ for $\Phi_x\in\Sigma^{1,\infty}(1)$, that is $\Phi(x)\in Sp^*(1)$ and indeed, the half of an intersection at $\mathbb{1}$ is counted differently for μ_{wind} and μ_{spec}: Define $M,m,\tilde{M},\tilde{m}$ as above, then consider the following three cases.

- $M,m>0$: $\mu_{wind}(\Phi_x)$ increases by 1 since we have the picture

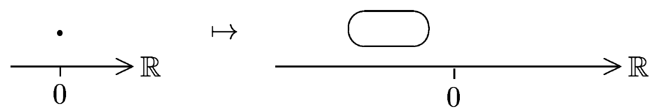


  and $\mu_{spec}(\Phi_x)$ increases by 2 since $-\tilde{M},-\tilde{m}<0$, hence two eigenvalues go from 0 to $(-\infty,0)$.

- $M>0>m$: $\mu_{wind}(\Phi_x)$ does not change since we have the picture

and $\mu_{spec}(\Phi_x)$ increases by 1 since $-\tilde{M} < 0 < -\tilde{m}$, hence one eigenvalue stays within $[0, \infty)$ and one goes from 0 to $(-\infty, 0)$.

- $M, m < 0$: $\mu_{wind}(\Phi_x)$ decreases by 1 since we have the picture



and $\mu_{spec}(\Phi_x)$ does not change since $-\tilde{M}, -\tilde{m} > 0$, hence both eigenvalues go from 0 to $[0, \infty)$.

One sees that the difference between $\mu_{wind}(\Phi_0)$ and $\mu_{spec}(\Phi_0)$ is exactly compensated by the above intersection formula for the half intersection at $\Phi_0 = \mathbb{1}$. All further intersections are full crossings of $\mathcal{C}^+$ and thus are counted as described on page 70. This is since we are only comparing the two indices where $\Phi(x) \in Sp^*$ and since $\Phi(1) \in Sp^*$. Thus we have seen in the nondegenerate case assumed here, that the winding interval construction actually defines the same index for $\Sigma^{1,\infty}(1)$ as does the construction using the spectral flow of the asymptotic operator.

Chapter 3

Special coordinates near generalized finite energy surfaces

The aim of this chapter is to construct a trivialization of the normal bundle of a finite energy surface into the manifold $\mathbb{R} \times M$. More precisely, we will prove the following theorem.

Theorem 3.1 ([12] Thm.4.7.)
Consider the immersed finite energy surface

$$\tilde{u} = (a, u) : (\dot{S}, j) \to (\mathbb{R} \times M, \tilde{J})$$

with the almost complex structure $\tilde{J}$ special or generalized. Assume the punctures are non-degenerate. Then for some neighbourhood $B_\varepsilon(0) \subset \mathbb{R}^2$ of the origin, there exists an immersion

$$\Phi : \dot{S} \times B_\varepsilon(0) \to \mathbb{R} \times M,$$

satisfying $\Phi(z, 0) = \tilde{u}(z)$ for all $z \in \dot{S}$. If $\tilde{u}$ is an embedding, then also Φ restricted to a sufficiently small neighbourhood of $\dot{S} \times \{0\}$ is an embedding.

Moreover, the almost complex structure

$$\bar{J} = T\Phi^{-1} \circ \tilde{J}(\Phi) \circ T\Phi$$

induced on $\dot{S} \times B_\varepsilon(0)$ splits along $\dot{S} \times \{0\}$ near the punctures,

$$\bar{J}(z, 0) = j(z) \oplus J_0 \ \in \mathcal{L}(T_z\dot{S} \times \mathbb{R}^2)$$

for all $z \in \dot{S}$ sufficiently close to a puncture.

Finally, introducing the cylindrical coordinates σ from (1.4) around a (positive) puncture $z_0 \in \Gamma \subset S$, the almost complex structure

$$\bar{J}_0(s, t, x, y) = (T\sigma \oplus \mathbb{1}_{\mathbb{R}^2})^{-1} \circ \bar{J}(\sigma(s, t), (x, y)) \circ (T\sigma \oplus \mathbb{1}_{\mathbb{R}^2})$$

converges as follows. For all derivatives D^α with respect to s, t, x and y we have

$$D^\alpha \Big(\bar{J}_0(s,t,x,y) - A(t,x,y)^{-1}\ \hat{J}_\infty(t,x,y)\ A(t,x,y) \Big) \overset{s\to\infty}{\longrightarrow} 0$$

uniformly on bounded sets of (t,x,y), where A and $\hat{J}_\infty$ are some smooth matrix functions.

In order to achieve the special behaviour of $\tilde{J}$ near the punctures, we will explicitly construct Φ near every puncture. Then we have to make sure that this is done in such a way that Φ can be extended to all of $\dot{S}$.

For the construction of a trivialization $\Phi_j : \dot{D}_j \times B_\varepsilon(0) \to \mathbb{R} \times M$ near a puncture $z_j \in \Gamma$ we use the special cylindrical holomorphic coordinates σ on $\dot{D}_j \subset \dot{S}$ introduced in (1.4). For simplicity we assume the puncture to be positive; negative punctures can be treated analogously.

According to the theorems 1.2 and 1.4 and since all punctures were assumed to be nondegenerate, we have $a(\sigma(s,\cdot)) \to \infty$ as $s \to \infty$, and $u(\sigma(s,\cdot))$ converging in $\mathcal{C}^\infty(S^1)$ to a nondegenerate periodic orbit of the Reeb vectorfield of M. The first convergence also implies that — after choosing D_j small enough — the almost complex structure $\tilde{J}$ has on all of $\tilde{u}(\dot{D}_j)$ the special form.

Furthermore, by the subsequent lemma, one has special coordinates in some tubular neighbourhood of the asymptotic orbit, and due to above convergence of u we can choose D_j sufficiently small such that $\tilde{u}(\dot{D}_j)$ lies within this tubular neighbourhood. That way we can use the following local coordinates of M for the construction of Φ_j.

Lemma 3.2 ([10] Lemma 2.3)
Let (M,λ) be a 3-dimensional contact manifold, and let $x(t)$ be a T-periodic solution of the corresponding Reeb vectorfield $\dot{x} = X_\lambda(x)$ on M. Let τ be the minimal period of x such that $T = k\tau$ for some positive integer k. Then there is an open neighbourhood $U \subset S^1 \times \mathbb{R}^2$ of $S^1 \times \{0\}$ and an open neighbourhood $V \subset M$ of $P = \{x(t) \mid t \in \mathbb{R}\}$ and a diffeomorphism $\psi : U \to V$ mapping $S^1 \times \{0\}$ onto P such that

$$\psi^*\lambda = f \cdot \lambda_0,$$

with $\lambda_0 = \mathrm{d}\theta + x\,\mathrm{d}y$ the canonical contact form on $S^1 \times \mathbb{R}^2$ and a positive smooth function $f : U \to \mathbb{R}$ satisfying

$$f(\theta,0,0) = \tau \qquad \textit{and} \qquad df(\theta,0,0) = 0$$

for all $\theta \in S^1$.

Let us recall the properties of these coordinates in more detail from [10]. Without loss of generality we work on the covering $\mathbb{R} \times \mathbb{R}^2$ of $S^1 \times \mathbb{R}^2$ with coordinates (θ, x, y). Then the function f is 1-periodic in the variable θ and the periodic orbit is $x(t) = (kt + d, 0, 0)$, $t \in [0, 1]$ for some $d \in \mathbb{R}$. The Reeb vector field X on M is represented by

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{f^2} \begin{pmatrix} f + x f_x \\ f_y - x f_\theta \\ -f_x \end{pmatrix}. \tag{3.1}$$

The contact plane ξ_m at $m = (\theta, x, y)$ is spanned by the vectors $(0, 1, 0)$ and $(-x, 0, 1)$ and the projection along X onto ξ is given by $\pi(v) = v - f\lambda_0(v)$.

From lemma 3.2 we also get local coordinates

$$\Psi := \mathbb{1}_{\mathbb{R}} \times \psi^{-1} \; : \; \mathbb{R} \times V \to \mathbb{R} \times U$$

of $\mathbb{R} \times M$ onto $\mathbb{R} \times S^1 \times \mathbb{R}^2$ or onto the covering $\mathbb{R}^4$. Using these and the cylindrical holomorphic coordinates σ on $\dot{S}$, we represent $\tilde{u}$ near the puncture by

$$\tilde{v} := \Psi \circ \tilde{u} \circ \sigma : \; (s_0, \infty) \times S^1 \to \mathbb{R} \times S^1 \times \mathbb{R}^2.$$

We will work on the covering $\mathbb{R}$ of S^1 and denote the coefficients of the map $\tilde{v}$ in the following way:

$$\begin{array}{rcll} \tilde{v}(s,t) & = & (a(s,t), v(s,t)) & \in \; \mathbb{R} \times \mathbb{R}^3 \\ & = & (a(s,t), \theta(s,t), z(s,t)) & \in \; \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3. \end{array}$$

According to [10], the functions a and z are 1-periodic in t, whereas θ satisfies $\theta(s, t+1) = \theta(s,t) + k$.

We also represent the almost complex structure $\tilde{J}$ in the local coordinates: it is lifted to

$$\hat{J} = T\Psi \circ \tilde{J}(\Psi^{-1}) \circ T\Psi^{-1} : \; T(\mathbb{R} \times \mathbb{R}^3) \to T(\mathbb{R} \times \mathbb{R}^3)$$

and has on $\tilde{v}((s_0, \infty) \times S^1)$ the special form

$$\hat{J}(a, m)(h, w) = (-f(m)\lambda_0(w), hX(m) + J(m)\pi w),$$

for $(h, w) \in T_a\mathbb{R} \times T_m\mathbb{R}^3 \cong T_{(a,m)}(\mathbb{R} \times \mathbb{R}^3)$. Here $J : \xi \to \xi$ is the complex multiplication compatible with $d(f\lambda_0)$, that the given almost complex sturcutre (also denoted by J) on $\xi \in TM$ is lifted to by Ψ.

The differential equation (1.6) for $\tilde{u}$ now corresponds to

$$\hat{J}(\tilde{v})\tilde{v}_s = \tilde{v}_t \tag{3.2}$$

or — using the explicit form of $\hat{J}$ —

$$\begin{array}{rcl} -f\lambda_0(v_s) & = & a_t, \\ a_s & = & f\lambda_0(v_t), \\ J(v)v_s & = & \pi v_t. \end{array} \tag{3.3}$$

In order to formulate the condition of nondegeneracy of the asymptotic orbit x in local coordinates, we have to linearize the flow ϕ_{Tt}, $t \in [0,1]$ of the Reeb vector field and project it onto the contact planes along the orbit $x(t) = (kt+d, 0, 0)$, which are the (x, y)-planes. These contact planes are equipped with the almost complex structure

$$J(t) := J(kt+d, 0, 0) : \xi_{x(t)} \to \xi_{x(t)}, \tag{3.4}$$

and the local coordinates provide a trivialization of $x^*\xi$. Therefore, in analogy to the construction in chapter 1, the linearization yields an arc $\Phi := \mathrm{d}\phi_{Tt}(x_0)|_\xi : [0,1] \to Sp(\mathbb{R}^2, J)$ of matrices $\Phi(t)$ which are symplectic with respect to $J(t)$.

Now in analogy to proposition 2.37, the orbit x is nondegenerate if and only if the boundary value problem for $z : [0,1] \to \mathbb{R}^2$,

$$\begin{cases} \dot{z}(t) &= F_\infty(t) z(t) \\ z(0) &= z(1) \end{cases}, \qquad \text{with} \quad F_\infty(t) = \dot{\Phi}(t)\Phi(t)^{-1}$$

admits only the trivial solution $z(t) \equiv 0$, or equivalently, if 0 is not included in the spectrum $\sigma(A_\infty)$ of the *asymptotic operator*

$$A_\infty = -J(t)\tfrac{\mathrm{d}}{\mathrm{d}t} + J(t)F_\infty(t) : W^{1,2}(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2).$$

In order to calculate F_∞, we first remark that due to the reparametrization to $t \in [0,1]$ we now have $\frac{\mathrm{d}}{\mathrm{d}t}\phi_{Tt}(x_0) = T \cdot X(\phi_{Tt}(x_0))$ and $\phi_{Tt}(x_0) = x(t)$, hence linearization yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\phi_{Tt}(x_0) = T \cdot DX(x(t))\, \mathrm{d}\phi_{Tt}(x_0).$$

From this we obtain

$$\begin{aligned} F_\infty(t) &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\phi_{Tt}(x_0)\right)\left(\mathrm{d}\phi_{Tt}(x_0)\right)^{-1}\big|_{\xi(x(t))} \\ &= T \cdot DX(x(t))|_{\xi(x(t))} \\ &= T \cdot DY(kt+d, 0, 0), \end{aligned} \tag{3.5}$$

where we have abbreviated $Y = (X_2, X_3)$.

The asymptotic operator is needed for the following theorem that describes the exponential convergence of $\tilde{v}$ to the asymptotic orbit.

Theorem 3.3 ([10] Thm.2.8.)
Consider a non-degenerate finite energy surface $\tilde{u} : \dot{S} \to \mathbb{R} \times M$ and assume $z_j \in \Gamma$ is a positive puncture. Near z_j, the map $\tilde{v} : \mathbb{R} \times S^1 \to \mathbb{R}^4$ defined as above, has, for large s, the following properties.

Either there exist constants $c, d \in \mathbb{R}$ such that

$$\tilde{v}(s,t) = (Ts + c, kt + d, 0, 0),$$

for all $(s,t) \in (s_0,\infty) \times S^1$*, or we have for some* $c, d \in \mathbb{R}$

$$\begin{aligned} a(s,t) &= Ts + c + \hat{a}(s,t), \\ \theta(s,t) &= kt + d + \hat{\theta}(s,t), \\ z(s,t) &= \mathrm{e}^{\int_{s_0}^{s} \mu(\rho)\mathrm{d}\rho}(e(t) + \hat{z}(s,t)). \end{aligned}$$

The functions $\hat{a}(s,\cdot)$ *and* $\hat{\theta}(s,\cdot)$ *converge to* 0 *exponentially as* $s \to \infty$ *in the following sense: for all derivatives* D^α, $\alpha \in \mathbb{N}_0 \times \mathbb{N}_0$*, with respect to* s *and* t *we have*

$$\begin{aligned} |D^\alpha \hat{a}(s,t)| &\leq M_\alpha \mathrm{e}^{-\lambda s}, \\ |D^\alpha \hat{\theta}(s,t)| &\leq M_\alpha \mathrm{e}^{-\lambda s} \end{aligned}$$

with constants M_α *and a constant* $\lambda > 0$*. Moreover,*

$$D^\alpha \hat{z}(s,t) \stackrel{s\to\infty}{\longrightarrow} 0$$

uniformly in t *for all derivatives* $\alpha \in \mathbb{N}_0 \times \mathbb{N}_0$*, and for all* $j \in \mathbb{N}_0$ *we have*

$$D^j(\mu(\rho) - \mu) \stackrel{\rho\to\infty}{\longrightarrow} 0,$$

where the number μ *is a negative eigenvalue of the asymptotic operator* A_∞ *and* $e(t) \in \mathbb{R}^2$ *is a corresponding eigenfunction.*

Actually, in [10] it is only proven that all derivatives of μ are bounded and that $\lim_{\rho\to\infty} \mu(\rho) = \mu$. But in view of the subsequent remark, this suffices in order to have all derivatives of μ converging to zero.

Remark 3.4 *Let* $\mu : (a,\infty) \to \mathbb{R}$ *with* $a \in \mathbb{R}$ *or* $a = -\infty$ *be a smooth function satisfying*

$$\mu(x) \stackrel{x\to\infty}{\longrightarrow} \mu \in \mathbb{R}$$

and

$$\left| \frac{\mathrm{d}^k \mu}{\mathrm{d}x^k}(x) \right| \leq C_k$$

for all $x \in (a,\infty)$*,* $k \in \mathbb{N}$ *and some finite constants* C_k*. Then it follows that*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}[\mu(x) - \mu] \stackrel{x\to\infty}{\longrightarrow} 0, \tag{3.6}$$

for all $k \in \mathbb{N}_0$*.*

Proof: Arguing by contradiction, we assume that there exists an $m \in \mathbb{N}$ such that (3.6) holds for $k = m-1$ but does not hold for $k = m$. (Note that (3.6) holds for $k = 0$ by assumption.) Hence there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset (a+1,\infty)$ with $x_n \stackrel{n\to\infty}{\longrightarrow} \infty$ and some $d > 0$ such that we have $|\mu^{(m)}(x_n)| \geq d$ for all $n \in \mathbb{N}$. Since $\mu^{(m+1)}$ is uniformly bounded, we find some $\varepsilon \in (0,1)$ such that $|\mu^{(m)}(x)| \geq \frac{1}{2}d$ holds for all $x \in \bigcup_{n\in\mathbb{N}}[a_n, b_n]$, where $a_n : s = x_n - \varepsilon, \beta_n := x_n + \varepsilon \in (a, \infty)$. Therefore, $\mu^{(m)}$ can not change its sign on any of the intervals $[a_n, b_n]$, and thus we have for all $n \in \mathbb{N}$

$$\begin{aligned}\left|\mu^{(m-1)}(b_n) - \mu^{(m-1)}(a_n)\right| &= \left|\int_{a_n}^{b_n} \mu^{(m)}(x)\,\mathrm{d}x\right| \\ &= \int_{a_n}^{b_n} |\mu^{(m)}(x)|\,\mathrm{d}x \\ &\geq \varepsilon \cdot d.\end{aligned}$$

On the other hand, (3.6) for $k = m-1$ implies that

$$\begin{aligned}0 < \varepsilon \cdot d &\leq \left|\mu^{(m-1)}(b_n) - \mu^{(m-1)}(a_n)\right| \\ &\leq \left|\left.\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\right|_{b_n}(\mu(x) - \mu)\right| + \left|\left.\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\right|_{a_n}(\mu(x) - \mu)\right| \overset{x\to\infty}{\longrightarrow} 0,\end{aligned}$$

a contradiction. $\square$

In both cases of theorem 3.3, the periodic orbit is obviously given by $x(t) = (kt + d, 0, 0)$.

The first alternative of the theorem is fully characterized since we have the following proposition for pseudoholomorphic curves, i.e. solutions of (1.3), in analogy to the theory of (classical) holomorphic maps.

Proposition 3.5 *If two pseudoholomorphic curves are defined on a connected surface, then they are identical if and only if they are identical on a subset which contains a cluster point.*

For a proof see §5 in [9].

This propositon implies that in the case of $\tilde{v}(s,t) = (Ts + c, kt + d, 0, 0)$ on the whole neighbourhood of a puncture, the finite energy surface $\tilde{u}$ is actually a part of the finite energy cylinder

$$\begin{aligned}\mathbb{R} \times S^1 &\to \mathbb{R} \times M \\ (s,t) &\mapsto (Ts + c, x(t))\end{aligned}$$

over a periodic orbit x. This very special case is of no interest in the theory and thus is left out of consideration in the following.

In the second alternative, we have $\pi T u(z) \neq 0$ for z sufficiently close to z_j, so we can choose D_j even smaller such that we have $\pi T u(z) \neq 0$ on all of $\dot{D}_j$. Using this we can define an explicit basis of the normal bundle of $\tilde{u}(\dot{D}_j)$ in $\mathbb{R} \times M$. In the case of $\tilde{u}$ only being an immersion, $\tilde{u}(\dot{D}_j)$ might not be a submanifold of $\mathbb{R} \times M$ and hence the normal bundle might not be well-defined. But for any $z \in \dot{D}_j$ we can find a neighbourhood $\mathcal{U} \subset \dot{D}_j$ on which $\tilde{u}$ is an embedding and thus a normal bundle of $\tilde{u}(\mathcal{U})$ exists, for which we can ask for a basis. We make this more precise by the following lemma, for which we recall from (1.2) the $\tilde{J}$-invariant metric $\langle\cdot,\cdot\rangle_J$ on $\mathbb{R}\times M$ and its associated norm $|\cdot|_J$.

Lemma 3.6 *Assume that z_j is a (positive) puncture, for which the second alternative of theorem 3.3 holds. Then in a sufficiently small punctured neighbourhood $\dot{D}_j$ of z_j we can define the following vectors in $T_{\tilde{u}(z)}(\mathbb{R}\times M) \cong \mathbb{R}\times T_{u(z)}M$,*

$$
\begin{aligned}
n(z) &= \frac{1}{|\tilde{u}_s|_J}\left(|\pi u_s|_J\,,\ -\lambda(u_t)\frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s)\frac{\pi u_t}{|\pi u_t|_J}\right),\\
m(z) &= \tilde{J}(\tilde{u}(z))\, n(z),
\end{aligned}
$$

such that for all $z \in \dot{D}_j$ we have

(i) $|n(z)|_J = |m(z)|_J = 1$,

(ii) $\langle n(z), m(z)\rangle_J = 0$,

(iii) $n(z), m(z) \in (T_{\tilde{u}(z)}\tilde{u}(\mathcal{U}))^{\perp} \subset T_{\tilde{u}(z)}(\mathbb{R}\times M)$ *for some neighbourhood* $\mathcal{U}\subset \dot{D}_j$ *of z, on which $\tilde{u}$ is an embedding.*

Moreover, let $|\cdot| = |\Psi^*\cdot|_J$ and

$$
\hat{e}(t) := \frac{e(t)}{|e(t)|}\,, \qquad \hat{f}(t) := J(t)\hat{e}(t),
$$

where $e(t) \in \xi_{x(t)} \equiv \mathbb{R}^2$ is the nonzero eigenfunction of the asymptotic operator from theorem 3.3 and $J(t)$ is given by (3.4). Then for the normal vectors in local coordinates,

$$
\begin{aligned}
n(s,t) &:= T_{\tilde{u}(\sigma(s,t))}\Psi\; n(\sigma(s,t)),\\
m(s,t) &:= T_{\tilde{u}(\sigma(s,t))}\Psi\; m(\sigma(s,t)) \;=\; \hat{J}(\tilde{v}(s,t))\, n(s,t),
\end{aligned}
$$

we have

$$
D^{\alpha}\left(n(s,t) - (0,0,\hat{e}(t))\right) \overset{s\to\infty}{\longrightarrow} 0
$$

and

$$
D^{\alpha}\left(m(s,t) - (0,0,\hat{f}(t))\right) \overset{s\to\infty}{\longrightarrow} 0
$$

uniformly in t for all derivatives D^{α} with respect to s and t.

Proof: The metric $\langle\cdot,\cdot\rangle_J$ on $\mathbb{R}\times M$ induces a metric in local coordinates,

$$
\begin{aligned}
\langle (a,v),(b,w)\rangle &= \langle \Psi^*(a,v), \Psi^*(b,w)\rangle_J\\
&= a\cdot b + \psi^*\lambda(v)\cdot\psi^*\lambda(w) + \psi^*\mathrm{d}\lambda(\pi v, (\psi^*J)\pi w)\\
&= a\cdot b + f^2\cdot\lambda_0(v)\cdot\lambda_0(w) + f\cdot \mathrm{d}\lambda_0(\pi v, J\pi w)
\end{aligned}
$$

for $(a,v),(b,w) \in T(\mathbb{R}\times U) \subset T(\mathbb{R}^4)$, where we have used the fact that $\xi = \ker(f\lambda_0) = \ker(\lambda_0)$ and hence

$$\psi^*\mathrm{d}\lambda|_\xi = \mathrm{d}\psi^*\lambda|_\xi = \mathrm{d}(f\lambda_0)|_\xi = f\cdot \mathrm{d}\lambda_0|_\xi + \mathrm{d}f\wedge\lambda_0|_\xi = f\cdot\mathrm{d}\lambda_0|_\xi$$

and we write J for ψ^*J as before. The associated norm on $\mathbb{R}^4$ is denoted by $|\cdot|$. Moreover, in the following we will denote by $B(s,t)$ various functions having all partial derivatives uniformly bounded and by $R(s,t)$ various functions which satisfy $D^\alpha R(s,t) \overset{s\to\infty}{\longrightarrow} 0$ uniformly in t.

Abbreviating by $z=(x,y)$ the $\mathbb{R}^2$-components of $v(s,t)$ and using theorem 3.3 and the fact that all partial derivatives of the function f from lemma 3.2 are uniformly bounded, we obtain as a first step

$$\begin{aligned}
f(v(s,t)) &= f(\theta,0) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\rho} f(\theta,\rho z)\,\mathrm{d}\rho \\
&= f(\theta,0) + \left[\int_0^1 Df(\theta,\rho z)\,\mathrm{d}\rho\right]\cdot z \\
&= \tau + \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}\left[\int_0^1 Df(\theta,\rho z)\,\mathrm{d}\rho\right]\cdot[e(t)+\hat{z}(s,t)] \\
&= \tau + \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t), \qquad (3.7)
\end{aligned}$$

where we have moreover used the fact that e is a smooth function on S^1, hence $e(t) = B(s,t)$. Analogously, we calculate for the Reeb vectorfield that is represented in (3.1) by partial derivatives of f,

$$\begin{aligned}
X_1(v(s,t)) &= X_1(\theta,0) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\rho} X_1(\theta,\rho z)\,\mathrm{d}\rho \\
&= \frac{1}{\tau} + \left[\int_0^1 DX_1(\theta,\rho z)\,\mathrm{d}\rho\right]\cdot z \\
&= \frac{1}{\tau} + \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}\left[\int_0^1 DX_1(\theta,\rho z)\mathrm{d}\rho\right]\cdot[e(t)+\hat{z}(s,t)] \\
&= \frac{1}{\tau} + \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t). \qquad (3.8)
\end{aligned}$$

For the component Y of the Reeb vector field, we have to take into account its connection (3.5) with the asymptotic operator. For this purpose we introduce

$$F(s,t) := \int_0^1 DY(\theta(s,t),\rho z(s,t))\,\mathrm{d}\rho$$

and remark that due to theorem 3.3 and the representation (3.1) of Y by partial derivatives of f, we have

$$F(s,t) = \int_0^1 DY(kt+d,0)\,\mathrm{d}\rho \; + R(s,t) \;=\; \tfrac{1}{T}F_\infty(t) + R(s,t).$$

Thus we obtain

$$\begin{aligned}
Y(v(s,t)) &= Y(\theta,0) + \left[\int_0^1 DY(\theta,\rho z)\mathrm{d}\rho\right]\cdot z \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} F(s,t)[e(t)+\hat{z}(s,t)] \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} \left[\tfrac{1}{T}F_\infty(t)e(t)+R(s,t)\right] \qquad (3.9)\\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t). \qquad (3.10)
\end{aligned}$$

Now we consider the projections of v_s and v_t onto the contact planes, that are explicitly given by

$$\begin{aligned}
\pi v_s &= v_s - f(v)\lambda_0(v_s)X(v) \\
&= v_s + a_t X \\
&= (\theta_s + a_t X_1, z_s + a_t Y)
\end{aligned}$$

and

$$\begin{aligned}
\pi v_t &= v_t - f(v)\lambda_0(v_t)X(v) \\
&= v_t - a_s X \\
&= (\theta_t - a_s X_1, z_t - a_s Y),
\end{aligned}$$

where we have used (3.3). Recalling $\mu(s)=\mu+R(s,t)$ and $a=Ts+c+R(s,t)$ from theorem 3.3, we calculate for the z-component of v_s

$$\begin{aligned}
z_s + a_t Y &\overset{(3.10)}{=} \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\mu(s)(e(t)+\hat{z}(s,t))+\hat{z}_s(s,t)] + a_t \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t) \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\mu\, e(t)+R(s,t)], \qquad (3.11)
\end{aligned}$$

and for the z- component of v_t we obtain

$$\begin{aligned}
z_t - a_s Y &\overset{(3.9)}{=} \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\dot{e}(t)+\hat{z}_t(s,t)] \\
&\quad -[T+R(s,t)]\mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\tfrac{1}{T}F_\infty(t)e(t)+R(s,t)] \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\dot{e}(t)-F_\infty(t)e(t)+R(s,t)] \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[J(t)\mu\, e(t)+R(s,t)]. \qquad (3.12)
\end{aligned}$$

Here we have used in the last step $J(t)^2 = -\mathbb{1}$ and the fact that e is the eigenfunction of A_∞ corresponding to the eigenvalue μ, i.e. we have

$$\mu\, e(t) = -J(t)[\dot{e}(t) - F_\infty(t)e(t)]. \qquad (3.13)$$

Noting that for $x \in \xi$ the norm simplifies to $|x|^2 = f\cdot \mathrm{d}\lambda_0(x,Jx)$ and that $\mathrm{d}\lambda_0 = \mathrm{d}x \wedge \mathrm{d}y$ only acts on the z-components, we now calculate using (3.11) and (3.12)

$$
\begin{aligned}
|\pi v_s|^2 &= f(v)\cdot \mathrm{d}\lambda_0(\pi v_s, J(v)\pi v_s) \\
&\overset{(3.3)}{=} f(v)\cdot \mathrm{d}\lambda_0(\pi v_s, \pi v_t) \\
&= f(v)\cdot \mathrm{d}\lambda_0(z_s + a_t Y, z_t - a_s Y) \\
&= f(v)\cdot \mathrm{e}^{2\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}[\,\mathrm{d}\lambda_0(\,\mu\, e(t), J(t)\mu\, e(t)\,) + R(s,t)\,] \\
&= \mathrm{e}^{2\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}[\mu^2|e(t)|^2 + R(s,t)],
\end{aligned}
$$

which implies in view of $\mu < 0$

$$
|\pi v_s| = \mathrm{e}^{\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}(-\mu)|e(t)|[1 + R(s,t)] = R(s,t). \tag{3.14}
$$

Here we also used $e(t) = B(s,t)$. Moreover, from the differential equation (3.13) it is clear that either $|e(t)| \neq 0$ for all $t \in S^1$ or $e \equiv 0$, in which case e would not be an eigenvector of A_∞. From this we see that $\pi v_s \neq 0$ for s sufficiently large, and hence also $\pi u_s \neq 0$ and $\pi u_t = J\pi u_s \neq 0$, so that n and m can be defined as stated.

In order to check (i) to (iii), we first remark that because of the $\tilde{J}$-invariance of the metric we have for all $v, w \in T(\mathbb{R}\times M)$

$$
\langle v, \tilde{J}w\rangle_J = \langle \tilde{J}v, \tilde{J}^2 w\rangle_J = \langle -\tilde{J}v, w\rangle_J
$$

and thus $\tilde{J}^{ad} = -\tilde{J}$ for the adjoint with respect to this metric. Now (ii) simply follows from

$$
\langle n(z), m(z)\rangle_J = \langle n(z), \tilde{J}n(z)\rangle_J = \langle -\tilde{J}n(z), n(z)\rangle_J = -\langle n(z), m(z)\rangle_J\,.
$$

(i) is easily checked for n by the subsequent calculation and then follows for m since the metric is $\tilde{J}$-invariant.

$$
\begin{aligned}
&|n(z)|_J^2 \\
&= \frac{1}{|\tilde{u}_s|_J^2}\Bigg[|\pi u_s|_J^2 + \mathrm{d}\lambda\Bigg(-\lambda(u_t)\frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s)\frac{\pi u_t}{|\pi u_t|_J}\,, \\
&\qquad\qquad J(u)\left\{-\lambda(u_t)\frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s)\frac{\pi u_t}{|\pi u_t|_J}\right\}\Bigg)\Bigg] \\
&\overset{(1.7)}{=} \frac{1}{|\tilde{u}_s|_J^2}\Bigg[|\pi u_s|_J^2 + \mathrm{d}\lambda\Bigg(-a_s\frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s)\frac{\pi u_t}{|\pi u_t|_J}\,, \\
&\qquad\qquad -a_s\frac{\pi u_t}{|\pi u_t|_J} + \lambda(u_s)\frac{-\pi u_s}{|\pi u_t|_J}\Bigg)\Bigg] \\
&= \frac{1}{|\tilde{u}_s|_J^2}\Bigg[|\pi u_s|_J^2 + a_s^2\frac{1}{|\pi u_s|_J^2}\mathrm{d}\lambda(\pi u_s, \pi u_t) + \lambda(u_s)^2\frac{1}{|\pi u_t|_J^2}\mathrm{d}\lambda(\pi u_t, -\pi u_s)\Bigg] \\
&\overset{(1.7)}{=} \frac{1}{|\tilde{u}_s|_J^2}\Bigg[|\pi u_s|_J^2 + a_s^2\frac{1}{|\pi u_s|_J^2}\mathrm{d}\lambda(\pi u_s, J(u)\pi u_s)
\end{aligned}
$$

$$+\lambda(u_s)^2 \frac{1}{|\pi u_t|_J^2} \mathrm{d}\lambda(\pi u_t, J(u)\pi u_t)\Big]$$

$$= \frac{1}{|\tilde{u}_s|_J^2} \left[|\pi u_s|_J^2 + a_s^2 + \lambda(u_s)^2\right] = 1.$$

For (iii) we calculate

$$\begin{aligned}
&\langle n(z), \tilde{u}_s(z)\rangle \\
&= \frac{1}{|\tilde{u}_s|_J} \left[|\pi u_s|_J \cdot a_s + \mathrm{d}\lambda\left(-\lambda(u_t)\frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s)\frac{\pi u_t}{|\pi u_t|_J}\, ,\, J(u)\pi u_s\right)\right] \\
&\overset{(1.7)}{=} \frac{1}{|\tilde{u}_s|_J} \left[|\pi u_s|_J \cdot \lambda(u_t) - \frac{\lambda(u_t)}{|\pi u_s|_J} \mathrm{d}\lambda\left(\pi u_s, J(u)\pi u_s\right)\right] = 0
\end{aligned}$$

and we analogously obtain $\langle n(z), \tilde{u}_t(z)\rangle = 0$, hence $n(z) \in (T_{\tilde{u}(z)}\tilde{u}(\mathcal{U}))^{\perp}$. Now the statement for $m(z)$ follows simply from the fact that $T_{\tilde{u}(z)}\tilde{u}(\mathcal{U})$ is $\tilde{J}$-invariant (since $\tilde{J}\tilde{u}_s = \tilde{u}_t$ holds by (1.6)) — and thus so is its orthogonal complement, i.e. for all $w \in T_{\tilde{u}(z)}\tilde{u}(\mathcal{U})$ we have

$$\langle m(z), w\rangle_J = \langle \tilde{J}(v(z))n(z), w\rangle = -\langle n(z), \tilde{J}(v(z))w\rangle = 0$$

and hence $m(z) = \tilde{J}(v(z))n(z) \in N_{\tilde{u}(z)}\tilde{u}(\mathcal{U})$.

Finally, we will prove the convergence of

$$n(s,t) \overset{(3.3)}{=} \frac{1}{|\tilde{v}_s|}\left(|\pi v_s|\, ,\, -a_s\frac{\pi v_s}{|\pi v_s|} - a_t\frac{\pi v_t}{|\pi v_t|}\right).$$

We start by noting that

$$|\tilde{v}_s| = T[1 + R(s,t)] \tag{3.15}$$

follows from theorem 3.3 and (3.14)

$$\begin{aligned}
|\tilde{v}_s|^2 &= a_s^2 + \lambda(v_s)^2 + |\pi v_s|^2 \\
&\overset{(3.3)}{=} a_s^2 + a_t^2 + |\pi v_s|^2 \\
&= T^2 + R(s,t).
\end{aligned}$$

Using again (3.14) we thus obtain for the $\mathbb{R}$-component of $n(s,t)$

$$\frac{|\pi v_s|}{|\tilde{v}_s|} = \frac{\mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}(-\mu)|e(t)|[1 + R(s,t)]}{T[1 + R(s,t)]} = \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t) = R(s,t),$$

as claimed. In order to show the convergence for the M-component of $n(s,t)$, we first calculate

$$\begin{aligned}
a_t(s,t) &\overset{(3.3)}{=} -f(v)\lambda_0(v_s) \\
&\overset{(3.7)}{=} -\tau(\theta_s + xy_s) - \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t)(\theta_s + xy_s) \\
&= -\tau\theta_s + \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} R(s,t) \;=\; R(s,t),
\end{aligned}$$

and from this obtain for the θ-component of πv_s

$$\begin{aligned}\theta_s + a_t X_1 &\overset{(3.8)}{=} \theta_s + \frac{a_t}{\tau} + a_t \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t) \\ &= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} R(s,t). \qquad (3.16)\end{aligned}$$

From (3.11), (3.14) and (3.16) we now obtain with respect to the splitting into the θ- and z-component

$$\begin{aligned}\frac{\pi v_s}{|\pi v_s|} &= \left(\frac{R(s,t)}{(-\mu)|e(t)|[1+R(s,t)]}\,,\ \frac{\mu\, e(t) + R(s,t)}{(-\mu)|e(t)|[1+R(s,t)]}\right) \\ &= (R(s,t)\,,\ -\hat{e}(t) + R(s,t)) \\ &= -\hat{e}(t) + R(s,t) \qquad (3.17) \\ &= B(s,t). \qquad (3.18)\end{aligned}$$

Due to (3.15) and theorem 3.3 we have

$$\frac{-a_s}{|\tilde{v}_s|} = \frac{-T + R(s,t)}{T[1+R(s,t)]} = -1 + R(s,t),$$

and together with (3.17) we infer

$$\frac{-a_s}{|\tilde{v}_s|}\frac{\pi v_s}{|\pi v_s|} = \hat{e}(t) + R(s,t)$$

for the first part of the M-component of $n(s,t)$.

Furthermore, we remark that $\hat{J}$ is a smooth function of (θ, z) and we thus have in view of theorem 3.3

$$\hat{J}(\tilde{v}(s,t)) = \hat{J}(kt+d,0) + R(s,t) \qquad (3.19)$$

mapping $(b,w) \in T_{\tilde{v}(s,t)}(\mathbb{R}\times\mathbb{R}^3)$ to

$$\begin{aligned}&\Big(-f(kt+d,0)\cdot\lambda_0(w)\,,\ b\cdot X(kt+d,0) + J(kt+d,0)\,\pi w\Big) + R(s,t) \\ =\ &\Big(-\tau\cdot\lambda_0(w)\,,\ b\cdot(\tau^{-1},0) + J(t)\,\pi w\Big) + R(s,t).\end{aligned}$$

Considering the second part, $-a_t\frac{\pi v_t}{|\pi v_t|}$, of the M-component of $n(s,t)$, we first remark that due to (3.3) and the $\hat{J}$-invariance of the norm we have $|\pi v_t| = |\hat{J}(v)\pi v_s| = |\pi v_s|$. Thus, using (3.15), (3.18) and (3.19), we infer

$$\begin{aligned}\frac{-a_t}{|\tilde{v}_s|}\frac{\pi v_t}{|\pi v_t|} &= \frac{R(s,t)}{T[1+R(s,t)]}(\hat{J}(kt+d,0) + R(s,t))|_\xi\, B(s,t) \\ &= R(s,t)[J(t)B(s,t) + R(s,t)] \\ &= R(s,t),\end{aligned}$$

so alltogether we have with respect to the $\mathbb{R} \times \mathbb{R}^3$-splitting

$$n(s,t) = (R(s,t), \hat{e}(t) + R(s,t))$$

as claimed. The convergence of $m(s,t)$ now follows easily from (3.19):

$$\begin{aligned}
m(s,t) &= \hat{J}(\tilde{v}(s,t))n(s,t) \\
&= [\hat{J}(kt+d,0) + R(s,t)]\ (\ R(s,t)\ ,\ \hat{e}(t) + R(s,t)\ ) \\
&= (\ -\tau \cdot \lambda_0(R(s,t))\ ,\ R(s,t)\cdot(\tau^{-1},0) + J(t)[\hat{e}(t) + R(s,t)]\ ) \\
&= (\ R(s,t)\ ,\ J(t)\hat{e}(t) + R(s,t)\ ) \\
&= (\ R(s,t)\ ,\ \hat{f}(t) + R(s,t)\ )\ .
\end{aligned}$$

□

Having established this lemma, we can now — as a start for the construction of Φ_j — define an immersion $\Gamma : (s_0,\infty) \times S^1 \times B_\varepsilon(0) \to \mathbb{R}^4$ for some $\varepsilon > 0$ by

$$\Gamma(s,t,x,y) = \tilde{v}(s,t) + x\,\tilde{n}(s,t) + y\,\tilde{m}(s,t),$$

where

$$\begin{aligned}
\tilde{n}(s,t) &= \mathrm{e}^{wt\hat{J}(\tilde{v}(s,t))}n(s,t) \\
&= \cos(wt)\,n(s,t) + \sin(wt)\,m(s,t), \\
\tilde{m}(s,t) &= \mathrm{e}^{wt\hat{J}(\tilde{v}(s,t))}m(s,t) \\
&= -\sin(wt)\,n(s,t) + \cos(wt)\,m(s,t).
\end{aligned}$$

In order to see that this is indeed an immersion, we first calculate the tangent map $T_p\Gamma$ abbreviating $p = (s,t,x,y)$. It has the following columns:

$$T_p\Gamma = (\ \tilde{v}_s + x\tilde{n}_s + y\tilde{m}_s \mid \tilde{v}_t + x\tilde{n}_t + y\tilde{m}_t \mid \tilde{n} \mid \tilde{m}\ ), \tag{3.20}$$

where all functions are evaluated at the point (s,t). From theorem 3.3 and the previous lemma we know that

$$\begin{aligned}
\tilde{v}_s(s,t) &\overset{s\to\infty}{\longrightarrow} (T,0,0) \\
\tilde{v}_t(s,t) &\overset{s\to\infty}{\longrightarrow} (0,k,0) \\
\tilde{n}(s,t) &\overset{s\to\infty}{\longrightarrow} (0,0,\cos(wt)\hat{e}(t) + \sin(wt)J(t)\hat{e}(t)) \\
\tilde{m}(s,t) &\overset{s\to\infty}{\longrightarrow} (0,0,-\sin(wt)\hat{e}(t) + \cos(wt)J(t)\hat{e}(t))
\end{aligned} \tag{3.21}$$

uniformly in t, thus for $x = y = 0$, the columns of $T_p\Gamma$ converge to linearly independent vectors as $s \to \infty$. Hence, if we choose s_0 sufficiently large, then on $(s_0,\infty) \times S^1 \times \{0\}$ we have $\det(T_p\Gamma)$ bounded away from zero. Moreover,

$\tilde{n}_s, \tilde{m}_s, \tilde{n}_t$ and $\tilde{m}_t$ are all uniformly bounded on $(s_0, \infty) \times S^1$, since we have the following uniform convergences:

$$
\begin{aligned}
\tilde{n}_s(s,t) &= \cos(wt) n_s(s,t) + \sin(wt) m_s(s,t) \overset{s\to\infty}{\longrightarrow} 0, \\
\tilde{m}_s(s,t) &= -\sin(wt) n_s(s,t) + \cos(wt) m_s(s,t) \overset{s\to\infty}{\longrightarrow} 0, \\
\tilde{n}_t(s,t) &= -\sin(wt) n(s,t) + \cos(wt) m(s,t) \\
&\quad + \cos(wt) n_t(s,t) + \sin(wt) m_t(s,t) \\
&\overset{s\to\infty}{\longrightarrow} -\sin(wt)\hat{e}(t) + \cos(wt)\hat{f}(t) \\
&\quad + \cos(wt)\hat{e}'(t) + \sin(wt)\hat{f}'(t), \\
\tilde{m}_t(s,t) &= -\cos(wt) n(s,t) - \sin(wt) m(s,t) \\
&\quad - \sin(wt) n_t(s,t) + \cos(wt) m_t(s,t) \\
&\overset{s\to\infty}{\longrightarrow} -\cos(wt)\hat{e}(t) - \sin(wt)\hat{f}(t) \\
&\quad - \sin(wt)\hat{e}'(t) + \cos(wt)\hat{f}'(t).
\end{aligned}
\tag{3.22}
$$

Therefore the perturbation of $\det(T_p\Gamma)$ by $(x,y) \in B_\varepsilon(0)$ is bounded on all of $(s_0, \infty) \times S^1$, so if we choose $\varepsilon > 0$ sufficiently small, then Γ indeed is an immersion.

Finally, since σ and Ψ are diffeomorphisms, we can use them to lift Γ to the required immersion

$$\Phi_j := \Psi^{-1} \circ \Gamma \circ (\sigma^{-1} \times \mathbb{1}_{\mathbb{R}^2}) \;:\; \dot{D}_j \times B_\varepsilon(0) \to \mathbb{R} \times M$$

that is explicitly given by

$$\Phi_j(\sigma(s,t),(x,y)) = \Psi^{-1}\left[\tilde{v}(s,t) + x\,\tilde{n}(s,t) + y\,\tilde{m}(s,t)\right].$$

The winding $e^{wt\hat{J}}$ in $\tilde{n}$ and $\tilde{m}$ has been introduced in order to adjust Φ_j by choosing $w \in \mathbb{Z}$, so that it can be extended to all of $\dot{S}$. This construction moreover obviously meets

$$\Phi_j(z,0) = \Psi^{-1}[\tilde{v}(\sigma^{-1}(z))] = \tilde{u}(z) \tag{3.23}$$

for all $z \in \dot{D}_j$.

In the case of $\tilde{u}$ being an embedding, note that also $\tilde{v}$ is an embedding and thus — since Ψ is a diffeomorphism — Φ_j is an embedding on some neighbourhood of $\dot{D}_j \times \{0\}$. This can only be a neighbourhood of the form $\dot{D}_j \times B_\varepsilon(0)$, if the asymptotic orbit is simply covered; if it is multiply covered, then the embedded neighbourhood of $\dot{D}_j \times \{0\}$ has to shrink to zero at the puncture.

We have thus found disks $\dot{D}_j$ around all punctures $z_j \in \Gamma$, on which we constructed trivializations Φ_j as required, leaving some freedom in order to make sure that these maps can be extended to all of $\dot{S}$. Moreover, since Γ

is finite, we can choose the disks small enough to not intersect one another. Now the punctured surface $\dot{S} = S \setminus \Gamma$ can be decomposed as follows:

$$\dot{S} = \tilde{S} \,\dot{\cup} \dot{\bigcup_{z_j \in \Gamma}} \dot{D}_j \qquad \text{where} \qquad \tilde{S} = S \setminus \dot{\bigcup_{z_j \in \Gamma}} D_j.$$

As next step we will prove that — for the surface $\tilde{S}$ away from the punctures — there exists a trivialization $\tilde{S} \times B_\varepsilon(0) \to \mathbb{R} \times M$ with the required properties.

Lemma 3.7 *With the previous assumptions and notation there exists for some $\varepsilon > 0$ an immersion*

$$\tilde{\Phi} : \tilde{S} \times B_\varepsilon(0) \to \mathbb{R} \times M$$

satisfying $\tilde{\Phi}(z, 0) = \tilde{u}(z)$ for all $z \in \tilde{S}$. Moreover, if $\tilde{u}$ is an embedding, then also $\tilde{\Phi}$ restricted to a sufficiently small neighbourhood of $\tilde{S} \times \{0\}$ is an embedding.

Proof: We introduce a Hermitian vector bundle over $\tilde{S}$,

$$E := \{(z, \xi) \mid z \in \tilde{S}, \xi \in N_{\tilde{u}(z)}\tilde{u}(U)\},$$

where $U \in \tilde{S}$ is a neighbourhood of z on which $\tilde{u}$ is an embedding, and $N_{\tilde{u}(z)}\tilde{u}(U)$ is the normal space of $\tilde{u}(U)$ in $\mathbb{R} \times M$ with respect to the metric $\langle \cdot, \cdot \rangle_J$ defined in (1.2). With the projection $(z, \xi) \mapsto z$ this obviously is a vector bundle over $\tilde{S}$. Furthermore, the almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ induces an almost complex structure on E. Indeed, as shown in the proof of lemma 3.6, $N_{\tilde{u}(z)}\tilde{u}(U)$ is invariant under $\tilde{J}$, and thus induces the almost complex structure

$$\begin{array}{rrcl} J : & E & \to & E \\ & (z, \xi) & \mapsto & (z, \tilde{J}(\tilde{u}(z))\xi) =: J(z)\,(z, \xi). \end{array}$$

According to [17] proposition 2.61, there exists a symplectic form on E that is compatible with J, hence E can be viewed as a Hermitian vector bundle. Moreover, $\tilde{S}$ is a compact surface since S was assumed to be compact and the disks D_j are open, and it has nonempty boundary since there is at least one puncture $z_1 \in \Gamma$, hence $\partial D_1 \subset \partial \tilde{S}$. Now in view of [17] proposition 2.64, there exists a unitary trivialization of E, i.e. we have a smooth bundle isomorphism

$$\begin{array}{rrcl} \phi^E : & \tilde{S} \times \mathbb{R}^2 & \to & E \\ & (z, \vec{v}) & \mapsto & (z, A(z)\vec{v}) \end{array}$$

such that the linear maps $A(z)$ satisfy

$$A(z)^* J(z) = J_0, \tag{3.24}$$

where J_0 is the canonical almost complex structure on $\mathbb{R}^2$.

If $\tilde{u}$ is an embedding, then E is just the lifting of the normal bundle $N\tilde{u}(\tilde{S})$ of $\tilde{u}(\tilde{S})$ in $\mathbb{R} \times M$, and in the above we could have used $N\tilde{u}(\tilde{S})$ instead of E. Therefore, identifying $E = \tilde{u}^* N\tilde{u}(\tilde{S})$ with $N\tilde{u}(\tilde{S})$, we have a diffeomorphism $\phi^E : \tilde{S} \times \mathbb{R}^2 \to N\tilde{u}(\tilde{S})$. Then by the tubular neighbourhood theorem (e.g. [7] §4 Thm.6.3), a neighbourhood $\mathcal{V}$ of the zero section of $N\tilde{u}(\tilde{S})$ is diffeomorphic to a neighbourhood of $\tilde{u}(\tilde{S})$ in $\mathbb{R} \times M$. Since $\tilde{S}$ is compact, there will be an $\varepsilon > 0$ such that $\phi^E(\tilde{S} \times B_\varepsilon(0)) \subset \mathcal{V}$, so composing the tubular neighbourhood diffeomorphism ψ with ϕ^E yields a diffeomorphism

$$\tilde{\Phi} := \psi \circ \phi^E : \tilde{S} \times B_\varepsilon(0) \to \mathcal{U} \subset \mathbb{R} \times M$$

onto some neighbourhood $\mathcal{U} \subset \mathbb{R} \times M$ of $\tilde{u}(\tilde{S})$. This map satisfies the requirement $\tilde{\Phi}(z,0) = \tilde{u}(z)$ for all $z \in \tilde{S}$ since

$$\phi^E(z,0) = (z, A(z)0) = (z,0) \cong 0 \in N_{\tilde{u}(z)}\tilde{u}(\tilde{S})$$

and the tubular neighbourhood diffeomorphism ψ leaves the zero section of $N\tilde{u}(\tilde{S})$ (that is identified with $\tilde{S}$) fix.

For the case of $\tilde{u}$ being only an immersion, we will to describe the construction of an analogue of the tubular neighbourhood in detail. By the Whitney embedding theorem (see e.g. [7] §1 Thm.3.5) we can think of $\mathbb{R} \times M$ as a submanifold of $\mathbb{R}^8$. We then consider the normal bundle $N(\mathbb{R} \times M) \subset T\mathbb{R}^8 \cong \mathbb{R}^8$ of $\mathbb{R} \times M$ with respect to the Euclidean metric on $\mathbb{R}^8$ and construct a diffeomorphism between a neighbourhood $\mathcal{V}$ of the zero section of this bundle and a tubular neighbourhood $\mathcal{U}$ of $\mathbb{R} \times M$ in $\mathbb{R}^8$:

$$\begin{array}{rcl} \tau : N(\mathbb{R} \times M) \supset \mathcal{V} & \to & \mathcal{U} \subset \mathbb{R}^8 \\ v \in N_p(\mathbb{R} \times M) & \mapsto & p + v. \end{array}$$

According to [7], this actually is a diffeomorphism that leaves $\mathbb{R} \times M$ (that is identified with the zero section of $N(\mathbb{R} \times M)$) fix. We now find a neighbourhood $\mathcal{W} \subset E$ of the zero section, such that for all $(z,\xi) \in \mathcal{W}$ we have $\tilde{u}(z) + \xi \in \mathcal{U}$. Then using τ we can define

$$\begin{array}{rrcl} \mathcal{T} : & E \supset \mathcal{W} & \to & \mathbb{R} \times M \\ & (z,\xi) & \mapsto & \mathrm{pr}(\tau^{-1}(\tilde{u}(z) + \xi)) \end{array},$$

where $\tilde{u}(z)$ and ξ are identified with vectors in $\mathbb{R}^8$, and pr is the bundle projection of $N(\mathbb{R} \times M)$.

Identifiying the tangent space of $N_{\tilde{u}(z)}\tilde{u}(U)$ in 0 with itself, we obtain for every $z \in \tilde{S}$ the splitting

$$T_{(z,0)}E = T_z\tilde{S} \oplus N_{\tilde{u}(z)}\tilde{u}(U). \tag{3.25}$$

We now want to determine how $T_{(z,0)}\mathcal{T}$ acts on this space. On the zero section of E (that is identified with $\tilde{S}$), we have for all $z \in \tilde{S}$

$$\mathcal{T}(z,0) = \mathrm{pr}(\tau^{-1}(\tilde{u}(z))) = \mathrm{pr}(0_{N_{\tilde{u}(z)}(\mathbb{R}\times M)}) = \tilde{u}(z)$$

and hence

$$T_{(z,0)}\mathcal{T}\big|_{T_z\tilde{S}} = T_z\tilde{u}.$$

Along the fibres $N_{\tilde{u}(z)}\tilde{u}(U)$ of E we obtain

$$T_{(z,0)}\mathcal{T}\big|_{N_{\tilde{u}(z)}\tilde{u}(U)} = \mathbb{1}.$$

In order to show this, we have to consider any tangent vector of E in $(z,0)$. This tangent vector is represented by some path $(-\varepsilon,\varepsilon) \ni t \mapsto (z,t\xi)$, where $\xi \in N_{\tilde{u}(z)}\tilde{u}(U)$ can also be seen as the tangent vector itself. We first note that the differential of $(z,\xi) \mapsto \tilde{u}(z)+\xi$ maps the above tangent vector to $\xi \in N_{\tilde{u}(z)}\tilde{u}(U) \subset T_{\tilde{u}(z)}(\mathbb{R}\times M)$. Moreover, since τ and hence also $\mathrm{pr}\circ\tau^{-1}$ leaves $\mathbb{R}\times M$ fix, we have $T_p(\mathrm{pr}\circ\tau^{-1})\big|_{T_p(\mathbb{R}\times M)} = \mathbb{1}$ for all $p \in \mathbb{R}\times M$.

Alltogether we obtain

$$T_{(z,0)}\mathcal{T} = T_z\tilde{u} \oplus \mathbb{1}$$

with respect to the splitting (3.25) of $T_{(z,0)}E$ and mapping onto

$$T_{\tilde{u}(z)}\tilde{u}(U) \oplus N_{\tilde{u}(z)}\tilde{u}(U) = T_{\tilde{u}(z)}(\mathbb{R}\times M).$$

Now $\tilde{u}$ was assumed to be an immersion, therefore the differential of $\mathcal{T}$ is nondegenerate all along $\tilde{S}\times\{0\}$ and hence $\mathcal{T}$ is an immersion on some neighbourhood of $\tilde{S}\times\{0\}$. As in the embedded case we can now define $\tilde{\Phi} := \mathcal{T}\circ\phi^E$, and we restrict this map to $\tilde{S}\times B_\varepsilon(0)$ for some $\varepsilon > 0$ such that $\mathcal{T}$ is an immersion on $\phi^E(\tilde{S}\times B_\varepsilon(0))$ (we have $\varepsilon > 0$ since $\tilde{S}$ is compact). This $\tilde{\Phi}$ is indeed an immersion and it meets

$$\tilde{\Phi}(z,0) = \mathcal{T}(z,0) = \tilde{u}(z)$$

as claimed. □

Of course, the $\tilde{\Phi}$ constructed above can still be homotoped in the set of immersions $\tilde{S}\times B_\varepsilon(0) \to \mathbb{R}\times M$ that are equal to $\tilde{u}$ on $\dot{S}\times\{0\}$. We will use this freedom to make $\tilde{\Phi}$ match Φ_j on ∂D_j for all $z_j \in \Gamma$. For this purpose, we only have to make sure that restricted to $\partial D_j \times B_\varepsilon(0)$, the maps Φ_j and $\tilde{\Phi}$ are homotopic within the set of immersions equalling $\tilde{u}$ on $\partial D_j \times \{0\}$. The homotopy class of $\tilde{\Phi}|_{\partial D_j\times B_\varepsilon(0)}$ is determined by its linearization on $\partial D_j \times \{0\}$, that is

$$\begin{aligned} T_{(z,0)}\tilde{\Phi} &= T_{(z,0)}\mathcal{T}\circ T_{(z,0)}\phi^E \\ &= (T_z\tilde{u}\oplus\mathbb{1})\circ(\mathbb{1}\oplus A(z)) \\ &= T_z\tilde{u}\oplus A(z). \end{aligned}$$

for $z = \sigma(s_0, t)$, $t \in S^1$.

For all Φ in the set that we want to homotope in, we have $\Phi(\cdot, 0) = \tilde{u}(\cdot)$ and hence $T_{(z,0)}\Phi\big|_{T_zS} = T_z\tilde{u}$, so for the homotopy degree only $T_{(\sigma(s_0,\cdot),0)}\Phi\big|_{T_0\mathbb{R}^2}$ is relevant.

From the construction we know that $T_{(\sigma(s_0,t),0)}\tilde{\Phi}|_{T_0\mathbb{R}^2} = A(\sigma(s_0,t))$ maps onto $N_{\tilde{u}(\sigma(s_0,t))}\tilde{u}(U)$, where U is a neighbourhood of $\sigma(s_0,t)$, and so does $T_{(\sigma(s_0,t),0)}\Phi_j\big|_{T_0\mathbb{R}^2}$. Indeed, we calculate from (3.23)

$$
\begin{aligned}
T_{(\sigma(s_0,t),0)}\Phi_j\,\frac{\partial}{\partial x} &= T_{\tilde{v}(s_0,t)}\Psi\ \mathrm{e}^{wt\hat{J}(\tilde{v}(s_0,t))}n(s_0,t)\\
&= \mathrm{e}^{wt\tilde{J}(\tilde{u}(\sigma(s_0,t)))}n(\sigma(s_0,t))\\
&= [\,\cos(wt)\mathbb{1} + \sin(wt)\tilde{J}(\tilde{u}(\sigma(s_0,t))\,]\; n(\sigma(s_0,t))\\
&= \cos(wt)\,n(\sigma(s_0,t)) + \sin(wt)\,m(\sigma(s_0,t)),\\
T_{(\sigma(s_0,t),0)}\Phi_j\,\frac{\partial}{\partial y} &= T_{\tilde{v}(s_0,t)}\Psi\ \mathrm{e}^{wt\hat{J}(\tilde{v}(s_0,t))}m(s_0,t)\\
&= \mathrm{e}^{wt\tilde{J}(\tilde{u}(\sigma(s_0,t)))}m(\sigma(s_0,t))\\
&= [\,\cos(wt)\mathbb{1} + \sin(wt)\tilde{J}(\tilde{u}(\sigma(s_0,t))\,]\; m(\sigma(s_0,t))\\
&= \cos(wt)\,m(\sigma(s_0,t)) - \sin(wt)\,n(\sigma(s_0,t)).
\end{aligned}
$$

Therefore, we can determine the required homotopy class by choosing the bases $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} = J_0\frac{\partial}{\partial x})$ of $T_0\mathbb{R}^2$ and $(n(z), m(z) = \tilde{J}(\tilde{u}(z))n(z))$ of $N_{\tilde{u}(z)}\tilde{u}(U)$.

From (3.24) we have $\tilde{J}(\tilde{u}(z))A(z) = A(z)J_0$, hence $A(z)$ preserves the orientation of above bases, and thus $T_{(\sigma(s_0,\cdot),0)}\tilde{\Phi}|_{T_0\mathbb{R}^2} = A(\sigma(s_0,\cdot))$ is represented by a loop of matrices in $Gl^+(\mathbb{R}^2)$. Hence it is homotopic to

$$
S^1 \ni\ t \mapsto \begin{pmatrix} \cos(kt) & -\sin(kt) \\ \sin(kt) & \cos(kt) \end{pmatrix}
$$

for some $k \in \mathbb{Z}$, but this is just $T_{(\sigma(s_0,\cdot),0)}\Phi_j\big|_{T_0\mathbb{R}^2}$ when we choose $w = k$. So we have shown that $\tilde{\Phi}$ can be homotoped in a small neighbourhood of each of the boundary components ∂D_j of $\tilde{S}$ such that it can be extended by Φ_j to $\dot{D}_j$. That way we obtain a map $\Phi : \dot{S} \times B_{\bar{\varepsilon}}(0) \to \mathbb{R} \times M$, where $\bar{\varepsilon} > 0$ is the finite minimum of the $\varepsilon > 0$ for which $\tilde{\Phi}$ and the Φ_j were constructed. As all of its constituents, Φ is an immersion, and in the case of $\tilde{u}$ being an embedding, it is an embedding on a small neighbourhood of $\dot{S} \times \{0\}$. Moreover, the above homotopy left $\tilde{\Phi}$ fix on $\dot{S} \times \{0\}$ and hence we also have $\Phi(z, 0) = \tilde{u}(z)$ for all $z \in \dot{S}$. This proves the first part of the theorem.

The rest of the theorem describes the behaviour of the almost complex structure near the punctures in our special coordinates Φ. Hence considering the neighbourhood of $z_j \in \Gamma$, we have Φ explicitly given by Φ_j. Now we calculate in this neighbourhood

$$
\bar{J}(z,0) \;=\; (T_{(z,0)}\Phi_j)^{-1} \circ \tilde{J}(\tilde{u}(z)) \circ T_{(z,0)}\Phi_j
$$

$$
\begin{aligned}
&= ((T_z\sigma^{-1})^{-1}\times \mathbb{1})\circ (T_{(\sigma^{-1}(z),0)}\Gamma)^{-1}\circ T_{\tilde v(\sigma^{-1}(z))}\Psi\circ \tilde J(\tilde u(z))\\
&\quad \circ (T_{\tilde v(\sigma^{-1}(z))}\Psi)^{-1}\circ T_{(\sigma^{-1}(z),0)}\Gamma\circ (T_z\sigma^{-1}\times \mathbb{1})\\
&= ((T_z\sigma^{-1})^{-1}\times \mathbb{1})\circ (T_{(\sigma^{-1}(z),0)}\Gamma)^{-1}\circ \hat J(\tilde v(\sigma^{-1}(z)))\\
&\quad \circ T_{(\sigma^{-1}(z),0)}\Gamma\circ (T_z\sigma^{-1}\times \mathbb{1})\\
&= ((T_z\sigma^{-1})^{-1}\times \mathbb{1})\circ (J_0\times J_0)\circ (T_z\sigma^{-1}\times \mathbb{1})\\
&= j(z)\oplus J_0 .
\end{aligned}
$$

Here we used the definition of $\hat J$, the fact that σ is holomorphic (see (1.5)) and

$$
(T_{(s,t,0)}\Gamma)^{-1}\circ \hat J(\tilde v(s,t))\circ T_{(s,t,0)}\Gamma = J_0\times J_0
$$

as an isomorphism on $T_{(s,t)}((s_0,\infty)\times S^1)\times T_0\mathbb{R}^2$. The latter holds since from (3.20) and (3.2) we obtain

$$
\begin{array}{lllllll}
T_{(s,t,0)}\Gamma\frac{\partial}{\partial s} &=& \tilde v_s(s,t) &=& -\hat J(\tilde v(s,t))\tilde v_t(s,t) &=& -\hat J(\tilde v(s,t))T_{(s,t,0)}\Gamma\frac{\partial}{\partial t},\\
T_{(s,t,0)}\Gamma\frac{\partial}{\partial t} &=& \tilde v_t(s,t) &=& \hat J(\tilde v(s,t))\tilde v_s(s,t) &=& \hat J(\tilde v(s,t))T_{(s,t,0)}\Gamma\frac{\partial}{\partial s},\\
T_{(s,t,0)}\Gamma\frac{\partial}{\partial x} &=& \tilde n(s,t) &=& -\hat J(\tilde v(s,t))\tilde m(s,t) &=& -\hat J(\tilde v(s,t))T_{(s,t,0)}\Gamma\frac{\partial}{\partial y},\\
T_{(s,t,0)}\Gamma\frac{\partial}{\partial y} &=& \tilde m(s,t) &=& \hat J(\tilde v(s,t))\,\tilde n(s,t) &=& \hat J(\tilde v(s,t))T_{(s,t,0)}\Gamma\frac{\partial}{\partial x}.
\end{array}
$$

For the last claim of theorem 3.1 we first note that the convergences in (3.21) and (3.22) even hold for all derivatives D^α with respect to s and t of the left and right hand side. Thus in view of (3.20), all entries of $T_{(s,t,x,y)}\Gamma$ converge with all derivatives with repect to s and t and uniformly in t to some smooth functions as $s\to\infty$. Moreover, x and y enter $T_{(s,t,x,y)}\Gamma$ only with linear dependence. Thus there is a smooth matrix function $A(t,x,y)$ such that

$$
D^\alpha(T_{(s,t,x,y)}\Gamma - A(t,x,y)) \overset{s\to\infty}{\longrightarrow} 0
$$

converges uniformly on bounded sets of (t,x,y) for all derivatives D^α with respect to s, t, x and y. The same holds, as $T_{(s,t,x,y)}\Gamma$ is bounded away from the noninvertible matrices, for $(T_{(s,t,x,y)}\Gamma)^{-1}$. Finally, in the same sense as above, the asymptotic behaviour of the map Γ itself is

$$
\Gamma(s,t,x,y) \overset{s\to\infty}{\longrightarrow} (\ \infty\ ,\ kt+d\ ,\ x\,\mathrm{e}^{wtJ(t)}\hat e(t)+y\,\mathrm{e}^{wtJ(t)}\hat f(t)\ )\ .
$$

Since the almost complex structure $\hat J$ does not depend on the first coordinate a, and depends smoothly on all other components, we deduce the existence of a smooth matrix function $\hat J_\infty(t,x,y)$ such that in the above sense of convergence

$$
\hat J(\Gamma(s,t,x,y)) \overset{s\to\infty}{\longrightarrow} \hat J_\infty(t,x,y).
$$

This finally proves the claim since

$$
\begin{aligned}
\bar J_0 &= (T\sigma\oplus \mathbb{1}_{\mathbb{R}^2})^{-1}\circ \bar J(\sigma\times \mathbb{1}_{\mathbb{R}^2})\circ (T\sigma\oplus \mathbb{1}_{\mathbb{R}^2})\\
&= (T\sigma\oplus \mathbb{1}_{\mathbb{R}^2})^{-1}\circ (T\Phi)^{-1}\circ \tilde J(\Phi)\circ T\Phi\circ (T\sigma\oplus \mathbb{1}_{\mathbb{R}^2})\\
&= (T\Gamma)^{-1}\circ T\Psi\circ \tilde J(\Phi)\circ (T\Psi)^{-1}\circ T\Gamma\\
&= (T\Gamma)^{-1}\circ \hat J(\Gamma)\circ T\Gamma .
\end{aligned}
$$

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