CONNECTED CERF THEORY

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Abstract. We review the theory of Cerf describing the decomposition of cobordisms into elementary cobordisms, and the Cerf moves between different decompositions. We put special emphasis on connectedness and define Cerf decompositions as decompositions of morphisms in the category of connected manifolds and connected cobordisms. In addition, we discuss the cyclic Cerf theory of Morse functions to S^1.

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1. Introduction

Let $X_\pm$ be compact oriented n-manifolds. Recall that a *cobordism* from X_- to X_+ is a compact, oriented $n+1$-manifold Y with boundary such that X_+ resp. X_- is the component of the boundary $\partial Y = X_- \cup X_+$ on which the given orientation agrees resp. disagrees with the orientation induced by the orientation on Y.

Definition 1.1. The *connected cobordism category* Cob_{n+1} is the category whose

(a) objects are compact, connected, oriented n-dimensional smooth manifolds;
(b) morphisms are connected $n+1$-dimensional cobordisms modulo the equivalence relation given by diffeomorphisms fixing the boundary;
(c) composition is given by gluing (see Remark 1.2 below).

Remark 1.2. (a) The identity morphism for a manifold X in Cob_{n+1} is the (equivalence class of the) trivial cobordism $X \times [0,1]$.

(b) Given two cobordisms Y_{01} from X_0 to X_1 and Y_{12} from X_1 to X_2, we may glue them together to a cobordism $Y_{01} \cup_{X_1} Y_{12}$ from X_0 to X_2.

For an explicit construction choose collar neighborhoods $\kappa_{01} : X_1 \times (-\epsilon, 0] \to Y_{01}$ and $\kappa_{12} : X_1 \times [0,\epsilon) \to Y_{12}$ and define $Y_{01} \cup_{X_1} Y_{12} := Y_{01} \sqcup Y_{12} \sqcup X_1 \times (-\epsilon,\epsilon)/\sim$ where $\sim$ is the obvious equivalence. Any two choices of collar neighbourhoods are isotopic, hence gluing is well-defined up to diffeomorphisms fixing the boundary; see e.g. [Mi, Thm.1.4].

By a *connected cobordism* we will mean a connected cobordism between connected manifolds as above, that is, a representative of a morphism in the category Cob_{n+1}.

Throughout the paper we will assume $n \geq 2$ since the connected $1+1$-dimensional connected cobordism category Cob_{1+1} has exceptional form: Its only object is S^1, however there are nontrivial connected 2-dimensional cobordisms that arise from composition of more elementary cobordisms between disconnected 1-manifolds. This obstructs the decomposition into elementary connected cobordisms that will be the result of our Cerf decomposition in higher dimensions.

Section 2 considers $\mathbb{R}$-valued Morse functions whose critical points can be separated by level sets and shows in Lemmas 2.5, 2.6 that such "Morse data" (see Definition 2.1) are equivalent to Cerf decompositions (see Definition 2.3). Section 3 then studies homotopies of Morse data to prove in Theorem 3.4 that Cerf decompositions are unique up to the Cerf given by Definition 3.2. Finally, Section 4 extends Cerf theory to the case of S^1-valued Morse functions $f : Y \to S^1$. We show in Lemma 4.4 that cyclic Cerf decompositions exist for any homotopy class $[f]$ such that

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$f_*: \pi_1(Y) \to \pi_1(S^1)$ is surjective. Theorem 4.7 then proves uniqueness of cyclic Cerf decompositions up to the same Cerf moves as in the $\mathbb{R}$-valued case.

2. Cerf decompositions of cobordisms

In this section we describe the decomposition of cobordisms and morphisms in the cobordism category into elementary pieces. In the following, let X_-, X_+ be compact, connected, oriented manifolds of dimension $n \geq 2$, and let Y be a compact, connected, oriented cobordism from X_- to X_+.

Definition 2.1. A *Morse datum* for Y consists of a pair $(f, \underline{b})$ of a Morse function $f : Y \to \mathbb{R}$ and an ordered tuple $\underline{b} = (b_0 < b_1 < \ldots < b_m) \subset \mathbb{R}^{m+1}$ such that

(i) $X_- = f^{-1}(b_0)$ and $X_+ = f^{-1}(b_m)$ are the sets of minima resp. maxima of f;
(ii) each level set $f^{-1}(b)$ for $b \in \mathbb{R}$ is connected;
(iii) f has distinct values at the (isolated) critical points, i.e. it induces a bijection $\mathrm{Crit} f \to f(\mathrm{Crit} f)$ between critical points and critical values;
(iv) $b_0, \ldots, b_m \in \mathbb{R} \setminus f(\mathrm{Crit} f)$ are regular values of f such that each interval (b_{i-1}, b_i) contains at most one critical value of f.

Definition 2.2. Y is an *elementary cobordism* if it admits a Morse datum $(f, \underline{b} = (\min f, \max f))$, that is f is a Morse function with at most one critical point. Y is a *cylindrical cobordism* if it admits a Morse datum $(f, \underline{b} = (\min f, \max f))$, where f is a Morse function with no critical point.

Definition 2.3. (a) A *Cerf decomposition* of a cobordism Y as above is a decomposition

$$Y = Y_1 \cup_{X_1} Y_2 \cup_{X_2} \ldots \cup_{X_{m-1}} Y_m$$

into a sequence $(Y_i \subset Y)_{i=1,\ldots m}$ of elementary cobordisms embedded in Y that are disjoint from each other and ∂Y except for $Y_1 \cap \partial Y = X_-$, $Y_m \cap \partial Y = X_+$, and the intersections $X_i := Y_i \cap Y_{i+1}$, which are also connected submanifolds in Y of codimension 1. As a consequence we have $\partial Y_i = X_{i-1} \sqcup X_i$ for $i = 1, \ldots, m$ with $X_0 = X_-$ and $X_m = X_+$.

(b) A *Cerf decomposition* of a morphism $[Y]$ in the connected cobordism category Cob_{n+1} is a sequence $([Y_i])_{i=1,\ldots m}$ of elementary morphisms that compose to

$$[Y] = [Y_1] \circ [Y_2] \circ \ldots \circ [Y_m].$$

Remark 2.4. (a) Any Morse datum $(f, \underline{b})$ for a cobordism Y as above induces a Cerf decomposition $Y = Y_1 \cup_{X_1} \ldots \cup_{X_{m-1}} Y_m$ into the sequence $(Y_i := f^{-1}([b_{i-1}, b_i]))_{i=1,\ldots m}$ of elementary cobordisms between the connected level sets $X_i = f^{-1}(b_i)$.

(b) Any Cerf decomposition $Y = Y_1 \cup_{X_1} \ldots \cup_{X_{m-1}} Y_m$ of a representative of a morphism $[Y]$ in Cob_{n+1} induces a Cerf decomposition $[Y] = [Y_1] \circ [Y_2] \circ \ldots \circ [Y_m]$ in the cobordism category.

(c) On the other hand, any Cerf decomposition $Y = Y_1 \cup_{X_1} \ldots \cup_{X_{m-1}} Y_m$ arises from a Morse datum. Indeed, by definition each elementary cobordism Y_i supports a Morse datum with just two levels (b_0, b_1), hence its trivial Cerf decomposition is induced by a Morse datum. Now we can iterate Lemma 2.6 below with $Y_- = Y_1 \cup \ldots \cup Y_k$ and $Y_+ = Y_{k+1}$ for $k = 1, \ldots, m = 1$ to construct a Morse datum on Y that induces exactly the given Cerf decomposition.

(d) Finally, any Cerf decomposition $[Y] = [Y_1] \circ [Y_2] \circ \ldots \circ [Y_m]$ in Cob_{n+1} arises from a Cerf decomposition of a representative. Indeed, pick representatives $Y_1, Y_2, \ldots, Y_m$ and collar neighbourhoods of the common boundaries $X_1, \ldots, X_{m-1}$, then the glued cobordism $Y' := Y_1 \cup_{X_1} Y_2 \cup_{X_2} \ldots \cup_{X_{m-1}} Y_m$ is a representative of $[Y]$. Y' has a Cerf decomposition given by $Y_1, \ldots, Y_m \hookrightarrow Y'$, and this induces exactly the given Cerf decomposition of $[Y]$.

Lemma 2.5. *Given a connected cobordism as above, a Morse function satisfying (i)-(iii) in Definition 2.1 always exists. Given such a Morse function f, $b_0 := \min f$, and $b_m := \max f$, for*

$m \geq \#\mathrm{Crit} f$, there evidently always exists a choice of $b_1 < \ldots < b_{m-1}$ satisfying condition (iv), hence making $(f, \underline{b})$ a Morse datum.

As a consequence, any such cobordism has a Cerf decomposition, and any morphism in Cob_{n+1} has a Cerf decomposition.

Proof. By [Mi, Theorem 8.1], there exists a Morse function $f : Y \to \mathbb{R}$ such that f is *self-indexing* in the sense that the critical points of index i have critical value i, for each $i = 0, \ldots, n+1$, and furthermore there are no critical points of index 0 or $n+1$, and such that $X_\pm$ is the set of global minima resp. maxima of f. After a small perturbation, we may assume that the critical values of f are distinct, by [Mi, Chapter 4], but still with the property that if y, y' are critical points with index of y less than that of y', then $f(y) < f(y)'$. We claim the fibers of f are connected. Indeed, each level set is obtained by attaching handles to lower level sets (see Remark 2.7); the level sets become disconnected by either attaching a handle of index 0, which does not exist by assumption, or by a handle of index n with disconnecting attaching cycle. Once a level set is disconnected, it can be become connected again only by attaching a handle of index one, with the points of the attaching cycle in different components. But since the Morse function is self-indexing and $n \geq 2$, the n-handles are attached after the 1-handles, so the existence of a disconnecting n-handle would imply that X_+ is disconnected, a contradiction. Given such a Morse function f, $b_0 := \min f$, and $b_m := \max f$, for $m \geq \#\mathrm{Crit} f$, there evidently always exists a choice of $b_1 < \ldots < b_{m-1}$ satisfying condition (iv), hence making $(f, \underline{b})$ a connected Morse datum. $\square$

Lemma 2.6. *Suppose that $Y = Y^- \cup_X Y^+$ is a decomposition into two connected cobordisms embedded in Y such that $Y^\pm \cap \partial Y = X_\pm$ and $X = Y^- \cap Y^+$ is a connected submanifold in Y of codimension 1. Let Morse data $(f^\pm, \underline{b}^\pm)$ on $Y^\pm$ be given, inducing Cerf decompositions $Y^\pm = Y_1^\pm \cup_{X_1^\pm} \ldots \cup_{X_{m^\pm - 1}} Y_{m^\pm}^\pm$. Then there exists a Morse datum on Y which induces the Cerf decomposition given by the two parts $Y = Y_1^- \cup_{X_1^-} \ldots \cup_{X_{m^- - 1}} Y_{m^-}^- \cup_X Y_1^+ \cup_{X_1^+} \ldots \cup_{X_{m^+ - 1}} Y_{m^+}^+$.*

Proof. This follows from [Mi, Theorem 1.4, Lemma 3.7], which uses [Mu, Lemma 6.1]. Here we give a somewhat alternative proof. First recall that both f^- and f^+ are constant on X, hence after shifting f^- by a constant (and simultaneously shifting $\underline{b}^-$ such that the Cerf decomposition is not affected) we can assume that $f^-|_X = f^+|_X =: c$. By Whitney's extension theorem, both functions also extend smoothly to all of Y. Unfortunately, they cannot simply be interpolated smoothly without creating new critical points. Instead, we use the flow of $\nabla f^- / |\nabla f^-|^2$ to construct an embedding $X \times [-\epsilon, \epsilon] \hookrightarrow Y$ to a neighbourhood of X such that after pullback $f^-(x,t) = c + t$. By choosing $\epsilon > 0$ small, we can moreover ensure that the embedding only intersects the first elementary cobordism $X \times [0, \epsilon] \hookrightarrow Y_1^+$ inside Y^+. Next, after pulling back f^+ to a function on $X \times [-\epsilon, \epsilon]$ as well, we use the flow of $\nabla f^+ / |\nabla f^+|^2$ to construct the germ of an embedding $\psi : X \times [0, \delta) \hookrightarrow X \times [-\epsilon, \epsilon]$ such that $\psi(x, 0) = (x, 0)$ and $\psi^* f^+(x,t) = c + t$. We claim that ψ can be extended to a diffeomorphism $\psi : X \times [0, \epsilon) \to X \times [0, \epsilon)$ that equals to the identity near $X \times \{\epsilon\}$. Assuming this extension, it can be transfered and trivially extended to a diffeomorphism of Y which is supported in $Y_1^+ \setminus X$, and hence keeps the given Cerf decomposition fixed. We then obtain a Morse datum by defining a smooth function $\tilde{f} : Y \to \mathbb{R}$ via $\tilde{f}|_{Y^-} = f^-$ on Y^- and $\tilde{f}|_{Y^+} = \psi^* f^+$, and using the union of levels $\underline{\tilde{b}} = \underline{b}^- \cup \underline{b}^+$. This induces the Cerf decomposition as claimed into elementary cobordisms Y_i^- and $\psi^{-1}(Y_i^+) = Y_i^+$.

In order to construct the required diffeomorphism $\psi : X \times [0, \epsilon) \to X \times [0, \epsilon)$ denote the coordinates by $(x, t) \in X \times [0, \epsilon)$ and fix a split metric $\mathrm{d}t^2 + g_X$. Then the gradient splits $\nabla f^+ = \nabla_X f^+ + (\partial_t f^+) \partial_t$, where by assumption $\nabla_X f^+|_{t=0} = 0$ and $\partial_t f|_{t=0} > 0$. Pick a cutoff function $\lambda : [0, \epsilon) \to [0, 1]$ equal to 1 near 0 and also supported near 0 and consider the vector field

$$V := \lambda \tfrac{\nabla f^+}{|\nabla f^+|^2} + (1 - \lambda) \partial_t = \lambda \tfrac{\nabla_X f^+}{|\nabla f^+|^2} + \Big(\lambda \tfrac{\partial_t f^+}{|\nabla f^+|^2} + 1 - \lambda \Big) \partial_t$$

on $X \times [0,\epsilon)$. By picking the support of λ small, we can make the X-component of V arbitrarily small and also ensure that the ∂_t-component is positive throughout. Now define $\psi : X \times [0,\delta) \hookrightarrow X \times [0,\epsilon]$ as the flow of V starting at $\psi(x,0) = (x,0)$. We will analyze the components of $\psi(x,s) = \big(\phi(x,s), \tau(x,s)\big) \in X \times [0,\epsilon)$ separately. The second component solves $\tau(x,0) = 0$, $\partial_s\tau(x,s) = \langle V(\psi(x,s)), \partial_t\rangle$, hence increases monotonely and for some small s_0 the value $\inf_{x\in X}\tau(x,s_0)$ will be outside the support of λ. Hence the second component for $s \geq s_0$ has the form $\tau(x,s) = \tau(x,s_0) + s - s_0$. When shrinking the support of λ to 0, we get $s_0 \to 0$ and $\tau(x,s_0) \to 0$, i.e. τ converges to the map $(x,s) \mapsto s$ in $\mathcal{C}^0$. The first component starts out as identity $\phi(x,0) = x$ and solves $\partial_s\phi(x,s) = \lambda(\tau(x,s))\frac{\nabla_X f^+}{|\nabla f^+|^2}(\tau(x,s),\phi(x,s))$. Here the right hand side converges uniformly to 0 as we shrink the support of λ to 0, hence the resulting family of maps $(\phi(\cdot,s) : X \to X)_{s\in[0,s_0]}$ converges in $\mathcal{C}^1$ to the identity. So for sufficiently small support of λ, the first component will be a smooth family of diffeomorphisms with $\phi(x,s) = \phi(x,s_0)$ for all $s \geq s_0$. We can moreover ensure that $s_0 \leq \frac{1}{3}\epsilon$ and $\sup_{x\in X}\tau(x,s_0) < \epsilon$ before beginning to adjust ψ on $X \times (s_0,\epsilon]$, where it is given by $\psi(s,t) = (\tau(x,s_0) + s - s_0, \phi(x,s_0))$. Since the second component already is a diffeomorphism in x, we can replace the first component on $X \times [s_0,\epsilon]$ by any smooth family over $x \in X$ of monotonely increasing functions equal to $\tau(x,s_0) + s - s_0$ near $s = s_0$ and equal to s for $s \geq \frac{2}{3}\epsilon$. Now the adjusted diffeomorphism ψ on $X \times (\frac{2}{3}\epsilon, \epsilon]$ has the form $(s,t) \mapsto (s, \phi(x,s_0))$, and we can replace the second component by an isotopy of diffeomorphisms ending at the identity near $X \times \{\epsilon\}$. This finishes the construction of the required diffeomorphism $\psi : X \times [0,\epsilon) \to X \times [0,\epsilon)$ and hence of the Lemma. □

Remark 2.7. The pieces Y_i of a Cerf decomposition have simple topological descriptions as follows If the elementary cobordism Y_i contains no critical point then it is in fact a cylindrical cobordism. In that case Y_i is diffeomorphic to the *cylinder* $X_i \times [b_{i-1}, b_i]$, and X_{i-1} is diffeomorphic to X_i. If Y_i contains a single critical point, then it has index $k \in \{1,\dots,n\}$. In that case Y_i is obtained from $X_{i-1} \times [b_{i-1}, b_i]$ by attaching a handle $B^k \times B^{n-k+1}$ along an attaching cycle $S^{k-1} \hookrightarrow X_{i-1}$. The attaching cycle can be obtained as the intersection of the unstable manifold (for some choice of a metric on Y_i) of the unique critical point with X_{i-1}, see Figure **??**. More precisely, to obtain Y_i one glues the handle on via an attaching map $S^{k-1} \times B^{n-k+1} \hookrightarrow X_{i-1} \times \{b_i\}$, whose image is a neighbourhood of the attaching cycle. Conversely, Y_i can be obtained from X_i by attaching a handle of opposite index to an attaching cycle in X_i.

Note that our definition of a Cerf decomposition differs from the standard handle decomposition in that we allow the elementary cobordisms Y_i to be cylindrical cobordisms and we do not keep track of the attaching cycles. This also simplifies the moves between different decompositions: Since we do not fix a metric or require the Smale condition of stable and unstable manifolds intersecting transversally, we need not consider the handle slide move discussed in [**?**, p. 40]. See Remark 3.3.

Recall that in the cobordism category, morphisms are equivalence classes of cobordisms modulo diffeomorphisms that act trivially on the boundary. In order for Cerf decompositions to make sense in this category, we need to ensure that equivalent cobordisms have equivalent decompositions – as made precise in the following Definition and Remark.

Definition 2.8. (a) A *diffeomorphism equivalence* between two Cerf decompositions $(Y_i)_{i=1,\dots m}$ and $(Y'_i)_{i=1,\dots m}$ of the same length m is a diffeomorphism $\psi : Y \to Y$ that is fixed on the boundary $\psi|_{X_\pm} = \mathrm{Id}_{X_\pm}$, such that $Y'_i = \psi(Y_i)$.

(b) A *diffeomorphism equivalence* between two Cerf decompositions $([Y_i])_{i=1,\dots m}$ and $([Y'_i])_{i=1,\dots m}$ of the same length m is a collection of diffeomorphisms $(\psi_i : Y_i \to Y'_i)_{i=1,\dots,m}$ satisfying the end conditions $\psi_1|_{X_-} = \mathrm{Id}_{X_-}$, $\psi_m|_{X_+} = \mathrm{Id}_{X_+}$ and the compatibility conditions $\psi_i|_{X_i} = \psi_{i+1}|_{X_i}$ for $i = 0,\dots,m-1$. Here X_i is the common boundary of Y_i and Y_{i+1} (which necessarily exists since these are composable morphisms in the cobordism category).

Remark 2.9. Given two representatives Y and Y' of the same morphism in Cob_{n+1} and a diffeomorphism $\psi : Y \to Y'$, any Morse datum $(f, \underline{b})$ for Y induces a Morse datum $(\psi^* f, \underline{b})$ for Y' by pullback. The Cerf decompositions of $[Y]$ induced by $(f, \underline{b})$ and $(\psi^* f, \underline{b})$ are then diffeomorphism equivalent via the collection $(\psi|_{Y_i})_{i=1,\dots,m}$ of diffeomorphisms.

3. Uniqueness of Cerf decompositions

In a first step towards Cerf theory, we show that homotopies of Morse data yield diffeomorphism equivalent Cerf decompositions. In fact, we have the following result for more general homotopies of pairs $(f, \underline{b})$ of functions on Y and tuples of regular level sets.

Lemma 3.1. *Suppose that $(f_s)_{s\in[0,1]}$ is a smooth family of smooth functions $f_s \in \mathcal{C}^\infty(Y, \mathbb{R})$ and $(\underline{b}^s)_{s\in[0,1]}$ is a smooth family of ordered tuples $\underline{b}^s = (b^s_0 < b^s_1 < \ldots < b^s_m) \subset \mathbb{R}^{m+1}$ such that for every $s \in [0,1]$*

(i) *$X_- = f_s^{-1}(b^s_0)$ and $X_+ = f_s^{-1}(b^s_m)$ are the sets of global minima resp. maxima of f_s;*
(iv') *$b^s_0, \dots, b^s_m \in \mathbb{R} \setminus f_s(\mathrm{Crit} f_s)$ are regular values of f_s.*

Then the decompositions $(Y^0_i := f_0^{-1}([b^0_{i-1}, b^0_i]))_{i=1,\dots m}$ and $(Y^1_i := f_1^{-1}([b^1_{i-1}, b^1_i]))_{i=1,\dots m}$ are diffeomorphism equivalent in the sense that there exists a diffeomorphism $\psi : Y \to Y$ which intertwines the cobordisms $\psi(Y^0_i) = Y^1_i$ as well as the level sets $\psi(X^0_i) = X^1_i$, where $X^k_i := f_k^{-1}(b^k_i)$ for $k = 0, 1$ and $i = 0, \dots, m$. We moreover have the following special cases.

(a) *If $(f_0, \underline{b}^0)$ and $(f_1, \underline{b}^1)$ are Morse data, then the two Cerf decompositions $(Y^0_i)_{i=1,\dots m}$ and $(Y^1_i)_{i=1,\dots m}$ are diffeomorphism equivalent in the sense of Definition 2.8.*
(b) *In fact, $\psi = \psi_1$ is part of a smooth isotopy $(\psi_s : Y \to Y)_{s\in[0,1]}$ of diffeomorphisms with $\psi_0 = \mathrm{Id}_Y$, providing diffeomorphism equivalences between (Y^0_i) and (Y^s_i) for each $s \in [0,1]$. If moreover $(f_s, \underline{b}_s)$ are Morse data for all $s \in [0,1]$, then there is a choice of (ψ_s) and an ambient isotopy $(\phi_s : \mathbb{R} \to \mathbb{R})_{s\in[0,1]}$ such that $\phi_s(b^0_i) = b^s_i$ and $f_s = \phi_s \circ f_0 \circ \psi_s^{-1}$.*

Proof. First note that, since b^s_i is never a critical value of f_s, each level set $X^s_i := f_s^{-1}(b^s_i)$ is smooth and each piece $Y^s_i := f_s^{-1}([b^s_{i-1}, b^s_i])$ is a smooth cobordism with boundary $X^s_{i-1}{}^- \cup X^s_i$ (since it consists of an open subset and its two smooth boundary components). Moreover, the union of level sets

$$\tilde{X}_i := \left\{(s, y) \in [0,1] \times Y \,\middle|\, f_s(y) = b^s_i\right\} = \bigcup_{s\in[0,1]} \{s\} \times f_s^{-1}(b^s_i),$$

is a smooth submanifold of $[0,1] \times Y$ since $(s, y) \mapsto f_s(y) - b^s_i$ is transverse to 0. Note that the extremal levels are the boundary components $\tilde{X}_0 = [0,1] \times X_-$ and $\tilde{X}_m = [0,1] \times X_+$. Moreover, $\tilde{X}_i \to [0,1]$ is a fiber bundle for each i (since it is a submersion and the level sets are compact), whose fibers are the level sets X^s_i.

Similarly, each piece $\bigcup_{s\in[0,1]} \{s\} \times f_s^{-1}([b^s_{i-1}, b^s_i])$ should form a fiber bundle over $[0,1]$ with boundary $\tilde{X}^-_{i-1} \cup \tilde{X}_i$, so that parallel transport along a connection provides the required bundle diffeomorphisms. However, instead of going into the notion of fiber bundles with boundary, we will explicitly construct a vector field on $[0,1] \times Y \to [0,1]$ that restricts to connection vector fields on each $\tilde{X}_i$. For that purpose we fix a metric on Y. Then each vector field

$$V_i(s, y) := \big(\partial_s \,,\, (\partial_s b^s_i - \partial_s f_s(y)) |\nabla f_s(y)|^{-2} \nabla f_s(y)\big)$$

is defined on the complement in $[0,1] \times Y$ of the union of critical sets of the functions f_s. In particular, V_i is defined in a neighborhood of each level set $\tilde{X}_i$ and by construction is tangent to each $\tilde{X}_i$. Note that in particular $V_0|_{\tilde{X}_0} = \partial_s$ since $\partial_s f_s|_{X_-} = \partial_s b^s_0$, and similarly $V_m|_{\tilde{X}_m} = \partial_s$. Now we interpolate between the V_i to define one vector field $\tilde{V}$ on $[0,1] \times Y$ as follows: Since the unions of level sets $\tilde{X}_i$ are disjoint for different i, we can pick a tuple of smooth cutoff functions

$h_i : [0,1] \times Y \to [0,1]$ that equal 1 on a neighbourhood of $\tilde{X}_i$ where V_i is defined, and such that the h_i for different i have disjoint support. Then

$$\tilde{V}(s,y) := \left(\partial_s \,\middle|\, \sum_{i=0}^{m} h_i(s,y) \frac{\partial_s b_i^s - \partial_s f_s(y)}{|\nabla f_s(y)|^2} \nabla f_s(y) \right)$$

defines a connection vector field on $[0,1] \times Y$ (i.e. a lift of ∂_s). Its flow takes the form $\Psi_s(0,y) = (s, \psi_s(y))$, inducing the parallel transport isotopy $(\psi_s : Y \to Y)_{s\in[0,1]}$ with $\psi_0 = \mathrm{id}_Y$. By construction, $\tilde{V}$ is parallel to each $\tilde{X}_i$, i.e. it restricts to connections $\tilde{V}|_{\tilde{X}_i} = V_i|_{\tilde{X}_i}$ on each $\tilde{X}_i$. Thus the flow of $\tilde{V}$ preserves $\tilde{X}_i$ and the parallel transport ψ_s restricts to diffeomorphisms $\psi_s|_{X_i^0} : X_i^0 \to X_i^s$ between the fibers. As a consequence, the flow of $\tilde{V}$ also preserves each intermediate piece $\bigcup_{s\in[0,1]}\{s\} \times f_s^{-1}([b_{i-1}^s, b_i^s])$ and thus induces diffeomorphisms $\phi_i^s := \psi_s|_{Y_i^0} : Y_i^0 \to Y_i^s$ between the fibers $Y_i^s = f_s^{-1}([b_{i-1}^s, b_i^s])$. By construction, these match on the intersections $Y_{i-1}^0 \cap Y_i^0 = X_i^0$ and restrict to the identity on the boundary components $X_0^s = X_-$ and $X_m^s = X_+$ (since $\tilde{V} = \partial_s$ on $\tilde{X}_0$ and $\tilde{X}_m$). This proves the main statement as well as (a).

To prove (b) suppose that $(f_s, \underline{b}_s)$ are Morse data for all $s \in [0,1]$. First, since the critical values of f_s and the regular values in $\underline{b}_s$ are distinct for all $s \in [0,1]$, we can post-compose with an ambient isotopy to arrange that the critical values and the chosen regular values are constant. More precisely, there is a smooth family $(\phi_s : \mathbb{R} \to \mathbb{R})_{s\in[0,1]}$ of diffeomorphisms of $\mathbb{R}$ such that $\phi_0 = \mathrm{Id}$, $\underline{b}_s = \phi_s(\underline{b}_0)$, and $f_s(\mathrm{Crit} f_s) = \phi_s(f_0(\mathrm{Crit} f_0)) = (\phi_s \circ f_0)\big(\mathrm{Crit}(\phi_s \circ f_0)\big)$. Then $(f_s, \underline{b}_s)$ and $(g_s := \phi_s^{-1} \circ f_s, \underline{b}_0)$ induce the same Cerf decompositions of Y.

If we can now refine our choice of connection $\tilde{V} = (\partial_s, V_s)$ on $[0,1]\times Y$ such that $\partial_s g_s + dg_s(V_s) \equiv 0$, then the induced isotopy ψ_s solves $\partial_s(g_s \circ \psi_s) = 0$, and hence $f_0 = g_s \circ \psi_s = \phi_s^{-1} \circ f_s \circ \psi_s$ as claimed. In the complement of critical points, our above construction $V_s(y) := \partial_s g_s(y))|\nabla g_s(y)|^{-2} \nabla g_s(y)$ gives the desired vector field $\tilde{V}$. Near every Morse critical point (s_0, y_0) with critical value $g_{s_0}(y_0) = c$ we can find coordinates $\theta_s : B_\epsilon \to Y$ for a neighbourhood of y_0, defined on an open ball $B_\epsilon \subset \mathbb{R}^n$, and varying smoothly with s near s_0 such that $g_s \circ \theta_s : (x_1, \ldots, x_n) \mapsto c + \sum \pm x_j^2$. (Here we are using the facts that all g_s are Morse and the critical value c is independent of s.) Now $V_s(y) := \frac{d}{dt}\big|_{t=0} \theta_{s+t}(\theta_s^{-1}(y))$ defines a vector field in a neighbourhood of (s_0, y_0), and for all $\underline{x} = (x_1, \ldots, x_n) \in B_\epsilon$ we have $\partial_s g_s(\theta_s(\underline{x})) + dg_s\big(\frac{d}{dt}\big|_{t=0} \theta_{s+t}(\underline{x})\big) = \frac{d}{dt}\big|_{t=0} g_{s+t}(\theta_{s+t}(\underline{x})) = 0$, as required. Finally, we can construct $\tilde{V}$ globally by patching the local vector fields above with a partition of unity, which finishes the proof. $\square$

We know by Lemma 2.5 that every connected cobordism has a Cerf decomposition. The subsequent Cerf Theorem 3.4 implies that Cerf decompositions are unique up to the Cerf moves, which will be defined in the following.

Definition 3.2. Let Y be a cobordism as before. A *Cerf move* from one Cerf decomposition $Y = Y_1 \cup_{X_1} \cup \ldots \cup_{X_{m-1}} Y_m$ to another $Y = Y_1' \cup_{X_1'} \cup \ldots \cup_{X_{m'-1}'} Y_{m'}'$ is one of the following operations.

(a) A *critical point cancellation* is the move

$$\text{from} \quad Y = \ldots Y_j \cup_{X_j} Y_{j+1} \ldots \quad \text{to} \quad Y = \ldots \big(Y_j \cup Y_{j+1}\big) \ldots,$$

where for some $j \in \{1, \ldots, m-1\}$ the two consecutive elementary cobordisms Y_j, Y_{j+1} compose to a cylindrical cobordism $Y_j \cup Y_{j+1} \subset Y$. More precisely, in this situation, critical point cancellation is the move from $(Y_i)_{i=1,\ldots,m}$ to $(Y_i')_{i=1,\ldots,m'}$ with $m' = m-1$, $Y_i' = Y_i$ for $i < j$, $Y_j' = Y_j \cup Y_{j+1}$, and $Y_i' = Y_{i+1}$ for $i > j$. A *critical point creation* is the same move with the roles of $(Y_i)_{i=1,\ldots,m}$ and $(Y_i')_{i=1,\ldots,m'}$ interchanged.

(b) A *critical point switch* is the move

$$\text{from} \quad Y = \ldots Y_j \cup_{X_j} Y_{j+1} \ldots \quad \text{to} \quad Y = \ldots Y_j' \cup_{X_j'} Y_{j+1}' \ldots,$$

where for some $j \in \{1, \dots, m-1\}$ the union $Y_j \cup Y_{j+1} \subset Y$ equals to the union $Y_j' \cup Y_{j+1}' \subset Y$, and the two Cerf decompositions $Y_j \cup_{X_j} Y_{j+1} = Y_j' \cup_{X_j'} Y_{j+1}'$ of the same cobordism are given by Morse data $(f, \underline{b})$ and $(f', \underline{b}')$ with unique critical points $y_{j(+1)} \in Y_{j(+1)}$ and $y_{j(+1)}' \in Y_{j(+1)}'$ in each part, whose attaching cycles (for some choice of a metric) switch in the following sense: The attaching cycles of y_j and y_{j+1} in X_j and those of y_j' and y_{j+1}' in X_j' are disjoint, while in $X_{j-1} = X_{j-1}'$ the attaching cycle of y_j is homotopic to that of y_{j+1}', and the attaching cycle of y_{j+1} is homotopic to that of y_j'; and analogously for the intersections of stable manifolds with $X_{j+1} = X_{j+1}'$. More precisely, in this situation, critical point switch is the move from $(Y_i)_{i=1,\dots,m}$ to $(Y_i')_{i=1,\dots,m'}$ with $m' = m$, $Y_i' = Y_i$ for $i < j$, $Y_j' \cup Y_{j+1}' = Y_j \cup Y_{j+1}$ as above, and $Y_i' = Y_i$ for $i > j+1$.

(c) A *cylinder cancellation* is the move

$$\text{from} \quad Y = \dots Y_j \cup_{X_j} Y_{j+1} \dots \quad \text{to} \quad Y = \dots \big(Y_j \cup Y_{j+1}\big) \dots,$$

where for some $j \in \{1, \dots, m-1\}$ one of the two consecutive elementary cobordisms Y_j, Y_{j+1} is cylindrical. Then the union $Y_j \cup Y_{j+1} \subset Y$ is an elementary cobordism as well. More precisely, in this situation, critical point cancellation is the move from $(Y_i)_{i=1,\dots,m}$ to $(Y_i')_{i=1,\dots,m'}$ with $m' = m-1$, $Y_i' = Y_i$ for $i < j$, $Y_j' = Y_j \cup Y_{j+1}$, and $Y_i' = Y_{i+1}$ for $i > j$. A *cylinder creation* is the same move with the roles of $(Y_i)_{i=1,\dots,m}$ and $(Y_i')_{i=1,\dots,m'}$ interchanged.

(d) A *diffeomorphism equivalence* is the move

$$\text{from} \quad Y = Y_1 \cup_{X_1} \dots \cup_{X_{m-1}} Y_m \quad \text{to} \quad Y = \psi(Y_1) \cup_{\psi(X_1)} \dots \cup_{\psi(X_{m-1})} \psi(Y_m),$$

where $\psi : Y \to Y$ is a diffeomorphism satisfying $\psi|_{X_\pm} = \mathrm{Id}_{X_\pm}$. More precisely, in this situation, diffeomorphism equivalence is the move from $(Y_i)_{i=1,\dots,m}$ to $(Y_i' = \psi(Y_i))_{i=1,\dots,m}$ between two diffeomorphism equivalent Cerf decompositions as in Definition 2.8.

Let $[Y]$ be a morphism in Cob_{n+1} and $[Y] = [Y_1] \circ \dots \circ [Y_m]$ a Cerf decomposition induced by a Cerf decomposition $Y = Y_1 \cup \dots \cup Y_m$. A Cerf move on $[Y] = [Y_1] \circ \dots \circ [Y_m]$ is an operation induced by a Cerf move on $Y = Y_1 \cup \dots \cup Y_m$.

Remark 3.3. Note that in our discussion of critical point switches above, we did use the notion of stable and unstable manifolds and attaching cycles for the two critical points involved. The reader who is familiar with handle decompositions may wonder how it is that we avoid handle slides as one of our Cerf moves, and the answer is that paying attention to handle slides is exchanged for paying attention to critical point switches. Handle slides are not meaningful as moves on Cerf decompositions; they are just isotopies of the gluing maps. Similarly, critical point switches are not meaningful as moves on handle decompositions. To relate these viewpoints, note that a convenient description of a critical point switch in which a higher critical point y_{j+1} drops below a lower critical point y_j is as a homotopy of the Morse function supported in a neighborhood of the unstable manifold of y_{j+1}. Thus, if we modify the gradient-like vector field so as to slide the handle for y_{j+1} over the handle for y_j, we will get a different critical point switch. In other words, two critical points can switch heights in many different ways, and these different ways can be related by handle slides. Furthermore, when discussing handle slides it is important to remember that the handle that is sliding needs to be above the handle that is being slid over, and thus that sequences of handle slides can require nontrivial sequences of critical point switches. The basic example of this phenomenon occurs in a loop of Morse functions on $\mathbb{R}^2$ obtained by perturbing the "monkey saddle", and is described carefully in [HW, p.202] as well as in [GK, Thm.4.10]. This loop involves two critical points of index 1, three critical point switches and three handle slides.

Theorem 3.4 (Cerf theory)**.** *Let Y be a cobordism as before, and let $Y = Y_1 \cup_{X_1} \cup \dots \cup_{X_{m-1}} Y_m$ be a Cerf decomposition. Then any other Cerf decomposition $Y = Y_1' \cup_{X_1'} \cup \dots \cup_{X_{m'-1}'} Y_{m'}'$ of Y*

can be obtained from $(Y_i)_{i=1,\ldots,m}$ by a finite sequence of Cerf moves. As a consequence, any Cerf decomposition of a morphism $[Y]$ in Cob_{n+1} *is unique up to Cerf moves.*

The proof builds on the following result from singularity theory for smooth families of functions on Y. For the purpose of smoothness and defining germs, these families will be viewed as maps $(f_s)_{s\in[0,1]} : [0,1]\times Y \to \mathbb{R}$. In the following, the germ of a smooth family $(f_s)_{s\in[0,1]} : [0,1]\times Y \to \mathbb{R}$ at (s_c, y_c) is called *equivalent to the local model* $(g_t)_{t\in(-1,1)} : (-1,1)\times\mathbb{R}^{n+1} \to \mathbb{R}$ (that is, its germ at $(0,0)$) if there exists a germ of a strictly increasing smooth function $t : [0,1] \to (-1,1)$ at $t(s_c) = 0$ and a germ of a smooth family of local diffeomorphism $(\tau_s)_{s\in[0,1]} : [0,1]\times Y \to \mathbb{R}^{n+1}$ at $\tau_{s_c}(y_c) = 0$ (i.e. $\tau_s : Y \to \mathbb{R}^{n+1}$ is a germ of a diffeomorphism for each s near s_c), and a germ of a smooth function $c : [0,1] \to \mathbb{R}$ at $c(s_c) = f_{s_c}(y_c) - g_0(0)$ such that in a neighbourhood of $(s_c, y_c) \in [0,1]\times Y$

$$f_s(y) = g_{t(s)}(\tau_s(y)) + c(s) \qquad \forall (s,y).$$

(See [Ma, p.151] for an equivalent definition.)

In the following, we will consider smooth functions $f : Y \to \mathbb{R}$ satisfying the boundary minimum/maximum condition (i) of Definition 2.1. Given two such functions f_0, f_1 we denote by $\mathcal{H}(f_0,f_1) \subset \mathcal{C}^\infty([0,1]\times Y,\mathbb{R})$ the space of smooth homotopies $F = (f_s : Y \to [-1,1])_{s\in[0,1]}$ between f_0 and f_1, such that f_s satisfies (i) for each $s \in [0,1]$. We equip $\mathcal{H}(f_0,f_1)$ with the $\mathcal{C}^\infty$-topology.

Theorem 3.5. *Let $f_0, f_1 \in \mathcal{C}^\infty(Y,\mathbb{R})$ satisfy condition (i) of Definition 2.1. Then there exists an open dense subset $\mathcal{H}^{\mathrm{reg}}(f_0,f_1) \subset \mathcal{H}(f_0,f_1)$ of* regular homotopies *such that for any $(f_s)_{s\in[0,1]} \in \mathcal{H}^{\mathrm{reg}}(f_0,f_1)$ and any critical point $y_c \in \mathrm{Crit}(f_{s_c})$ at any $s_c \in [0,1]$, the germ of (f_s) at (s_c,y_c) is equivalent to one of the following local models:*

(a) *a* Morse singularity $g_t(z_1,\ldots,z_{n+1}) = \pm z_1^2 \pm \ldots \pm z_{n+1}^2$;
(b) *a* cusp singularity $g_t(z_1,\ldots,z_{n+1}) = z_1^3 \pm t z_1 \pm z_2^2 \pm \ldots \pm z_{n+1}^2$.

A proof of Theorem 3.5 can be found in e.g. [Ma, p.167]. Note that these local models for a *regular homotopy* imply that the critical point set forms a smooth 1–manifold

$$\overline{\mathrm{Crit}}(f_s) := \{(s,y) \mid y \in \mathrm{Crit(f_s)}\} \;\subset\; [0,1]\times \mathrm{Y}.$$

It is moreover useful to think in terms of the function $F : [0,1]\times Y \to [0,1]\times\mathbb{R}$ defined by $F(s,y) = (s, f_s(y))$. For the regular homotopies of Theorem 3.5, the restriction of F to $\overline{\mathrm{Crit}}(f_s)$ is an immersion with cusps. Its image $F(\overline{\mathrm{Crit}}(f_s)) \subset [0,1]\times\mathbb{R}$ is called the *Cerf graphic*.

The following result was noted as a corollary in [C] without regard to connectedness of fibres.

Theorem 3.6. *Let $f_0, f_1 \in \mathcal{C}^\infty(Y,\mathbb{R})$ be two Morse functions satisfying (i)-(iii) in Definition 2.1. Then there exists a homotopy $(f_s)_{s\in[0,1]}$ in $\mathcal{H}^{\mathrm{reg}}(f_0,f_1)$ such that there are only finitely many special values $0 < c_1 < \ldots < c_\ell < 1$ for which f_s fails to be a Morse function satisfying (i)-(iii) in Definition 2.1. Moreover, for each of the special values $s = c_j$, exactly one of the following holds:*

(a) *f_{c_j} is a Morse function on Y and satisfies (i) and (ii). The values at the critical points of f_{c_j} are pairwise distinct except for one pair of critical points with the same value. These two critical points extend to smooth paths $y_j^\pm : (c_j-\epsilon, c_j+\epsilon) \to Y$ of critical points $y_j^\pm(s) \in$* $\mathrm{Crit(f_s)}$ *such that $f_s(y_j^+(s)) > f_s(y_j^-(s))$ for $s < c_j$ and $f_s(y_j^+(s)) < f_s(y_j^-(s))$ for $s > c_j$.*
(b) *f_{c_j} is a Morse function on the complement of a single point $y_j \in Y \setminus \partial Y$ and satisfies (i)–(iii). At (c_j, y_j) the family (f_s) is equivalent to a cusp singularity, and the values at the nondegenerate critical points of f_{c_j} are distinct from the degenerate critical value $f_{c_j}(y_j)$.*

Proof. To simplify the constructions we consider linearly rescaled Morse functions $g_i(y) = a_i f_i(y) + b_i$ for $i = 0,1$ so that $g_i(Y) \subset [-1,1]$ and $g_i^{-1}(\pm 1) = X_\pm$. These have the same level sets and merely rescaled critical values, so they still satisfy (i)–(iii) with $b_0 = -1$, $b_m = 1$. If $(g_s)_{s\in[0,1]}$ is a homotopy between these with all the claimed properties, then these properties are preserved by subtracting the linear family of constants $sb_0 + (1-s)b_1$ and by multiplication with the family of positive constants $sa_0^{-1} + (1-s)a_1^{-1}$. The result is a homotopy between f_0 and f_1 (given by $f_s(y) = \big(sa_0^{-1} + (1-s)a_1^{-1}\big)\big(g_s(y) - sb_0 - (1-s)b_1\big)\big)$ that satisfies all claimed properties.

Thus we have reduced the proof to the case $\max f_0 = \max f_1 = 1$, $\min f_0 = \min f_1 = -1$. In particular, we can now simply use a linear homotopy $sf_0 + (1-s)f_1$ to satisfy (i) for every $s \in [0,1]$. Next, Theorem 3.6 provides a small perturbation of f_s that still satisfies (i) and whose critical points are given by smooth families of Morse singularities (a) with isolated cusps (b). By compactness, the number of cusps is finite.

To arrange that the level sets are connected throughout, we can use the same arguments as in [GK, Thm.1.4]. First, note that property (i) excludes critical points of indices 0 or $n+1$. Moreover, the level sets below and above a critical point of index k are related by surgery along an S^{k-1} in the level set below. So the only way in which one of the level sets is disconnected while the other one is connected, occurs when the level set below a critical point of index n is connected and the S^n is a disconnecting hypersurface. Since the top level set X_+ is connected throughout, and the surgeries associated to further index n critical points cannot reconnect level sets, we can exclude disconnected level sets by ensuring that the index n critical values are larger than all critical values of index $k < n$. Let us call a Morse function on Y with this property and satisfying (i) an *ordered* Morse function. We just explained why this guarantees the connectedness property (ii) for the level sets. For that purpose it suffices to show how to lower an arc of critical values of index $k < n$ below an arc of critical values of index n. The general principle is that if, with respect to some metric, on some time interval $[s_0, s_1]$, the descending manifolds for a component A of $\overline{\mathrm{Crit}}(f_s)$ are disjoint from the ascending manifolds for another component B, then $F(A)$ can be moved below $F(B)$. However, while f_0, f_1 by assumption have connected level sets, they may not be ordered. So it remains to homotope each f_i within the class of Morse functions satisfying (i) and (ii) to an ordered Morse function.

Finally, to satisfy (iii) and (a), (b) we will use small deformations to ensure that cusps of $F(\overline{\mathrm{Crit}}(f_s))$ appear at different times s and to make the self-intersections of $F|_{\overline{\mathrm{Crit}}(f_s)}$ transverse, avoiding the cusps, with only double points, and at distinct times s from each other and the cusps. This can obviously be done as long as we can (a) move any cusp forward or backward in time and (b) add a small perturbation function to the values of any short arc of non-cusp critical values.

Up to reparametrization (n.b. this reparametrization is special in that the reparametrization of s can not depend on position in Y, whereas the reparametrization in Y may depend on s) in domain and range, there is a chart on $[0,1] \times Y$ near each cusp on which F looks like $(s, z_1, \dots, z_n) \mapsto (s, z_1^3 + sz_1 \pm z_2^2 \pm \dots \pm z_{n+1}^2)$. We move this cusp forward or backward in time by modifying this function inside such a chart so as to interpolate from this form near the boundary of the chart to $(s, z_1^3 + (s+\epsilon)z_1 \pm z_2^2 \pm \dots \pm z_{n+1}^2)$ in a ball at the center of the chart, without introducing any new singularities. This should involve two bump functions, one a function of s and one a function of $(z_1, \dots, z_n)$, i.e. the interpolating function will have the form $(s, z_1^3 + (s + \epsilon\beta(s)\beta(z_1^2 + \dots + z_{n+1}^2))z_1 \pm z_2^2 \pm \dots \pm z_{n+1}^2)$, for some appropriate bump function β. The trick, of course, is to control the derivatives of β so as not to introduce extra critical points.

Up to the same kind of reparametrization as above, in a neighborhood of a short arc of non-cusp critical points F looks like $(s, z_1, \dots, z_{n+1}) \mapsto (s, \pm z_1^2 \pm \dots \pm z_{n+1}^2)$, for $s \in [-\epsilon, \epsilon]$. In this chart, $\overline{\mathrm{Crit}}(f_s) = [-\epsilon, \epsilon] \times \{(0, \dots, 0)\}$ and $F(\overline{\mathrm{Crit}}(f_s)) = [\epsilon, \epsilon] \times \{0\}$. Given a perturbation function $p(s)$, we want to modify f_s so that $\overline{\mathrm{Crit}}(f_s)$ is unchanged but $F(\overline{\mathrm{Crit}}(f_s)) = \{(s, \delta\beta(s)p(s))\}$, where $\delta > 0$ is a small constant and $\beta(s)$ is a bump function on $[-\epsilon, \epsilon]$. Simply adding $\delta\beta(s)\beta(z_1^2 + \dots + z_{n+1}^2)$ should do this without adding new critical points, as long as δ and/or the derivatives of β are kept small.

□

Lemma 3.7. *In Theorem 3.6 above, we can assume (i.e. homotope the homotopy $(f_s)_{s\in[0,1]}$ so as to arrange) that the events (a) and (b) are supported in arbitrarily small balls in the following sense:*

(a) *In case (a) above, we can arrange that the two paths $y^{\pm}$ are constant and that, given any descending disk D for y^+ on $f^{-1}_{c_j-\epsilon}([f_{c_j-\epsilon}(y^-), f_{c_j-\epsilon}(y^+)])$ and any open neighborhood U of D, f_s is independent of s outside D, for $s \in [c_j - \epsilon, c_j + \epsilon]$.*

(b) *In case (b) above, in any ball around y_j, we can arrange for coordinates $(z_1, \dots, z_{n+1})$ around y_j and some $\epsilon > 0$ such that, for $s \in [c_j - \epsilon, c_j + \epsilon]$, f_s is independent of s outside an ϵ–ball neighborhood of c_j in these local coordinates and, inside this ball, $f_s = g_s(z_1) \pm z_2^2 \pm \dots \pm z_{n+1}^2$, where $g_s : [-\epsilon, \epsilon] \to [-\epsilon, \epsilon]$ is a smooth function equal to $x^3 \pm sx$ for $x \in [-\epsilon/3, \epsilon/3]$, equal to x for $x \in [-\epsilon, -2\epsilon/3] \cup [2\epsilon/3, \epsilon]$ and with positive derivative on $[-2\epsilon/3, -\epsilon/3] \cup [\epsilon/3, 2\epsilon/3]$.*

Corollary 3.8. (a) *Suppose that the regular homotopy $(f_s)_{s\in[0,1]} \in \mathcal{H}^{\mathrm{reg}}(f_0, f_1)$ from Theorem 3.5 has no cusp singularity at $s_c \in [0,1]$. Then for some $\delta > 0$ there exist smooth paths $y_1, \dots, y_N : (s_c - \delta, s_c + \delta) \cap [0,1] \to Y$ such that $\mathrm{Crit}(f_s) = \{y_1(s), \dots, y_N(s)\}$ is the disjoint union of these points.*

(b) *Suppose that the regular homotopy $(f_s)_{s\in[0,1]} \in \mathcal{H}^{\mathrm{reg}}(f_0, f_1)$ from Theorem 3.5 has a single cusp singularity at $s_c \in [0,1]$, namely at (s_c, y_c) with local model $z_1^3 - tz_1 + \sum \pm z_j^2$. Then for some $\delta > 0$ there exist smooth paths $y_1, \dots, y_N : (s_c - \delta, s_c + \delta) \cap [0,1] \to Y$ and $y_0^-, y_0^+ : (s_c, s_c + \delta) \cap [0,1] \to Y$ with $\lim_{s\to s_c} y_0^{\pm}(s) = y_c$ such that the critical sets $\mathrm{Crit}(f_s) = \{y_0^-(s), y_0^+(s), y_1(s), \dots, y_N(s)\}$ for $s > s_c$, $\mathrm{Crit}(f_s) = \{y_c, y_1(s), \dots, y_N(s)\}$ for $s = s_c$, and $\mathrm{Crit}(f_s) = \{y_1(s), \dots, y_N(s)\}$ for $s < s_c$ are the disjoint union of these points.*

(c) *If in (b) the single cusp singularity has local model $z_1^3 + tz_1 + \sum \pm z_j^2$, then there exist $y_1, \dots, y_N$ as above and $y_0^-, y_0^+ : (s_c - \delta, s_c] \cap [0,1] \to Y$ that analogously parametrize the critical sets for $s < s_c$, $s = s_c$, and $s > s_c$.*

Proof. In the local model of a Morse singularity, the critical point is given by the constant path $y_i(t) = 0 \in \mathbb{R}^{n+1}$ for each $t \in (-1,1)$, which is equivalent to the germ of a smooth path in Y. In the local model of a cusp singularity $z_1^3 - tz_1 + \sum \pm z_j^2$ the critical sets for fixed $t \in (-1,1)$ are two nondegenerate critical points $y_0^{\pm}(t) = (\pm\sqrt{t/3}, 0, \dots, 0) \in \mathbb{R}^{n+1}$ for $t > 0$, a degenerate critical point $\lim_{t\to 0} y_0^{\pm}(t) = 0 \in \mathbb{R}^{n+1}$ for $t = 0$, and none for $t < 0$. These are equivalent to two germs of smooth paths in Y defined for $s > s_c$, with coinciding limits for $s \to s_c$, and disjoint for $s > s_c$. Similarly, in the local model of a cusp singularity $z_1^3 + tz_1 + \sum \pm z_j^2$ the critical points are $(\pm\sqrt{-t/3}, 0, \dots, 0)$ for $t \le 0$.

In this way we define paths $y_0, y_1, \dots, y_N : (s_c - \delta, s_c + \delta) \cap [0,1] \to Y$ for some common $\delta > 0$, one for each critical point $y_0(0), y_1(0), \dots, y_N(0)$ of f_{s_c}. We moreover find disjoint open neighbourhoods of $(s_c, y_\iota(0)) \in [0,1] \times Y$ for each $\iota = 0, \dots, N$ in which the only critical points are those given by the path y_ι. It remains to find a possibly smaller $\delta > 0$ such that the paths $y_0, \dots, y_N$ are the only critical points of f_s with $|s - s_c| < \delta$. If there was no such δ, then we would find a sequence of critical points $x^\nu \in \mathrm{Crit} f_{s^\nu} \setminus \{y_0(s^\nu), \dots, y_N(s^\nu)\}$ for $s^\nu \to s_c$ as $\nu \to \infty$. By compactness of Y we can find a convergent subsequence $x^{\nu_j} \to x^\infty \in Y$. By continuity of ∇f, the limit x^∞ is a critical point of f_{s_c}, and hence coincides with some $y_\iota(0)$. For sufficiently large j, the points x^{ν_j} now lie in the neighbourhood of $y_\iota(0)$ where the only critical points are given by y_ι, in contradiction to $x^{\nu_j} \neq y_\iota(s^{\nu_j})$. This finishes the proof. □

Proof of Theorem 3.4. By Remark 2.4 we can find Morse data $(f, \underline{b})$ and $(f', \underline{b}')$ for Y which induce the given Cerf decompositions, that is $Y_i = f^{-1}[b_{i-1}, b_i]$, $X_i = f^{-1}(b_i)$, and $Y_i' = f'^{-1}[b_{i-1}', b_i']$, $X_i' = f'^{-1}(b_i')$. Next, we choose a homotopy $(f_s)_{s\in[0,1]}$ of functions between $f_0 = f$ and $f_1 = f'$ as in Theorem 3.6. Then we pick times $s_0, \dots, s_{2\ell+1}$ with

$$0 = s_0 < c_1 < s_1 < s_2 < c_2 < \dots < s_{2j} < c_j < s_{2j+1} < \dots < s_{2\ell} c_\ell < s_{2\ell+1} = 1$$

such that s_{2j}, s_{2j+1} is close to c_j in a sense to be specified below. For each intermediate time s_j, $j = 1, \dots, \ell - 1$ we can pick a tuple $\underline{b}_j$ of regular levels as in Remark 2.5, making $(f_{s_j}, \underline{b}_j)$ a Morse datum, and hence inducing a Cerf decomposition of Y. This provides a finite sequence of Morse data relating $(f, \underline{b})$ for $j = 0$ to $(f', \underline{b}')$ for $j = \ell$. It now suffices to prove that the two Cerf decompositions of Y induced by any two consecutive Morse data in this sequence are related by a sequence of Cerf moves. That is, it suffices to prove the Theorem under the assumption that there exists a homotopy $(f_s)_{s\in[0,1]}$ of functions between $f_0 = f$ and $f_1 = f'$ as in Theorem 3.6 with at most one special time.

If the homotopy (f_s) has no special time, then it is in fact a homotopy within the class of Morse functions with distinct values at the critical points, and we know from Corollary 3.8 that the critical points and hence critical values are tuples of disjoint points resp. distinct real values that vary smoothly with $s \in [0, 1]$. We may hence extend the tuple $\underline{b} = (b_i(0))_{i=0,\dots,m}$ of regular values separating the critical points of f_0 to a tuple of smooth functions $(b_i : [0, 1] \to \mathbb{R})_{i=0,\dots,m}$ such that $\min f_s = b_0(s) < b_1(s) < \dots < b_m(s) = \max f_s$ are regular values of f_s for each $s \in [0, 1]$. In particular, the families (f_s) and $(\underline{b}(s))$ satisfy the assumptions of Lemma 3.1, and hence the two Cerf decompositions induced by $(f_0 = f, \underline{b}(0) = \underline{b})$ and $(f_1 = f', \underline{b}(1))$ are diffeomorphism equivalent, i.e. related by Cerf move (d). Finally, let $\underline{b}'' = \underline{b}(1) \cup \underline{b}'$ be the strictly ordered tuple obtained from the union of the two tuples of regular values. Then $(f', \underline{b}'')$ is yet another Morse datum for Y, and now the Cerf decompositions induced by $(f', \underline{b}')$ as well as $(f', \underline{b}(1))$ can be obtained from the Cerf decomposition induced by $(f', \underline{b}'')$ by a sequence of cylinder cancellations, i.e. Cerf moves (c). More precisely, for each $i = 1, \dots m'$ we have $b'_{i-1} = b''_j, b'_i = b''_{j+k}$ for some integers j, k, and

$$f'^{-1}[b'_{i-1}, b'_i] = f'^{-1}[b''_j, b''_{j+1}] \cup_{f'^{-1}(b''_{j+1})} \dots \cup_{f'^{-1}(b''_{j+k-1})} f'^{-1}[b''_{j+k-1}, b''_{j+k}],$$

where at most one morphism on the right hand side is not cylindrical. The analogous relation holds with $\underline{b}'$ replaced by $\underline{b}(1)$. Summarizing, we obtain the $(f', \underline{b}')$ Cerf decomposition of Y from the $(f, \underline{b})$ Cerf decomposition by a diffeomorphism equivalence, followed by a sequence of cylinder creations and a sequence of cylinder cancellations.

It remains to consider a homotopy (f_s) with one special time $0 < c = c_1 < 1$ and either one of the cases in Theorem 3.6 – a critical point crossing as in (a) or a cusp singularity as in (b). In both cases, let us first drop entries from the tuple $\underline{b}$ to define $\underline{b}(0)$ such that each interval $[b_{i-1}(0), b_i(0)]$ contains exactly one critical point of $f_0 = f$. On the level of Cerf decompositions, this is reflected in a sequence of cylinder cancellations. Now extending $\underline{b}(0)$ as above to a smooth family $s \mapsto \underline{b}(s)$ of ordered tuples $\underline{b}(s) = (\min f_s = b_0(s) < b_1(s) < \dots < b_m(s) = \max f_s)$ of regular values separating the critical points of f_s is possible for $s \in [0, c)$.

In case (a) of critical point crossing, for $s \to c$ one of the critical levels $b_k(s) \in \big(f_s(y^-(s)), f_s(y^+(s))\big)$ is forced to converge to a critical value $b_k(c) = f_s(y^+(c)) = f_s(y^-(c))$. All other levels $b_i(s)$, $i \neq k$ can be extended smoothly as regular levels to $s \geq c$. For the one level meeting the critical value, we may choose $b_k(s) = \frac{1}{2}(f_s(y^+(c)) + f_s(y^-(c)))$ for $|s - c|$ small. Then for $s > c$ this again becomes a regular level at least for a short time. That way we construct $\underline{b}(s)$ for small $s - c > 0$ that are again ordered tuples of regular values, with exactly one critical value in each interval $[b_{i-1}(s), b_i(s)]$. As such they can be extended smoothly to $(c, 1]$.

Next, dropping the level $b_k(s)$, the families (f_s) and $\underline{b}^\vee(s) := (b_1(s), \dots, b_{k-1}(s), b_{k+1}(s), \dots, b_m(s))$ are no Morse data anymore, but still satisfy the assumptions of Lemma 3.1 and hence induce a diffeomorphism $\psi : Y \to Y$ which preserves the level sets $\psi(X_i^0) = X_i^1$ for $i = 0, \dots, k-1, k+1, \dots, m$, where $X_i^s := f_s^{-1}(b_i(s))$. Let us denote $Y_i^0 := f_0^{-1}[b_{i-1}(0), b_i(0)]$ and $Y_i^1 := f_1^{-1}[b_{i-1}(1), b_i(1)]$, then ψ restricts to diffeomorphisms $Y_i^0 \to Y_i^1$ for $i = 1, \dots, k-1, k+2, \dots, m$ and a diffeomorphism

$$Y_k^0 \cup_{X_k^0} Y_{k+1}^0 = f_0^{-1}[b_{k-1}(0), b_{k+1}(0)] \xrightarrow{\sim} f_1^{-1}[b_{k-1}(1), b_{k+1}(1)] = Y_k^1 \cup_{X_k^1} Y_{k+1}^1$$

preserving the boundaries $\psi(X_i^0) = X_i^1$ for $i = k \pm 1$. Note here that $\psi^{-1}(Y_k^1)$ and $\psi^{-1}(Y_{k+1}^1)$ are both simple cobordisms since they support the Morse function $\psi^* f_1$. Hence we obtain a new Cerf decomposition

$$Y = Y_1^0 \cup_{X_1^0} \ldots \cup_{X_{k-1}^0} \psi^{-1}(Y_k^1) \cup_{\psi^{-1}(X_k^1)} \psi^{-1}(Y_{k+1}^1) \cup_{X_{k+1}^1} \ldots \cup_{X_{m-1}^0} Y_m^0. \tag{1}$$

This differs from the decomposition $Y = Y_1^0 \cup_{X_1^0} \ldots \cup_{X_{m-1}^0} Y_m^0$ induced by $(f_0, \underline{b}(0))$ by a critical point switch on $Y_k^0 \cup_{X_k^0} Y_{k+1}^0$. Indeed, on this cobordism the decompositions are given by the Morse data $(f_0, (b_{k-1}(0), b_k(0), b_{k+1}(0)))$ and $(\psi^* f_1, (b_{k-1}(1), b_k(1), b_{k+1}(1)))$. Each of these has a unique critical point $y_k \in Y_k^0, y_{k+1} \in Y_{k+1}^0, y_k' \in \psi^{-1}(Y_k^1), y_{k+1}' \in \psi^{-1}(Y_{k+1}^1)$ in each elementary piece. The critical set of f_s contains smooth paths from y_k to $\psi(y_{k+1}')$, and from y_{k+1} to $\psi(y_k')$. Now fix a metric g on Y. We may assume, possibly after shrinking the time interval around c, that the stable and unstable manifolds of the two families of critical points are disjoint, since they are disjoint at time c. In particular, the attaching cycles are disjoint in the middle surfaces X_k^0 as well as in X_k^1 and hence the attaching cycles of y_k' and y_{k+1}' with respect to $\psi^* g$ are disjoint in $\psi^{-1}(X_k^1)$. The families of stable and unstable manifolds induce smooth families of attaching cycles in X_{k-1}^s and X_{k+1}^s, which can be pulled back by $\psi_s : X_{k\pm1}^0 \to X_{k\pm1}^s$ to provide homotopies between e.g. the attaching cycle of y_k with respect to g and the attaching cycle of y_{k+1}' with respect to $\psi^* g$. For sufficiently small time interval around c, the metrics $g, \psi^* g$ are C^1-close and so an isotopy of metrics induces an isotopy of attaching cycles without introducing intersection points.

Next, the Cerf decomposition (1) differs from the decomposition $Y = Y_1^1 \cup_{X_1^1} \ldots \cup_{X_{m-1}^1} Y_m^1$ induced by $(f_1, \underline{b}(1))$ by the diffeomorphism equivalence for the diffeomorphism $\psi : Y \to Y$. Finally, the Cerf decompositions induced by the Morse data $(f_1, \underline{b}(1))$, $(f_1, \underline{b}(1) \cup \underline{b}')$, and $(f_1 = f', \underline{b}')$ are – as above – obtained from each other by sequences of cylinder cancellations and cylinder creations. Summarizing, in this case we obtain the $(f', \underline{b}')$ Cerf decomposition of Y from the $(f, \underline{b})$ Cerf decomposition by cylinder cancellations, a critical point switch (b), a diffeomorphism equivalence, cylinder creations, and cylinder cancellations.

In case (b) of a cusp singularity, suppose for now that the local model is $z_1^3 - tz_1 + \sum \pm z_j^2$. As before, we find a family $s \mapsto \underline{b}(s)$ of ordered tuples $\underline{b}(s) = (\min f_s = b_0(s) < b_1(s) < \ldots < b_m(s) = \max f_s)$ of regular values separating the critical points of f_s for $s \in [0, c)$. Now as $s \to c$, there is one regular value $b_k(s) \in \big(f_s(y_0^-(s)), f_s(y_0^+(s))\big)$ that is forced to converge to the degenerate critical value $b_k(c) = f_s(y_c)$. All other levels $b_i(s)$, $i \neq k$ can be extended smoothly as regular levels to $s \geq c$. Here $y_0^\pm : [0, c) \to Y$ are the smooth paths of critical points converging to the degenerate point $\lim_{s\to c} y_0^\pm(s) = y_c$, as given by Corollary 3.8. Using the local model we also see that these two critical values for $s < c$ are

$$f_s(y_0^\pm(s)) = g_{t(s)}(\pm\sqrt{t(s)/3}, 0, \ldots, 0) = \mp\tfrac{2}{3} t(s)\sqrt{t(s)/3} + c(s)$$

for some smooth, strictly increasing $t : [0,1] \to (-1,1)$ with $t(c) = 0$ and a smooth function $c : [0,1] \to \mathbb{R}$. We may hence choose the intermediate level as $b_k(s) = c(s)$, depending smoothly on s for $|s - c|$ small. For $s > c$ this again becomes a regular level at least for a short time. That way we construct $\underline{b}(s)$ for small $s - c > 0$ that are again ordered tuples of regular values. As such they can be extended smoothly to $(c, 1]$. For $i \neq k, k+1$ there again is exactly one critical value in each interval $[b_{i-1}(s), b_i(s)]$. However, there is no critical value in the intervals $[b_{k-1}(s), b_{k+1}(s)]$. So with the tuple $\underline{b}^\vee(s) := (b_1(s), \ldots, b_{k-1}(s), b_{k+1}(s), \ldots, b_m(s))$ we have further Morse data $(f_s, \underline{b}^\vee(s))$ for $s \geq c$.

Now consider the decomposition $(f^{-1}[b_{i-1}(0), b_i(0)])_{i=1,\ldots,k-1,k+1,\ldots,m}$ of Y induced by $(f_0 = f, \underline{b}^\vee(0))$. Despite this not being a Morse datum, we will see that it is a Cerf decomposition since $Y_k^0 := f^{-1}[b_{k-1}(0), b_{k+1}(0)]$ is a cylindrical cobordism. For this purpose note that the families (f_s) and $(\underline{b}^\vee(s))$ satisfy the assumptions of Lemma 3.1 and hence induce a diffeomorphism $\psi : Y \to Y$ which preserves the level sets. In particular, it restricts to a diffeomorphism $Y_k^0 = f_0^{-1}[b_{k-1}^\vee(0), b_k^\vee(0)] \to f_1^{-1}[b_{k-1}^\vee(1), b_k^\vee(1)] =: Y_k^1$ preserving the boundaries $\psi(f_0^{-1}(b_i^\vee(0)) = f_1^{-1}(b_i^\vee(1))$ for $i = k-1, k$. Recall that $(f_1, (b_{k-1}^\vee(1), b_k^\vee(1)))$ is a Morse datum for Y_k^1, hence the pullback $(\psi^* f_1, (b_{k-1}^\vee(1), b_k^\vee(1)))$ is a Morse datum on Y_k^0, showing that it indeed is cylindrical. Hence the two Cerf decompositions $(f^{-1}[b_{i-1}(0), b_i(0)])_{i=1,\dots,m}$ and $(f^{-1}[b_{i-1}(0), b_i(0)])_{i=1,\dots,k-1,k+1,\dots,m}$ are related by critical point cancellation, i.e. Cerf move (a). Lemma 3.1 moreover implies that the Cerf decompositions $(f^{-1}[b_{i-1}(0), b_i(0)])_{i=1,\dots,k-1,k+1,\dots,m} = (f_0^{-1}[b_{i-1}^\vee(0), b_i^\vee(0)])_{i=1,\dots,m-1}$ and $(f_1^{-1}[b_{i-1}^\vee(1), b_i^\vee(1)])_{i=1,\dots,m-1}$ are diffeomorphism equivalent. Finally, the Cerf decompositions induced by the Morse data $(f_1, \underline{b}^\vee(1))$, $(f_1, \underline{b}^\vee(1) \cup \underline{b}')$, and $(f_1 = f', \underline{b}')$ are obtained from each other by sequences of cylinder cancellations and cylinder creations. Summarizing, in this case we obtain the $(f', \underline{b}')$ Cerf decomposition of Y from the $(f, \underline{b})$ Cerf decomposition by cylinder cancellations, a critical point cancellation (a), a diffeomorphism equivalence, cylinder creations, and cylinder cancellations.

In case (b) of a cusp singularity with local model $z_1^3 + tz_1 + \sum \pm z_j^2$ we reverse the process in $s \in [0,1]$ and start by dropping unnecessary levels from $\underline{b}'$ to define $\underline{b}'(1)$, then continue the regular levels to $\underline{b}'(s)$ for $s > c$, smoothly continue them through $s = c$, extend as regular levels to $s < c$, and drop one level to define $\underline{b}^\vee(s)$ such that $(f_s, \underline{b}^\vee(s))$ is a Morse datum for $s \le c$. Then we obtain the $(f', \underline{b}')$ Cerf decomposition of Y from the $(f, \underline{b})$ Cerf decomposition by cylinder creations and cylinder cancellations from $(f = f_0, \underline{b})$ to $(f_0, \underline{b}^\vee(0))$, a diffeomorphism equivalence from $(f_0, \underline{b}^\vee(0))$ to $(f_1, \underline{b}^\vee(1))$, a critical point creation (a) from $(f_1, \underline{b}^\vee(1))$ to $(f_1, \underline{b}'(1))$, and cylinder creations from $(f_1, \underline{b}'(1))$ to $(f' = f_1, \underline{b}')$. □

4. Cyclic Cerf decompositions

In this section we discuss the Cerf theory arising from S^1-valued Morse functions on closed manifolds. In the following, Y is a closed, connected, oriented manifold of dimension $n+1$.

Definition 4.1. A *cyclic Morse datum* for Y consists of a pair $(f, \underline{b})$ of a smooth function $f : Y \to S^1 \cong \mathbb{R}/\mathbb{Z}$ and a tuple of *levels* $\underline{b} = (b_1, \dots, b_m) \subset (S^1)^m$ such that

- (i) The function $f : Y \to S^1$ is Morse in the sense that its critical points are nondegenerate.
- (ii) The levels $\underline{b}$ form a cyclically ordered tuple in the following double sense: There exist representatives $(b_i)_{i=1,\dots,m} \subset \mathbb{R}^m$ such that $b_1 < \ldots < b_m < b_1 + 1$. Moreover, we identify the tuple $\underline{b} = (b_1, \dots, b_m)$ with $(b_{k+1}, \dots, b_m, b_1, \dots, b_k)$ for any $k \in \{1, \dots, m-1\}$, and for a fixed tuple we will use the notation b_i for $i \in \mathbb{N}$ modulo m, that is $b_{i+m} = b_i$.
- (iii) Each level set $f^{-1}(b)$ for $b \in S^1$ is connected.
- (iv) f has distinct values at the (isolated) critical points, i.e. it induces a bijection $\mathrm{Crit} f \to f(\mathrm{Crit} f)$ between critical points and critical values.
- (v) $b_1, \dots, b_m \in S^1 \setminus f(\mathrm{Crit} f)$ are regular values of f such that each interval $(b_{i-1}, b_i) \subset S^1$ for $i \in \mathbb{N}$ modulo m contains at most one critical value of f.

Definition 4.2. A *cyclic Cerf decomposition* of a closed manifold Y as above is a decomposition $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_m$ into a cyclic sequence $(Y_i \subset Y)_{i \in \mathbb{N} \bmod m}$ of elementary cobordisms embedded in Y that are disjoint from each other except for the intersections $X_i := Y_i \cap Y_{i+1}$ for $i \in \mathbb{N}$ modulo m, which are also connected submanifolds in Y of codimension 1. As a consequence we have $\partial Y_i = X_{i-1} \sqcup X_i$. We also denote this decomposition by

$$Y = Y_1 \cup_{X_1} Y_2 \cup_{X_2} \ldots \cup_{X_{m-1}} Y_m \cup_{X_m} \ldots \quad .$$

Remark 4.3. As in Remark 2.4, any cyclic Morse datum $(f, \underline{b})$ for Y induces a cyclic Cerf decomposition $Y = Y_1 \cup_{X_1} \ldots \cup_{X_{m-1}} Y_m \cup_{X_m} \ldots$ into a sequence $(Y_i := f^{-1}([b_{i-1}, b_i]))_{i=1,\dots m}$ of elementary cobordisms between connected level sets $X_i = f^{-1}(b_i)$. On the other hand, any cyclic Cerf decomposition arises from a cyclic Morse datum. To prove the latter, one first views Y as cobordism from X_m to X_m and constructs a real valued Morse datum using Lemma 2.6. In a last step, one shifts and rescales to achieve $f(Y) = [b_1 = 0, b_m = 1]$ and then uses the constructions of Lemma 2.6 to glue f into a smooth S^1-valued Morse function.

The space of homotopy classes of maps from Y to S^1 is in bijection with $H^1(Y, \mathbb{Z})$, since S^1 is an Eilenberg-Maclane space $K(\mathbb{Z}, 1)$. Given a Morse datum $(f, \underline{b})$ we will refer to $[f] \in [Y, S^1]$ as its homotopy class.

Lemma 4.4. *Given any homotopy class $\rho \in [Y, S^1]$ such that the induced map $\rho_* : \pi_1(Y) \to \pi_1(S^1)$ is surjective, there exists a cyclic Morse datum $(f, \underline{b})$ on Y with $[f] = \rho$.*

Proof. The only issue in generalizing Lemma 2.5 to this cyclic case is to arrange for connected level sets, and it is easy to see that π_1–surjectivity is a necessary condition. This proof is just a slight refinement of the Thom-Pontrjagin construction and appears in detail in [GK, Thm.1.3]; here we present a sketch. First we represent ρ by a map with at least one connected regular level set and then appeal to Lemma 2.5 to construct the map on the complementary connected cobordism. To see that there exists a map with one connected regular level set, choose an arbitary generic smooth map $g : Y \to S^1$ representing ρ with regular level set $F = g^{-1}(1) \subset Y$. By π_1–surjectivity, we can connect the components of F in Y with disjoint arcs that project to homotopically trivial loops in S^1. We then show how to homotope g in a neighborhood of each arc so as to modify $g^{-1}(1)$ via connect sums (tubing) along the arcs. We also observe that this homotopy does not introduce new components of $g^{-1}(1)$ at intermediate times during the homotopy; this is important for the proof of uniqueness up to Cerf moves in Theorem 4.7. □

Remark 4.5. Under Poincaré duality $[Y, S^1] \cong H^1(X, \mathbb{Z}) \cong H_2(X, \mathbb{Z})$ the homotopy class $[f]$ of a cyclic Morse datum corresponds to the homology class $[f^{-1}(b_i)]$, which is independent of the choice of regular level b_i. If $[f]$ is π_1-surjective, then the complement $Y \setminus f^{-1}(b_i)$ of any regular level set is connected.

Conversely, if $Y = Y_1 \cup_{X_1} \ldots \cup_{X_{m-1}} Y_m \cup_{X_m} \ldots$ is a cyclic Cerf decomposition, then the homology class $[X_j]$ is independent of j and via Poincaré duality determines the homotopy class of any cyclic Morse datum inducing this Cerf decomposition. More generally, any embedded (not necessarily connected) surface $X \hookrightarrow Y$ determines a cohomology class $[X]$ and hence a homotopy class $\rho_X \in [Y, S^1]$. In this setting, connectedness of $Y \setminus X$ is equivalent to ρ_X being π_1-surjective.

Lemma 4.4 shows that π_1-surjectivity of $\rho \in [Y, S^1]$ is in fact equivalent to the existence of a connected, embedded surface $X \hookrightarrow Y$ with $\rho = \rho_X$.

Definition 4.6. Let Y be a closed manifold as before. A *Cerf move* from one Cerf decomposition $Y = Y_1 \cup_{X_1} \cup \ldots \cup_{X_{m-1}} Y_m \cup_{X_m} \ldots$ to another $Y = Y'_1 \cup_{X'_1} \cup \ldots \cup_{X'_{m'-1}} Y'_{m'} \cup_{X'_{m'}} \ldots$ is one of the following operations.

(a) A *critical point cancellation or creation* analogous to Definition 3.2.
(b) A *critical point switch* analogous to Definition 3.2.
(c) A *cylinder cancellation* analogous to Definition 3.2.
(d) A *diffeomorphism equivalence* is the move

$$\begin{aligned} \text{from} \quad & Y = Y_1 \cup_{X_1} \ldots \cup_{X_{m-1}} Y_m \cup_{X_m} \ldots \\ \text{to} \quad & Y = \psi(Y_1) \cup_{\psi(X_1)} \ldots \cup_{\psi(X_{m-1})} \psi(Y_m) \cup_{\psi(X_m)} \ldots, \end{aligned}$$

where $\psi : Y \to Y$ is a diffeomorphism. More precisely, in this situation, diffeomorphism equivalence is the move from $(Y_i)_{i=1,\ldots,m}$ to $(Y'_i = \psi(Y_i))_{i=1,\ldots,m}$.

Theorem 4.7 (cyclic Cerf theory)**.** *Let Y be a closed manifold as before, and let $Y = Y_1 \cup_{X_1} \cup \ldots \cup_{X_{m-1}} Y_m \cup_{X_m} \ldots$ be a cyclic Cerf decomposition. Then any other cyclic Cerf decomposition*

$Y = Y'_1 \cup_{X'_1} \cup \ldots \cup_{X'_{m'-1}} Y'_{m'} \cup_{X'_{m'}} \ldots$ *of* Y *with* $[X_1] = [X'_1]$ *can be obtained from* $(Y_i)_{i=1,\ldots,m}$ *by a finite sequence of Cerf moves.*

Proof. Let $f, f' : Y \to S^1$ be Morse functions associated with the given Cerf decompositions. Then the existence of a sequence of Cerf moves between them is the content of [GK, Thm.1.4] and, again, we just sketch the proof here. Choose a generic homotopy f_s from $f_0 = f$ to $f_1 = f'$ and assume that, for each s, either 1 or -1 is a regular value of f_s. Note that the argument from the proof of Lemma 4.4 can be improved so that on overlapping time intervals $[s_0 = 0, s_2], [s_1, s_3], [s_2, s_4], \ldots [s_{k-2}, s_k = 1]$ we alternately arrange for $f_s^{-1}(1)$ and $f_s^{-1}(-1)$ to be connected and stationary in Y. This is based on the observation that the homotopy in Lemma 4.4 can be chosen to not add extraneous components at intermediate times. Then a *zig-zag argument* discussed in the proof of [GK, Thm.1.4] shows how to go back and forth between $f_s^{-1}(1)$ and $f_s^{-1}(-1)$, arranging that successive cobordisms between these level sets are connected with connected level sets. Theorem 3.4 then gives the rest of the proof. We can of course always arrange that there are at least three cobordisms in each decomposition, to avoid complications associated with nontrivial monodromy. □

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