Anti-self-dual instantons with Lagrangian boundary conditions I: Elliptic theory – with corrected proofs

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Abstract: We study nonlocal Lagrangian boundary conditions for anti-self-dual instantons on 4-manifolds with a space-time splitting of the boundary. We establish the basic regularity and compactness properties (assuming L^p-bounds on the curvature for $p > 2$) as well as the Fredholm theory in a compact model case. The motivation for studying this boundary value problem lies in the construction of an instanton Floer homology for 3-manifolds with boundary. The present paper is part of a program proposed by Salamon for the proof of the Atiyah-Floer conjecture for homology-3-spheres. The proofs required some minor corrections due to corrections in the analysis for Banach space valued holomorphic curves [W2].

1. Introduction

Let X be a manifold with boundary, let G be a compact Lie group, and consider a principal G-bundle $P \to X$. The natural boundary condition for the Yang-Mills equation $\mathrm{d}_A^* F_A = 0$ on P is $*F_A|_{\partial X} = 0$. (These are the Euler-Lagrange equations for the energy functional $\int |F_A|^2$.) For this boundary value problem there are regularity and compactness results, see for example [U1, U2, W1]. Every solution is gauge equivalent to a smooth solution, and Uhlenbeck compactness holds: Every sequence of solutions with L^p-bounded curvature (where $p > \frac{1}{2}\dim X$) contains a subsequence that is $\mathcal{C}^\infty$-convergent up to a sequence of gauge transformations.

On an oriented 4-manifold, the anti-self-dual instantons, i.e. connections satisfying $F_A + *F_A = 0$, are special first order solutions of the Yang-Mills equation. An important application of Uhlenbeck's theorem is the compactification of the moduli space of anti-self-dual instantons over a manifold without boundary, leading to the Donaldson invariants of smooth 4-manifolds [D1] and to the instanton Floer homology groups of 3-manifolds [Fl].

On a 4-manifold with boundary the boundary condition $*F_A|_{\partial X} = 0$ for anti-self-dual instantons implies that the curvature vanishes altogether at the boundary. This is an overdetermined boundary value problem comparable to Dirichlet boundary conditions for holomorphic maps. As in the latter case it is natural to consider weaker Lagrangian boundary conditions. The Cauchy-Riemann equation becomes elliptic when augmented with Lagrangian or more generally totally real boundary conditions. We consider a version of such Lagrangian boundary conditions for anti-self-dual instantons on a 4-manifold with a space-time splitting of the boundary, and prove that they suffice to obtain the analogue of the above mentioned regularity and compactness results for Yang-Mills connections.

More precisely, we consider oriented 4-manifolds X such that each connected component of the boundary ∂X is diffeomorphic to $\mathcal{S} \times \Sigma$, where $\mathcal{S}$ is a 1-manifold and Σ is a closed Riemann surface. We shall study a boundary value problem associated to a gauge invariant Lagrangian submanifold $\mathcal{L}$ of the space of connections on Σ: The restriction of the anti-self-dual instanton to each time-slice of the boundary is required to belong to $\mathcal{L}$. This boundary condition arises naturally from examining the Chern-Simons functional on a 3-manifold Y with boundary Σ. Namely, the Lagrangian boundary condition renders the Chern-Simons 1-form on the space of connections closed, see [S]. The resulting gradient flow equation leads to the boundary value problem studied in this paper (for the case $X = \mathbb{R} \times Y$). Besides the regularity and compactness properties on noncompact manifolds we also establish the Fredholm theory for the compact model case $X = S^1 \times Y$.

One motivation for studying the present boundary value problem lies in the Atiyah-Floer conjecture for Heegaard splittings of a homology-3-sphere: A Heegaard splitting $Y = Y_0 \cup_\Sigma Y_1$ of a homology 3-sphere Y into two handlebodies Y_0 and Y_1 with common boundary Σ gives rise to two Floer homologies (i.e. generalized Morse homologies) as follows: Firstly, the moduli space M_Σ of gauge equivalence classes of flat connections on the trivial $\mathrm{SU}(2)$-bundle over Σ is a symplectic manifold (with singularities) and the moduli spaces L_{Y_i} of flat connections over Σ that extend to a flat connection over Y_i are (singular) Lagrangian submanifolds of M_Σ as explained in [W2]. The symplectic Floer homology $\mathrm{HF}_*^{\mathrm{symp}}(M_\Sigma, L_{Y_0}, L_{Y_1})$ is now generated by the intersection points of the Lagrangian submanifolds, and the generalized connecting orbits (that define the boundary operator) are pseudoholomorphic strips with boundary values in the two Lagrangian submanifolds. (In view of the singularities of M_Σ, an appropriate generalization of the concept of pseudoholomorphic strips will be required to give a strict definition of this Floer homology.) It was conjectured by Atiyah [A2] and Floer that this should be isomorphic to the instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y)$. For the latter, the critical points are the flat $\mathrm{SU}(2)$-connections over Y. These are the actual critical points of the Chern-Simons functional, and the connecting orbits are given by its generalized flow lines, i.e. anti-self-dual instantons on $\mathbb{R} \times Y$.

The program by Salamon [S] for the proof of this conjecture is to define the instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y, L)$ for 3-manifolds with boundary $\partial Y = \Sigma$ using boundary conditions associated to a Lagrangian submanifold $L \subset M_\Sigma$. Then the conjectured isomorphism might be established in two steps via the intermediate Floer homology $\mathrm{HF}_*^{\mathrm{inst}}([0,1] \times \Sigma, L_{Y_0} \times L_{Y_1})$, as described in the outlook below.

Boundary value problems for (Hermitian) Yang-Mills connections were also used by Donaldson [D2], who considered connections induced by Hermitian holomorphic bundles with a Dirichlet boundary condition on the metric.

Fukaya [Fu] was the first to suggest the use of Lagrangian boundary conditions for anti-self-dual instantons in order to define a Floer homology for 3-manifolds with boundary. He studies a slightly different equation, involving a degeneration of the metric in the anti-self-duality equation, and uses $\mathrm{SO}(3)$-bundles that are nontrivial over the boundary Σ, so the moduli space M_Σ becomes smooth. However, when working on handlebodies as 3-manifolds, or when considering the Lagrangian submanifold L_Y as in the Atiyah-Floer conjecture, then one necessarily deals with the trivial bundle (or a non-connected Lie group).

The present paper sets up the basic analysis for a construction of $\mathrm{HF}_*^{\mathrm{inst}}(Y,L)$ as outlined in [S], using trivial $\mathrm{SU}(2)$-bundles. We will only consider trivial G-bundles for general compact Lie groups G. However, our main theorems A, B, and C below generalize directly to nontrivial bundles – just the notation becomes more cumbersome. The main theorems are described below; they are proven in sections 2 and 3. The appendix reviews the regularity theory for the Neumann and Dirichlet problem in the weak formulation that will be needed throughout this paper. Here we moreover introduce a technical tool for extracting regularity results for single components of a 1-form from weak equations that are related to a combination of Neumann and Dirichlet problems.

1.1. Notation and main results. Throughout this paper, we consider the trivial G-bundle over a 4-manifold X. Here G is a compact Lie group with Lie algebra $\mathfrak{g}$. We denote the Lie bracket on $\mathfrak{g}$ by $[\cdot,\cdot]$, and we equip $\mathfrak{g}$ with a G-invariant inner product $\langle\,\cdot,\cdot\,\rangle$. A connection on the trivial bundle $\mathrm{G}\times X$ is a $\mathfrak{g}$-valued 1-form $A\in\Omega^1(X;\mathfrak{g})$. We denote the space of smooth connections by $\mathcal{A}(X):=\Omega^1(X;\mathfrak{g})$. Associated to a connection $A\in\mathcal{A}(X)$ one has the exterior derivative d_A on $\mathfrak{g}$-valued differential forms given by

$$\mathrm{d}_A\eta=\mathrm{d}\eta+[A\wedge\eta]\qquad\forall\eta\in\Omega^k(X;\mathfrak{g}).$$

Here the Lie bracket indicates how the values of the differential forms are paired. Now $\mathrm{d}_A\circ\mathrm{d}_A$ does not necessarily vanish, but it is a zeroeth order operator, $\mathrm{d}_A\mathrm{d}_A\eta=[F_A\wedge\eta]$ given by the curvature

$$F_A=\mathrm{d}A+\tfrac{1}{2}[A\wedge A]\;\in\Omega^2(X;\mathfrak{g}).$$

So $\mathrm{d}_A\circ\mathrm{d}_A=0$ if and only if the connection is flat, that is its curvature vanishes.

The gauge group $\mathcal{G}(X):=\mathcal{C}^\infty(X,\mathrm{G})$ represents the smooth bundle isomorphisms. So a gauge transformation $u\in\mathcal{G}(X)$ acts on $A\in\mathcal{A}(X)$ by pullback,

$$u^*A=u^{-1}Au+u^{-1}\mathrm{d}u.$$

On a compact base manifold M and for $k\in\mathbb{N}_0$ and $1\le p\le\infty$ we denote the Sobolev spaces of connections and gauge transformations by

$$\begin{aligned}\mathcal{A}^{k,p}(M)&:=W^{k,p}(M,\mathrm{T}^*M\otimes\mathfrak{g}),\\ \mathcal{G}^{k,p}(M)&:=W^{k,p}(M,\mathrm{G}).\end{aligned}$$

For $kp>\dim M$ the latter is well-defined via an embedding $\mathrm{G}\subset\mathbb{R}^\ell$, and it forms a group $\mathcal{G}^{k,p}(M)$ that acts smoothly on $\mathcal{A}^{k-1,p}(M)$, see e.g. [W1, Appendix B]. For non-compact base manifolds X we denote by $\mathcal{A}^{k,p}_{\mathrm{loc}}(X)$ and $\mathcal{G}^{k,p}_{\mathrm{loc}}(X)$ the spaces of sections and maps for which the regularity holds on all compact subsets of X.

Next, we describe the class of 4-manifolds that we will be considering. Here and throughout all Riemann surfaces are closed oriented 2-dimensional manifolds. Moreover, unless otherwise mentioned, all manifolds are allowed to have a smooth boundary. Then the interior of a submanifold $X'\subset X$ is to be understood with respect to the relative topology, i.e. $\mathrm{int}\,X':=X\setminus\mathrm{cl}(X\setminus X')$ might intersect ∂X.

Definition 1.1. *A* **4-manifold with a boundary space-time splitting** *is a pair* (X,τ) *with the following properties:*

(i) X *is an oriented* 4*-manifold (with boundary) which can be exhausted by a nested sequence*

$$X = \bigcup_{k\in\mathbb{N}} X_k,$$

where all X_k *are compact submanifolds and deformation retracts of* X *such that* $X_k \subset \operatorname{int} X_{k+1}$ *for all* $k\in\mathbb{N}$.

(ii) $\tau = (\tau_1,\dots,\tau_n)$ *is an* n*-tuple of embeddings* $\tau_i : \mathcal{S}_i\times\Sigma_i \to X$ *with disjoint images, where* Σ_i *is a Riemann surface and* $\mathcal{S}_i$ *is either an open interval in* $\mathbb{R}$ *or is equal to* $S^1=\mathbb{R}/\mathbb{Z}$.

(iii) The boundary ∂X *is the union*

$$\partial X = \bigcup_{i=1}^{n} \tau_i(\mathcal{S}_i\times\Sigma_i).$$

Definition 1.2. *Let* (X,τ) *be a* 4*-manifold with a boundary space-time splitting. A Riemannian metric* g *on* X *is called* **compatible** *with* τ *if for each* $i=1,\dots n$ *there exists a neighbourhood* $\mathcal{U}_i\subset\mathcal{S}_i\times[0,\infty)$ *of* $\mathcal{S}_i\times\{0\}$ *and an extension of* τ_i *to an embedding* $\bar\tau_i : \mathcal{U}_i\times\Sigma_i\to X$ *such that*

$$\bar\tau_i^* g = \mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}.$$

Here $g_{s,t}$ *is a smooth family of metrics on* Σ_i *and we denote by* s *the coordinate on* $\mathcal{S}_i$ *and by* t *the coordinate on* $[0,\infty)$.

We call a triple (X,τ,g) *with these properties a* **Riemannian 4-manifold with a boundary space-time splitting**.

Remark 1.3. In definition 1.2 the extended embeddings $\bar\tau_i$ are uniquely determined by the metric as follows. The restriction $\bar\tau_i|_{t=0}=\tau_i$ to the boundary is prescribed, and the paths $t\mapsto\bar\tau_i(s,t,z)$ are normal geodesics.

Example 1.4. Let $X:=\mathbb{R}\times Y$, where Y is a compact oriented 3-manifold with boundary $\partial Y=\Sigma$, and let $\tau:\mathbb{R}\times\Sigma\to X$ be the obvious inclusion. Given any two metrics g_- and g_+ on Y there exists a metric g on X such that $g=\mathrm{d}s^2+g_-$ for $s\le -1$, $g=\mathrm{d}s^2+g_+$ for $s\ge 1$, and (X,τ,g) satisfies the conditions of definition 1.2. However, the metric g cannot necessarily be chosen in the form $\mathrm{d}s^2+g_s$ (one has to homotope the embeddings and the metrics).

Now let (X,τ,g) be a Riemannian 4-manifold with a boundary space-time splitting and consider a trivial G-bundle over X for a compact Lie group G.

Let $p>2$, then for each $i=1,\dots,n$ the Banach space of connections $\mathcal{A}^{0,p}(\Sigma_i)$ carries the symplectic form $\omega(\alpha,\beta)=\int_{\Sigma_i}\langle\,\alpha\wedge\beta\,\rangle$. Note that the Hodge $*$ operator for any metric on Σ_i is an ω-compatible complex structure on $\mathcal{A}^{0,p}(\Sigma_i)$, since $\omega(\cdot,*\cdot)$ defines a positive definite inner product – the L^2-metric. We call a Banach submanifold $\mathcal{L}\subset\mathcal{A}^{0,p}(\Sigma_i)$ Lagrangian if for all $A\in\mathcal{L}$ its tangent space is isotropic, $\omega|_{\mathrm{T}_A\mathcal{L}}\equiv 0$, and coisotropic in the following sense: If $\alpha\in\mathcal{A}^{0,p}(\Sigma_i)$ satisfies $\omega(\alpha,\mathrm{T}_A\mathcal{L})=\{0\}$, then $\alpha\in\mathrm{T}_A\mathcal{L}$.

We fix an n-tuple $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_n)$ of Lagrangian submanifolds $\mathcal{L}_i \subset \mathcal{A}^{0,p}(\Sigma_i)$ that are contained in the space of flat connections and that are gauge invariant,

$$\mathcal{L}_i \subset \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma_i) \qquad \text{and} \qquad u^*\mathcal{L}_i = \mathcal{L}_i \quad \forall u \in \mathcal{G}^{1,p}(\Sigma_i). \tag{1}$$

Here $\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma_i)$ is the space of weakly flat L^p-connections on Σ_i as introduced in [W2, Section 3]. For our purposes it is enough to know that this space coincides with $\mathcal{G}^{1,p}(\Sigma_i)^*\mathcal{A}_{\mathrm{flat}}(\Sigma_i)$, the set of connections that is $W^{1,p}$-gauge equivalent to a smooth flat connection, $A \in \mathcal{A}(\Sigma_i)$ with $F_A = 0$. Moreover, we recall from [W2, Lemma 4.2] the fact that the above assumptions on the $\mathcal{L}_i$ imply that they are totally real with respect to the Hodge $*$ operator for any metric on Σ_i, i.e. for all $A \in \mathcal{L}_i$ one has the topological sum

$$\mathcal{A}^{0,p}(\Sigma_i) = \mathrm{T}_A\mathcal{L}_i \oplus *\mathrm{T}_A\mathcal{L}_i.$$

We consider the following boundary value problem for connections $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$

$$\begin{cases} *F_A + F_A = 0, \\ \tau_i^* A|_{\{s\}\times\Sigma_i} \in \mathcal{L}_i \quad \forall s \in \mathcal{S}_i,\ i = 1, \ldots, n. \end{cases} \tag{2}$$

Observe that the above boundary condition is meaningful since for every neighbourhood $\mathcal{U} \times \Sigma \subset \mathcal{S} \times [0,\infty) \times \Sigma$ of a boundary slice $\{s\} \times \{0\} \times \Sigma$ one has the continuous embedding $W^{1,p}(\mathcal{U} \times \Sigma) \subset W^{1,p}(\mathcal{U}, L^p(\Sigma)) \hookrightarrow \mathcal{C}^0(\mathcal{U}, L^p(\Sigma))$. The first nontrivial observation is that every connection in $\mathcal{L}_i$ is gauge equivalent to a smooth connection on Σ_i and hence $\mathcal{L}_i \cap \mathcal{A}(\Sigma)$ is dense in $\mathcal{L}_i$, as shown in [W2, Theorem 3.1]. Moreover, the $\mathcal{L}_i$ are modelled on L^p-spaces, and every $W^{1,p}_{\mathrm{loc}}$-connection on X satisfying the boundary condition in (2) can be locally approximated by smooth connections satisfying the same boundary condition, see [W2, Corollaries 4.4, 4.5].

Note that the present boundary value problem is a first order equation with first order boundary conditions (flatness in each time-slice). Moreover, the boundary conditions contain some crucial nonlocal Lagrangian information. We moreover emphasize that while $\mathcal{L}_i$ is a smooth Banach submanifold of $\mathcal{A}^{0,p}(\Sigma_i)$, the quotient $\mathcal{L}_i/\mathcal{G}^{1,p}(\Sigma_i)$ is not required to be a smooth submanifold of the moduli space $M_{\Sigma_i} := \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma_i)/\mathcal{G}^{1,p}(\Sigma_i)$, which itself might be singular.

An example for these Lagrangians is $\mathcal{L}_i = \mathcal{L}_Y$, the space of flat connections on Σ_i that extend to flat connections on a handlebody Y with $\partial Y = \Sigma_i$. The nonlocal Lagrangian information in this case is the extensibility condition, which is equivalent to the vanishing of the holonomies along those paths in Σ_i that are contractible in Y. See [W2, Lemma 4.6] for a detailed discussion of this example. To overcome the difficulties arising from the singularities in the quotient, we work with the (smooth) quotient by the based gauge group.

The following two theorems are the regularity and compactness results for solutions of (2) generalizing the regularity theorem and the Uhlenbeck compactness for Yang-Mills connections on 4-manifolds without boundary. They will be proven in section 2.

Theorem A (Regularity)
Let $p > 2$. Then every solution $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$ of the boundary value problem (2) is gauge equivalent to a smooth solution, that is there exists a gauge transformation $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(X)$ such that $u^*A \in \mathcal{A}(X)$ is smooth.

Theorem B (Compactness)
Let $p > 2$ and let g^ν be a sequence of metrics compatible with τ that uniformly converges with all derivatives on every compact subset to a smooth metric. Suppose that $A^\nu \in \mathcal{A}^{1,p}_{\text{loc}}(X)$ is a sequence of solutions of (2) with respect to the metrics g^ν such that for every compact subset $K \subset X$ there is a uniform bound on the curvature $\|F_{A^\nu}\|_{L^p(K)}$.

Then there exists a subsequence (again denoted A^ν) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{\text{loc}}(X)$ such that $u^{\nu\,*}A^\nu$ converges uniformly with all derivatives on every compact subset to a smooth connection $A \in \mathcal{A}(X)$.

The difficulty of these results lies in the global nature of the boundary condition. This makes it impossible to directly generalize the proof of the regularity and compactness theorems for Yang-Mills connections, where one chooses suitable local gauges, obtains the higher regularity and estimates from an elliptic boundary value problem, and then patches the gauges together. With the present global boundary condition one cannot obtain local regularity results.

Thus we generalize a more global approach by Salamon to manifolds with boundary: Firstly, Uhlenbeck's weak compactness theorem yields a weakly convergent subsequence. Its limit serves as reference connection with respect to which one can achieve a global relative Coulomb gauge for a further subsequence. Then it remains to establish elliptic estimates and regularity results for the given boundary value problem together with the relative Coulomb gauge equations. The crucial point in this step is the regularity for the Σ-component of the connections in a neighbourhood $\mathcal{U} \times \Sigma$ of a boundary component. Here one deals with a Cauchy-Riemann equation on $\mathcal{U}$ with values in the Banach space $\mathcal{A}^{0,p}(\Sigma)$ and with Lagrangian boundary conditions. The regularity results for this boundary value problem are provided by [W2] in the general framework of a Cauchy-Riemann equation for functions with values in a complex Banach space and with totally real boundary conditions.

The case $2 < p \leq 4$, when $W^{1,p}$-functions are not automatically continuous, poses some special difficulties in this last step. Firstly, in order to obtain regularity results from the Cauchy-Riemann equation, one has to straighten out the Lagrangian submanifold by going to suitable coordinates. This requires a $\mathcal{C}^0$-convergence of the connections, which in case $p > 4$ is given by a standard Sobolev embedding. In case $p > 2$ one still obtains a special compact embedding $W^{1,p}(\mathcal{U} \times \Sigma) \hookrightarrow \mathcal{C}^0(\mathcal{U}, L^p(\Sigma))$ that suits our purposes. Secondly, the straightening of the Lagrangian introduces a nonlinearity in the Cauchy-Riemann equation that already poses some problems in case $p > 4$. In case $p \leq 4$ this forces us to also deal with the Cauchy-Riemann equation with values in an L^2-Hilbert space and then use some interpolation inequalities for Sobolev norms.

For the definition of the standard instanton Floer homology it suffices to prove a compactness result like theorem B for $p = \infty$. In our case however the bubbling analysis [W3] requires the compactness result for some $p < 3$. This is why we have taken some care to deal with this case.

Our third main result is the Fredholm theory in section 3. It is a step towards proving that the moduli space of finite energy solutions of (2) is a manifold whose components have finite (but possibly different) dimensions. This also exemplifies our hope that the further analytical details of Floer theory will work out along the usual lines once the right analytic setup has been found in the proof of theorems A and B.

In the context of Floer homology and in Floer-Donaldson theory it is important to consider 4-manifolds with cylindrical ends. This requires an analysis of the asymptotic behaviour which will be carried out elsewhere. Here we shall restrict the discussion of the Fredholm theory to the compact case. The crucial point is the behaviour of the linearized operator near the boundary; in the interior we are dealing with the usual anti-self-duality equation. Hence it suffices to consider the following model case. Let Y be a compact oriented 3-manifold with boundary $\partial Y = \Sigma$ and suppose that $(g_s)_{s\in S^1}$ is a smooth family of metrics on Y such that

$$X = S^1 \times Y, \qquad \tau : S^1 \times \Sigma \to X, \qquad g = \mathrm{d}s^2 + g_s$$

satisfy the assumptions of definition 1.2. Here the space-time splitting τ of the boundary is the obvious inclusion $\tau : S^1 \times \Sigma \hookrightarrow \partial X = S^1 \times \Sigma$, where $\Sigma = \bigcup_{i=1}^n \Sigma_i$ might be a disjoint union of an n-tuple of connected Riemann surfaces Σ_i. An n-tuple of Lagrangian submanifolds $\mathcal{L}_i \subset \mathcal{A}^{0,p}(\Sigma_i)$ as in (1) then constitutes a gauge invariant Lagrangian submanifold $\mathcal{L} := \mathcal{L}_1 \times \ldots \times \mathcal{L}_n$ of the symplectic Banach space $\mathcal{A}^{0,p}(\Sigma) = \mathcal{A}^{0,p}(\Sigma_1) \times \ldots \times \mathcal{A}^{0,p}(\Sigma_n)$ such that $\mathcal{L} \subset \mathcal{A}^{0,p}_{\text{flat}}(\Sigma)$.

In order to linearize the boundary value problem (2) together with the local slice condition, fix a smooth connection $A+\Phi\mathrm{d}s \in \mathcal{A}(S^1\times Y)$ such that $A_s := A(s)|_{\partial Y} \in \mathcal{L}$ for all $s \in S^1$. Here $\Phi \in \mathcal{C}^\infty(S^1\times Y, \mathfrak{g})$ and $A \in \mathcal{C}^\infty(S^1\times Y, \mathrm{T}^*Y\otimes\mathfrak{g})$ is an S^1-family of 1-forms on Y (not a 1-form on X as previously). Now let $E^{1,p}_A$ be the space of S^1-families of 1-forms $\alpha \in W^{1,p}(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ that satisfy the boundary conditions

$$*\alpha(s)|_{\partial Y} = 0 \qquad \text{and} \qquad \alpha(s)|_{\partial Y} \in \mathrm{T}_{A_s}\mathcal{L} \qquad \text{for all } s \in S^1. \tag{3}$$

Then the linearized operator

$$D_{(A,\Phi)} : E^{1,p}_A \times W^{1,p}(S^1 \times Y, \mathfrak{g}) \longrightarrow L^p(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g}) \times L^p(S^1 \times Y, \mathfrak{g})$$

is given with $\nabla_s = \partial_s + [\Phi, \cdot]$ by

$$D_{(A,\Phi)}(\alpha,\varphi) \;=\; \bigl(\nabla_s\alpha - \mathrm{d}_A\varphi + *\mathrm{d}_A\alpha \,,\, \nabla_s\varphi - \mathrm{d}_A^*\alpha\bigr).$$

The second component of this operator is $-\mathrm{d}^*_{A+\Phi\mathrm{d}s}(\alpha + \varphi\mathrm{d}s)$, and the first boundary condition is $*(\alpha + \varphi\mathrm{d}s)|_{\partial X} = 0$, corresponding to the choice of a local slice at $A + \Phi\mathrm{d}s$. In the first component of $D_{(A,\Phi)}$ we have used the global space-time splitting of the metric on $S^1 \times Y$ to identify the self-dual 2-forms $*\gamma_s - \gamma_s \wedge \mathrm{d}s$ with families γ_s of 1-forms on Y. The vanishing of this component is equivalent to the linearization $\mathrm{d}^+_{A+\Phi\mathrm{d}s}(\alpha + \varphi\mathrm{d}s) = 0$ of the anti-self-duality equation. Furthermore, the boundary condition $\alpha(s)|_{\partial Y} \in \mathrm{T}_{A_s}\mathcal{L}$ is the linearization of the Lagrangian boundary condition in the boundary value problem (2).

Theorem C (Fredholm properties)
Let Y be a compact oriented 3-manifold with boundary $\partial Y = \Sigma$ and let $S^1 \times Y$ be equipped with a product metric $\mathrm{d}s^2 + g_s$ that is compatible with the embedding $\tau : S^1 \times \Sigma \to S^1 \times Y$. Let $A + \Phi\mathrm{d}s \in \mathcal{A}(S^1 \times Y)$ such that $A(s)|_{\partial Y} \in \mathcal{L}$ for all $s \in S^1$. Then the following holds for all $p > 2$.

(i) *$D_{(A,\Phi)}$ is Fredholm.*

(ii) *There is a constant C such that for all $\alpha \in E^{1,p}_A$ and $\varphi \in W^{1,p}(S^1 \times Y, \mathfrak{g})$*

$$\|(\alpha,\varphi)\|_{W^{1,p}} \leq C\bigl(\|D_{(A,\Phi)}(\alpha,\varphi)\|_{L^p} + \|(\alpha,\varphi)\|_{L^p}\bigr).$$

(iii) Let $q \in (1,2)\cup(2,\infty)$. There is a constant C such that the following holds. Suppose that $\beta \in L^q(S^1\times Y, \mathrm{T}^*Y\otimes\mathfrak{g})$ and $\zeta \in L^q(S^1\times Y,\mathfrak{g})$, and assume that there exists a constant c such that for all $\alpha \in \mathcal{C}^\infty(S^1\times Y, \mathrm{T}^*Y\otimes\mathfrak{g})$ satisfying (3) and for all $\varphi\in\mathcal{C}^\infty(S^1\times Y,\mathfrak{g})$

$$\left|\int_{S^1\times Y}\langle\, D_{(A,\Phi)}(\alpha,\varphi)\,,\,(\beta,\zeta)\,\rangle\right| \;\le\; c\,\|(\alpha,\varphi)\|_{L^{q^*}}\,.$$

Then β and ζ are of class $W^{1,q}$ and $\|(\beta,\xi)\|_{W^{1,q}}\le Cc$.

Here and throughout we use the notation $\frac{1}{p}+\frac{1}{p^*}=1$ for the conjugate exponent p^* of p. The above inner product $\langle\,\cdot,\cdot\,\rangle$ is the pointwise inner product in $(\mathrm{T}^*Y\otimes\mathfrak{g})\times\mathfrak{g}$.

Theorem C (ii) actually extends to an L^2-estimate for $W^{1,p}$-regular (α,ϕ), that can be proven by more elementary methods than the general case, as will be shown in section 3. In fact, this estimate was already stated in [S] as an indication for the well-posedness of the boundary value problem (2).

The reason for our assumption $q\neq 2$ in theorem C (iii) is a technical problem in dealing with the singularities of $\mathcal{L}/\mathcal{G}^{1,p}(\Sigma)$. We resolve them by dividing only by the based gauge group. This leads to coordinates of $L^p(\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})$ in a Banach space, part of which is a based Sobolev space $W^{1,p}_z(\Sigma,\mathfrak{g})$ of functions vanishing at a fixed point $z\in\Sigma$. So these coordinates that straighten out $\mathrm{T}\mathcal{L}$ along $A|_{S^1\times\partial Y}$ are well-defined only for $p>2$. We prove theorem C (iii) by using such coordinates either for β or for the test 1-forms α, so we assume that either $q>2$ or $q^*>2$. This is completely sufficient for our purposes – proving the Fredholm property in theorem C (i) for $p>2$.

The Fredholm property of a generalized operator $D_{(A,\Phi)}$ for $p=2$ follows from more general Hilbert space techniques, that will be carried out elsewhere.

1.2. Outlook. We give a brief sketch of Salamon's program for the proof of the Atiyah-Floer conjecture (for more details see [S]) in order to point out the significance of the present results for the whole program.

The first step of the program is to define the instanton Floer homology $\mathrm{HF}^{\mathrm{inst}}_*(Y,L)$ of a 3-manifold Y with boundary $\partial Y=\Sigma$ and a (singular) Lagrangian submanifold $L=\mathcal{L}/\mathcal{G}^{1,p}(\Sigma)\subset M_\Sigma$ in the moduli space of flat connections. The Floer complex will be generated by the gauge equivalence classes of irreducible flat connections $A\in\mathcal{A}(Y)$ with Lagrangian boundary conditions $A|_\Sigma\in\mathcal{L}$. [1] For any two such connections A^+, A^- one then has to study the moduli space of Floer connecting orbits,

$$\mathcal{M}(A^-,A^+)=\bigl\{\tilde{A}\in\mathcal{A}(\mathbb{R}\times Y)\bigm|\tilde{A}\text{ satisfies }(2),\ \lim_{s\to\pm\infty}\tilde{A}=A^\pm\bigr\}/\mathcal{G}(\mathbb{R}\times Y).$$

Theorem A shows that the boundary value problem (2) is well-posed. In particular, the spaces of smooth connections and gauge transformations in the definition of the above moduli space can be replaced by the Sobolev completions $\mathcal{A}^{1,p}_{\mathrm{loc}}$ and $\mathcal{G}^{2,p}_{\mathrm{loc}}$. The next step in the construction of the Floer homology groups is the analysis of the asymptotic behaviour of the finite energy solutions of (2) on $\mathbb{R}\times Y$, which will be carried out elsewhere. Combining this with theorem C one obtains an appropriate Fredholm theory and

[1] A connection $A\in\mathcal{A}_{\mathrm{flat}}(Y)$ is called irreducible if its isotropy subgroup of $\mathcal{G}(Y)$ (the group of gauge transformations that leave A fixed) is discrete, i.e. $\mathrm{d}_A|_{\Omega^0}$ is injective. There should be no reducible flat connections with Lagrangian boundary conditions other than the gauge orbit of the trivial connection. This will be guaranteed by certain conditions on Y and L, for example this is the case when $L=L_{Y'}$ for a handlebody Y' with $\partial Y'=\bar{\Sigma}$ such that $Y\cup_\Sigma Y'$ is a homology-3-sphere.

proves that for a suitably generic perturbation the spaces $\mathcal{M}(A^-, A^+)$ are smooth manifolds. In the monotone case the connections in the k-dimensional part $\mathcal{M}^k(A^-, A^+)$ have a fixed energy $\int |F_{\tilde{A}}|^2$.

Theorem B is a major step towards a compactification of the spaces $\mathcal{M}^k(A^-, A^+)$. It proves their compactness under the assumption of an L^p-bound on the curvature for $p > 2$, whereas a priori the L^2-norm is bounded due to the fixed energy. So the key remaining analytic task is an analysis of the possible bubbling phenomena. This is carried out in [W3] and draws upon the techniques developed in this paper. When this is understood, the construction of the Floer homology groups should be routine. In particular, for the metric independence note that one can interpolate between different metrics on Y as in example 1.4, and theorem B allows for the variation of metrics on X. So this paper sets up the basic analytic framework for the Floer theory of 3-manifolds with boundary.

Moreover, the consideration of general 4-manifolds X (rather than just $X = \mathbb{R}\times Y$) in theorems A and B will allow for the definition of a product structure on this new Floer homology.

The further steps in the program for the proof of the Atiyah-Floer conjecture are to consider a Heegaard splitting $Y = Y_0 \cup_\Sigma Y_1$ of a homology 3-sphere, and identify $\mathrm{HF}^{\mathrm{inst}}_*([0,1]\times\Sigma, L_{Y_0}\times L_{Y_1})$ with $\mathrm{HF}^{\mathrm{inst}}_*(Y)$ and $\mathrm{HF}^{\mathrm{symp}}_*(M_\Sigma, L_{Y_0}, L_{Y_1})$ respectively. (These isomorphisms should also intertwine the ring structures on all three Floer homologies.) In both cases, the Floer complexes can be identified by elementary arguments, so the main task is to identify the connecting orbits.

In the case of the two instanton Floer homologies, the idea is to choose an embedding $(0,1)\times\Sigma \hookrightarrow Y$ starting from a tubular neighbourhood of $\Sigma \subset Y$ at $t = \frac{1}{2}$ and shrinking $\{t\}\times\Sigma$ to the 1-skeleton of Y_t for $t = 0, 1$. Then the anti-self-dual instantons on $\mathbb{R}\times Y$ pull back to anti-self-dual instantons on $\mathbb{R}\times[0,1]\times\Sigma$ with a degenerate metric for $t = 0$ and $t = 1$. On the other hand, one can consider anti-self-dual instantons on $\mathbb{R}\times[\varepsilon, 1-\varepsilon]\times\Sigma$ with boundary values in $\mathcal{L}_{Y_0}$ and $\mathcal{L}_{Y_1}$. As $\varepsilon\to 0$, one should be able to pass from this genuine boundary value problem to solutions on the closed manifold Y. This is a limit process for the boundary value problem studied in this paper.

The identification of the instanton and symplectic Floer homologies requires an adaptation of the adiabatic limit argument in [DS] to boundary value problems for anti-self-dual instantons and pseudoholomorphic curves respectively. Here one again deals with the boundary value problem (2) studied in this paper. As the metric on Σ is scaled to zero, the solutions, i.e. anti-self-dual instantons on $\mathbb{R}\times[0,1]\times\Sigma$ with Lagrangian boundary conditions in $\mathcal{L}_{Y_0}$ and $\mathcal{L}_{Y_1}$ should be in one-to-one correspondence with connections on $\mathbb{R}\times[0,1]\times\Sigma$ that descend to pseudoholomorphic strips in M_Σ with boundary values in L_{Y_0} and L_{Y_1}. The basic elliptic properties of the boundary value problem (2) that are established in this paper will also play an important role in this adiabatic limit analysis.

2. Regularity and compactness

The aim of this section is to prove the regularity theorem A and the compactness theorem B. Both theorems are dealing with a noncompact base manifold X that is exhausted by compact submanifolds X_k. We shall use an extension argument by Donaldson and Kronheimer [DK, Lemma 4.4.5] to reduce the problem to compact base manifolds. For the following special version of this argument a detailed proof can be found in [W1, Propositions 7.6,9.8]. At this point, the assumption that the exhausting compact submanifolds X_k are deformation retracts of X comes in crucially. It ensures that every gauge transformation on X_k can be extended to X, which is a central point in the argument due to Donaldson and Kronheimer.

Proposition 2.1. *Let the* 4*-manifold* $\tilde{M} = \bigcup_{k\in\mathbb{N}} M_k$ *be exhausted by compact submanifolds* $M_k \subset \operatorname{int} M_{k+1}$ *that are deformation retracts of* $\tilde{M}$*, and let* $p > 2$.

(i) Let $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(\tilde{M})$ *and suppose that for each* $k \in \mathbb{N}$ *there exists a gauge transformation* $u_k \in \mathcal{G}^{2,p}(M_k)$ *such that* $u_k^* A|_{M_k}$ *is smooth. Then there exists a gauge transformation* $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(\tilde{M})$ *such that* u^*A *is smooth.*

(ii) Let a sequence of connections $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}_{\mathrm{loc}}(\tilde{M})$ *be given and suppose that the following holds:*
For every $k \in \mathbb{N}$ *and every subsequence of* $(A^\nu)_{\nu\in\mathbb{N}}$ *there exist a further subsequence* $(\nu_{k,i})_{i\in\mathbb{N}}$ *and gauge transformations* $u^{k,i} \in \mathcal{G}^{2,p}(M_k)$ *such that*

$$\sup_{i\in\mathbb{N}} \left\| u^{k,i\,*} A^{\nu_{k,i}} \right\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall \ell \in \mathbb{N}.$$

Then there exists a subsequence $(\nu_i)_{i\in\mathbb{N}}$ *and a sequence of gauge transformations* $u^i \in \mathcal{G}^{2,p}_{\mathrm{loc}}(\tilde{M})$ *such that*

$$\sup_{i\in\mathbb{N}} \left\| u^{i\,*} A^{\nu_i} \right\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall k \in \mathbb{N}, \ell \in \mathbb{N}.$$

So in order to prove theorem A it suffices to find smoothing gauge transformations on the compact submanifolds X_k in view of proposition 2.1 (i). For that purpose we shall use the so-called local slice theorem. The following version is proven e.g. in [W1, Theorem 8.1]. Note that we are dealing with trivial bundles, so we will be using the product connection as reference connection in the definition of the Sobolev norms of connections.

Proposition 2.2. (Local Slice Theorem)
Let M *be a compact* 4*-manifold, let* $p > 2$*, and let* $q > 4$ *be such that* $\frac{1}{q} > \frac{1}{p} - \frac{1}{4}$ *(or* $q = \infty$ *in case* $p > 4$*). Fix* $\hat{A} \in \mathcal{A}^{1,p}(M)$ *and let a constant* $c_0 > 0$ *be given. Then there exist constants* $\varepsilon > 0$ *and* C_{CG} *such that the following holds. For every* $A \in \mathcal{A}^{1,p}(M)$ *with*

$$\|A - \hat{A}\|_q \leq \varepsilon \qquad \textit{and} \qquad \|A - \hat{A}\|_{W^{1,p}} \leq c_0$$

there exists a gauge transformation $u \in \mathcal{G}^{2,p}(M)$ *such that*

$$\begin{cases} \mathrm{d}^*_{\hat{A}}(u^*A - \hat{A}) = 0, \\ *(u^*A - \hat{A})|_{\partial M} = 0, \end{cases} \qquad \textit{and} \qquad \begin{array}{ll} \|u^*A - \hat{A}\|_q & \leq C_{CG}\|A - \hat{A}\|_q, \\ \|u^*A - \hat{A}\|_{W^{1,p}} & \leq C_{CG}\|A - \hat{A}\|_{W^{1,p}}. \end{array}$$

Remark 2.3.

(i) If the boundary value problem in proposition 2.2 is satisfied one says that u^*A is in Coulomb gauge relative to $\hat{A}$. This is equivalent to $v^*\hat{A}$ being in Coulomb gauge relative to A for $v = u^{-1}$, i.e. the boundary value problem can be replaced by

$$\begin{cases} \mathrm{d}^*_A(v^*\hat{A} - A) = 0, \\ *(v^*\hat{A} - A)|_{\partial M} = 0. \end{cases}$$

(ii) The assumptions in proposition 2.2 on p and q guarantee that one has a compact Sobolev embedding
$$W^{1,p}(M) \hookrightarrow L^q(M).$$

(iii) One can find uniform constants for varying metrics in the following sense. Fix a metric g on M. Then there exist constants $\varepsilon, \delta > 0$, and C_{CG} such that the assertion of proposition 2.2 holds for all metrics g' with $\|g - g'\|_{\mathcal{C}^1} \leq \delta$.

In the following we outline the proof of theorem A. Given a solution $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$ of (2) one proves the assumption of proposition 2.1 (i) for each of the exhausting submanifold X_k as follows. One finds a sufficiently large compact submanifold $M \subset X$ with $X_k \subset M$. Then one chooses a smooth connection $A_0 \in \mathcal{A}(M)$ sufficiently $W^{1,p}$-close to A and applies the local slice theorem with the reference connection $\hat{A} = A$ to find a gauge transformation that puts A_0 into relative Coulomb gauge with respect to A. This is equivalent to finding a gauge transformation that puts A into relative Coulomb gauge with respect to A_0. We denote this gauge transformed connection again by $A \in \mathcal{A}^{1,p}(M)$. It satisfies the following boundary value problem:

$$\begin{cases} \mathrm{d}^*_{A_0}(A - A_0) = 0, \\ *F_A + F_A = 0, \\ *(A - A_0)|_{\partial M} = 0, \\ \tau_i^* A|_{\{s\}\times\Sigma_i} \in \mathcal{L}_i \quad \forall s \in \mathcal{S}_i,\, i = 1, \ldots, n. \end{cases} \tag{4}$$

More precisely, the Lagrangian boundary condition only holds for those $s \in \mathcal{S}_i$ and $i \in \{1, \ldots n\}$ for which $\tau_i(\{s\} \times \Sigma_i)$ is entirely contained in ∂M. If M is chosen large enough (in particular, it has to contain the full boundary slice $\tau_i(\{s\}\times\Sigma_i)$ whenever this intersects X_k at all), then the regularity theorem 2.6 below will assert the smoothness of A on X_k.

The proof of theorem B goes along similar lines. We will use proposition 2.1 (ii) to reduce the problem to compact base manifolds. On these, we shall use the following weak Uhlenbeck compactness theorem (see [U1], [W1, Theorem 7.1]) to find a subsequence of gauge equivalent connections that converges $W^{1,p}$-weakly.

Proposition 2.4. (Weak Uhlenbeck Compactness)
Let M be a compact 4-manifold and let $p > 2$. Suppose that the sequence of connections $A^\nu \in \mathcal{A}^{1,p}(M)$ is such that $\|F_{A^\nu}\|_p$ is uniformly bounded. Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence $u^\nu \in \mathcal{G}^{2,p}(M)$ of gauge transformations such that $u^{\nu\,*}A^\nu$ weakly converges in $\mathcal{A}^{1,p}(M)$.

The limit A_0 of the convergent subsequence then serves as reference connection $\hat{A}$ in the local slice theorem, proposition 2.2, and this way one obtains a $W^{1,p}$-bounded sequence of connections $\tilde{A}^\nu$ that solve the boundary value problem (4). This makes crucial use of the compact Sobolev embedding $W^{1,p} \hookrightarrow L^q$ on compact 4-manifolds (with q from the local slice theorem). The estimates in the subsequent theorem 2.6 then provide the higher $W^{k,p}$-bounds on the connections that will imply the compactness. One difficulty in the proof of this regularity theorem is that due to the global nature of the boundary conditions one has to consider the Σ-components of the connections near the boundary as maps into the Banach space $\mathcal{A}^{0,p}(\Sigma)$ that solve a Cauchy-Riemann equation with Lagrangian boundary conditions. In order to prove a regularity result for such maps one has to straighten out the Lagrangian submanifold by using coordinates in $\mathcal{A}^{0,p}(\Sigma)$. (This is done in [W2].) Thus on domains $\mathcal{U}\times\Sigma$ at the boundary a crucial assumption is that the Σ-components of the connections all lie in one such coordinate chart, that is one needs the connections to converge strongly in the $L^\infty(\mathcal{U},L^p(\Sigma))$-norm. In the case $p>4$ this is ensured by the compact embedding $W^{1,p}\hookrightarrow L^\infty$ on $\mathcal{U}\times\Sigma$. To treat the case $2<p\le 4$ we shall make use of the following special Sobolev embedding.

Lemma 2.5. *Let M,N be compact manifolds and let $p>\max(\dim M,\dim N)$. Then the following embedding is compact,*

$$W^{1,p}(M\times N)\hookrightarrow L^\infty(M,L^p(N)).$$

Proof of lemma 2.5:
Since M is compact it suffices to prove the embedding in (finitely many) coordinate charts. These can be chosen as either balls $B_2\subset\mathbb{R}^m$ in the interior or half balls $D_2=B_2\cap\mathbb{H}^m$ in the half space $\mathbb{H}^m=\{x\in\mathbb{R}^m\,\big|\,x_1\ge 0\}$ at the boundary of M. We can choose both of radius 2 but cover M by balls and half balls of radius 1. So it suffices to consider a bounded set $\mathcal{K}\subset W^{1,p}(B_2\times N)$ and prove that it restricts to a precompact set in $L^\infty(B_1,L^p(N))$, and similarly with the half balls. Here we use the Euclidean metric on $\mathbb{R}^m$, which is equivalent to the metric induced from M.

For a bounded subset $\mathcal{K}\subset W^{1,p}(D_2\times N)$ of functions over the half ball we define the subset $\mathcal{K}'\subset W^{1,p}(B_2\times N)$ by extending every function $u\in\mathcal{K}$ to $B_2\setminus\mathbb{H}^m$ via $u(x_1,x_2,\dots,x_m):=u(-x_1,x_2,\dots,x_m)$ for $x_1<0$. The thus extended function is still $W^{1,p}$-regular with twice the norm of u. So $\mathcal{K}'$ also is a bounded subset, and if this restricts to a precompact set in $L^\infty(B_1,L^p(N))$, then also $\mathcal{K}\subset L^\infty(D_1,L^p(N))$ is precompact. Hence it suffices to consider the interior case of the full ball.

The claimed embedding is continuous by the standard Sobolev estimates – check for example in the proof of [Ad, Theorem 5.4,] that the estimates generalize directly to functions with values in a Banach space. In fact, one obtains an embedding

$$W^{1,p}(B_2\times N)\subset W^{1,p}(B_2,L^p(N))\hookrightarrow\mathcal{C}^{0,\lambda}(B_2,L^p(N))$$

into some Hölder space with $\lambda=1-\frac{m}{p}>0$. One can also use this Sobolev estimate for $W^{1,p}(N)$ with $\lambda'=1-\frac{n}{p}>0$ combined with the inclusion $L^p\hookrightarrow L^1$ on B_2 to obtain a continuous embedding

$$W^{1,p}(B_2\times N)\subset L^p(B_2,W^{1,p}(N))\hookrightarrow L^p(B_2,\mathcal{C}^{0,\lambda'}(N))\subset L^1(B_2,\mathcal{C}^{0,\lambda'}(N)).$$

Now consider a bounded subset $\mathcal{K}\subset W^{1,p}(B_2\times N)$. The first embedding ensures that the functions $\mathcal{K}\ni u:B_2\to L^p(N)$ are equicontinuous. For some constant C

$$\|u(x)-u(y)\|_{L^p(N)}\le C|x-y|^\lambda\qquad\forall u\in\mathcal{K},x,y\in B_2.\tag{5}$$

The second embedding asserts that for some constant C'

$$\int_{B_2}\|u\|_{\mathcal{C}^{0,\lambda'}(N)}\le C'\qquad\forall u\in\mathcal{K}.\tag{6}$$

In order to prove that $\mathcal{K}\subset L^\infty(B_1,L^p(N))$ is precompact we now fix any $\varepsilon>0$ and show that $\mathcal{K}$ can be covered by finitely many ε-balls.

Pick $J\in\mathcal{C}^\infty(\mathbb{R}^m,[0,\infty))$ with $\operatorname{supp}J\subset B_1$ and $\int_{\mathbb{R}^m}J=1$. Then the functions $J_\delta(x):=\delta^{-m}J(x/\delta)$ are mollifiers for $\delta>0$ with $\operatorname{supp}J_\delta\subset B_\delta$ and $\int_{\mathbb{R}^m}J_\delta=1$.

Let $\delta \leq 1$, then $J_\delta * u|_{B_1} \in \mathcal{C}^\infty(B_1, L^p(N))$ is well-defined. Moreover, choose $\delta > 0$ sufficiently small such that for all $u \in \mathcal{K}$

$$\begin{aligned}\big\|J_\delta * u - u\big\|_{L^\infty(B_1,L^p(N))} &= \sup_{x\in B_1}\Big\|\int_{B_\delta} J_\delta(y)\,(u(x-y)-u(x))\,\mathrm{d}^m y\Big\|_{L^p(N)} \\ &\leq \sup_{x\in B_1}\int_{B_\delta} J_\delta(y)\,C|y|^\lambda\,\mathrm{d}^m y \;\leq\; C\delta^\lambda \;\leq\; \tfrac14\varepsilon.\end{aligned}$$

Now it suffices to prove the precompactness of $\mathcal{K}_\delta := \{J_\delta * u \,\big|\, u \in \mathcal{K}\}$. If this holds then $\mathcal{K}_\delta$ can be covered by $\frac12\varepsilon$-balls around $J_\delta * u_i$ with $u_i \in \mathcal{K}$ for $i = 1,\dots,I$ [2] and above estimate shows that $\mathcal{K}$ is covered by the ε-balls around the u_i. Indeed, for each $u \in \mathcal{K}$ one has $\|J_\delta * u - J_\delta * u_i\|_{L^\infty(B_1,L^p(N))} \leq \frac{\varepsilon}{2}$ for some $i = 1,\dots,I$ and thus

$$\|u-u_i\| \;\leq\; \|u - J_\delta * u\| + \|J_\delta * u - J_\delta * u_i\| + \|J_\delta * u_i - u_i\| \;\leq\; \varepsilon.$$

The precompactness of $\mathcal{K}_\delta \subset L^\infty(B_1, L^p(N))$ will follow from the Arzéla-Ascoli theorem (see e.g. [L, IX §4]). Firstly, the smoothened functions $J_\delta * u$ are still equicontinuous on B_1. For all $u \in \mathcal{K}$ and $x, y \in B_1$ use (5) to obtain

$$\begin{aligned}\|(J_\delta * u)(x) - (J_\delta * u)(y)\|_{L^p(N)} &\leq \int_{B_\delta} J_\delta(z)\,\|u(x-z)-u(y-z)\|_{L^p(N)}\,\mathrm{d}^m z \\ &\leq \int_{B_\delta} J_\delta(z)\,C|x-y|^\lambda\,\mathrm{d}^m z \;=\; C|x-y|^\lambda.\end{aligned}$$

Secondly, the L^∞-norm of the smoothened functions is bounded by the L^1-norm of the original ones, so for fixed $\delta > 0$ one obtains a uniform bound from (6) : For all $u \in \mathcal{K}$ and $x \in B_1$

$$\|(J_\delta * u)(x)\|_{\mathcal{C}^{0,\lambda'}(N)} \;\leq\; \int_{B_2} J_\delta(x-y)\,\|u(y)\|_{\mathcal{C}^{0,\lambda'}(N)}\,\mathrm{d}^m y \;\;\leq\;\; C'\|J_\delta\|_\infty.$$

Now the embedding $\mathcal{C}^{0,\lambda'}(N) \hookrightarrow L^p(N)$ is a standard compact Sobolev embedding, so this shows that the subset $\{(J_\delta * u)(x) \,\big|\, u \in \mathcal{K}\} \subset L^p(N)$ is precompact for all $x \in B_1$. Thus the Arzéla-Ascoli theorem asserts that $\mathcal{K}_\delta \subset L^\infty(B_1, L^p(N))$ is compact, and this finishes the proof of the lemma. □

In the proof of theorem B, the weak Uhlenbeck compactness together with the local slice theorem and this lemma will put us in the position to apply the following main regularity theorem that also is the crucial point in the proof of theorem A.

Theorem 2.6. *Let (X, τ, g_0) be a Riemannian 4-manifold with a boundary space-time splitting. For every compact subset $K \subset X$ there exists a compact submanifold $M \subset X$ such that $K \subset M$ and the following holds for all $p > 2$.*

(i) Suppose that $A \in \mathcal{A}^{1,p}(M)$ solves (4). Then $A|_K \in \mathcal{A}(K)$ is smooth.[3]

[2] If a subset $K \subset (X, \mathrm{d})$ of a metric space is precompact, then for fixed $\varepsilon > 0$ one firstly finds $v_1,\dots,v_I \in X$ such that for each $x \in K$ one has $\mathrm{d}(x, v_i) \leq \varepsilon$ for some v_i. For each v_i choose one such $x_i \in K$, or simply drop v_i if this does not exist. Then K is covered by 2ε-balls around the x_i: For each $x \in K$ one has $\mathrm{d}(x, x_i) \leq \mathrm{d}(x, v_i) + \mathrm{d}(v_i, x_i)$ for some $i = 1,\dots,I$.

[3] More precisely, there is an open neighbourhood of $K \subset X$ on which A is smooth.

(ii) Fix a smooth connection $A_0 \in \mathcal{A}(M)$ such that $\tau_i^* A_0|_{\{s\}\times\Sigma_i} \in \mathcal{L}_i$ for all $s \in \mathcal{S}_i$ and $i = 1, \ldots, n$. Moreover, fix $\mathcal{V} = \bigcup_{i=1}^n \bar\tau_{0,i}(\mathcal{U}_i \times \Sigma_i)$, a compact neighbourhood of $K \cap \partial X$. (Here $\bar\tau_{0,i}$ denotes the extension of τ_i given by the geodesics of g_0.) Then for every given constant C_1 there exist constants $\delta > 0$, $\delta_k > 0$, and C_k for all $k \geq 2$ such that the following holds:
Fix $k \geq 2$ and let g be a metric on M that is compatible with τ and satisfies $\|g - g_0\|_{\mathcal{C}^{k+2}(M)} \leq \delta_k$. Suppose that $A \in \mathcal{A}^{1,p}(M)$ solves the boundary value problem (4) with respect to the metric g and satisfies

$$
\begin{aligned}
\|A - A_0\|_{W^{1,p}(M)} &\leq C_1, \\
\|\bar\tau_{0,i}^*(A - A_0)|_{\Sigma_i}\|_{L^\infty(\mathcal{U}_i,\mathcal{A}^{0,p}(\Sigma_i))} &\leq \delta \quad \forall i = 1, \ldots, n.
\end{aligned}
$$

Then $A|_K \in \mathcal{A}(K)$ is smooth by (i) and

$$
\|A - A_0\|_{W^{k,p}(K)} \leq C_k.
$$

We first give some preliminary results for the proof of theorem 2.6. The interior regularity as well as the regularity of the $\mathcal{U}_i$-components on a neighbourhood $\mathcal{U}_i \times \Sigma_i$ of a boundary component $\mathcal{S}_i \times \Sigma_i$ will be a consequence of the following regularity result for Yang-Mills connections. The proof is similar to that of lemma A.2 and can be found in full detail in [W1, Proposition 9.5]. Here M is a compact Riemannian manifold with boundary ∂M and outer unit normal ν. One then deals with two different spaces of test functions,

$$
\begin{aligned}
\mathcal{C}^\infty_\delta(M,\mathfrak{g}) &:= \bigl\{\phi \in \mathcal{C}^\infty(M,\mathfrak{g}) \bigm| \phi|_{\partial M} = 0\bigr\}, \\
\mathcal{C}^\infty_\nu(M,\mathfrak{g}) &:= \bigl\{\phi \in \mathcal{C}^\infty(M,\mathfrak{g}) \bigm| \tfrac{\partial\phi}{\partial\nu}\big|_{\partial M} = 0\bigr\}.
\end{aligned}
$$

Proposition 2.7. *Let (M, g) be a compact Riemannian 4-manifold. Fix a smooth reference connection $A_0 \in \mathcal{A}(M)$. Let $X \in \Gamma(\mathrm{T}M)$ be a smooth vector field that is either perpendicular to the boundary, i.e. $X|_{\partial M} = h \cdot \nu$ for some $h \in \mathcal{C}^\infty(\partial M)$, or is tangential, i.e. $X|_{\partial M} \in \Gamma(\mathrm{T}\partial M)$. In the first case let $\mathcal{T} = \mathcal{C}^\infty_\delta(M,\mathfrak{g})$, in the latter case let $\mathcal{T} = \mathcal{C}^\infty_\nu(M,\mathfrak{g})$. Moreover, let $N \subset \partial M$ be an open subset such that X vanishes in a neighbourhood of $\partial M \setminus N \subset M$. Let $1 < p < \infty$ and $k \in \mathbb{N}$ be such that either $kp > 4$ or $k = 1$ and $2 < p < 4$. In the first case let $q := p$, in the latter case let $q := \frac{4p}{8-p}$. Then there exists a constant C such that the following holds.*

Let $A = A_0 + \alpha \in \mathcal{A}^{k,p}(M)$ be a connection. Suppose that it satisfies

$$
\begin{cases} \mathrm{d}_{A_0}^*\alpha = 0, \\ *\alpha|_{\partial M} = 0 \quad \text{on } N \subset \partial M, \end{cases} \tag{7}
$$

and that for all 1-forms $\beta = \phi \cdot \iota_X g$ with $\phi \in \mathcal{T}$

$$
\int_M \langle F_A \,,\, \mathrm{d}_A\beta \rangle = 0. \tag{8}
$$

Then $\alpha(X) \in W^{k+1,q}(M,\mathfrak{g})$ and

$$
\|\alpha(X)\|_{W^{k+1,q}} \leq C\left(1 + \|\alpha\|_{W^{k,p}} + \|\alpha\|^3_{W^{k,p}}\right).
$$

Moreover, the constant C can be chosen such that it depends continuously on the metric g and the vector field X with respect to the $\mathcal{C}^{k+1}$-topology.

Remark 2.8. In the case $k = 1$ and $2 < p < 4$ the iteration of proposition 2.7 also allows to obtain $W^{2,p}$-regularity and -estimates from initial $W^{1,p}$-regularity and -estimates.

To see this remark, first note that the Sobolev embedding $W^{2,q} \hookrightarrow W^{1,p'}$ holds with $p' = \frac{4q}{4-q}$ since $q < 4$. Now as long as $p' < 4$ one can iterate the proposition and Sobolev embedding to obtain regularity and estimates in W^{1,p_i} with $p_0 = p$ and

$$p_{i+1} \;=\; \frac{4q_i}{4-q_i} \;=\; \frac{2p_i}{4-p_i} \;\geq\; \theta p_i \;>\; p_i .$$

Since $\theta := \frac{2}{4-p} > 1$ this sequence terminates after finitely many steps at some $p_N \geq 4$. Now in case $p_N > 4$ the proposition even yields W^{2,p_N}-regularity and -estimates. In case $p_N = 4$ one only uses W^{1,p_N} for some smaller $p'_N > \frac{8}{3}$ in order to conclude $W^{2,p'_{N+1}}$-regularity and -estimates for $p'_{N+1} > 4$.

Similarly, in case $k = 1$ and $p = 4$ one only needs two steps to reach $W^{2,p'}$ for some $p' > 4$.

The above proposition and remark can be used on all components of the connections in theorem 2.6 except for the Σ-components in small neighbourhoods $\mathcal{U} \times \Sigma$ of boundary components $\mathcal{S} \times \Sigma$. For the regularity of their higher derivatives in Σ-direction we shall use the following lemma. The crucial regularity of the derivatives in the direction of $\mathcal{U}$ of the Σ-components will then follow from the general regularity theory for Cauchy-Riemann equations in [W2].

Lemma 2.9. *Let $k \in \mathbb{N}_0$ and $1 < p < \infty$. Let Ω be a compact manifold, let Σ be a Riemann surface, and equip $\Omega \times \Sigma$ with a product metric $g_\Omega \oplus g$, where $g = (g_x)_{x\in\Omega}$ is a smooth family of metrics on Σ. Then there exists a constant C such that the following holds:*

Suppose that $\alpha \in W^{k,p}(\Omega \times \Sigma, \mathrm{T}^*\Sigma)$ such that both $\mathrm{d}_\Sigma\alpha$ and $\mathrm{d}^*_\Sigma\alpha$ are of class $W^{k,p}$ on $\Omega \times \Sigma$. Then $\nabla_\Sigma\alpha$ also is of class $W^{k,p}$ and one has the following estimate on $\Omega \times \Sigma$

$$\|\nabla_\Sigma\alpha\|_{W^{k,p}} \leq C\bigl(\|\mathrm{d}_\Sigma\alpha\|_{W^{k,p}} + \|\mathrm{d}^*_\Sigma\alpha\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}\bigr).$$

Here ∇_Σ denotes the family of Levi-Civita connections on Σ that is given by the family of metrics g. Moreover, for every fixed family of metrics g one finds a $\mathcal{C}^k$-neighbourhood of metrics for which this estimate holds with a uniform constant C.

Proof of lemma 2.9:
We first prove this for $k = 0$, i.e. suppose that $\alpha \in L^p(\Omega\times\Sigma, \mathrm{T}^*\Sigma)$ and that $\mathrm{d}_\Sigma\alpha, \mathrm{d}^*_\Sigma\alpha$ (defined as weak derivatives) are also of class L^p. We introduce the following functions

$$f := \mathrm{d}^*_\Sigma\alpha \;\in\; L^p(\Omega \times \Sigma), \qquad g := - *_\Sigma \mathrm{d}_\Sigma\alpha \;\in\; L^p(\Omega \times \Sigma),$$

and choose sequences $f^\nu, g^\nu \in \mathcal{C}^\infty(\Omega\times\Sigma)$, and $\alpha^\nu \in \mathcal{C}^\infty(\Omega\times\Sigma, \mathrm{T}^*\Sigma)$ that converge to f, g, and α respectively in the L^p-norm. Note that $\int_\Sigma f = \int_\Sigma g = 0$ in $L^p(\Omega)$, so the f^ν and g^ν can be chosen such that their mean value over Σ also vanishes at every $x \in \Omega$. Then fix $z \in \Sigma$ and find $\xi^\nu, \zeta^\nu \in \mathcal{C}^\infty(\Omega \times \Sigma)$ such that

$$\begin{cases} \Delta_\Sigma\xi^\nu = f^\nu, \\ \xi^\nu(x,z) = 0 \quad \forall x \in \Omega, \end{cases} \qquad\qquad \begin{cases} \Delta_\Sigma\zeta^\nu = g^\nu, \\ \zeta^\nu(x,z) = 0 \quad \forall x \in \Omega. \end{cases}$$

These solutions are uniquely determined since $\Delta_\Sigma : W^{j+2,p}_z(\Sigma) \to W^{j,p}_m(\Sigma)$ is a bounded isomorphism for every $j \in \mathbb{N}_0$ depending smoothly on the metric, i.e. on $x \in \Omega$. Here $W^{j,p}_m(\Sigma)$ denotes the space of $W^{j,p}$-functions with mean value zero and $W^{j+2,p}_z(\Sigma)$ consists of those functions that vanish at $z \in \Sigma$.

Furthermore, let $\pi_x : \Omega^1(\Sigma) \to h^1(\Sigma, g_x)$ be the projection of the smooth 1-forms to the harmonic part $h^1(\Sigma) = \ker \Delta_\Sigma = \ker \mathrm{d}_\Sigma \cap \ker \mathrm{d}^*_\Sigma$ with respect to the metric g_x on Σ. Then π is a family of bounded operators from $L^p(\Sigma, \mathrm{T}^*\Sigma)$ to $W^{j,p}(\Sigma, \mathrm{T}^*\Sigma)$ for any $j \in \mathbb{N}_0$, and it depends smoothly on $x \in \Omega$. So the harmonic part of $\tilde{\alpha}^\nu$ is also smooth, $\pi \circ \tilde{\alpha}^\nu \in \mathcal{C}^\infty(\Omega \times \Sigma, \mathrm{T}^*\Sigma)$. Now consider

$$\alpha^\nu := \mathrm{d}_\Sigma \xi^\nu + *_\Sigma \mathrm{d}_\Sigma \zeta^\nu + \pi \circ \tilde{\alpha}^\nu \;\in\; \mathcal{C}^\infty(\Omega \times \Sigma, \mathrm{T}^*\Sigma).$$

We will show that the sequence α^ν of 1-forms converges to α in the L^p-norm and that moreover $\nabla_\Sigma \alpha^\nu$ is an L^p-Cauchy sequence. For that purpose we will use the following estimate. For all 1-forms $\beta \in W^{1,p}(\Sigma, \mathrm{T}^*\Sigma)$ abbreviating $\mathrm{d}_\Sigma = \mathrm{d}$

$$\begin{aligned} \|\beta\|_{W^{1,p}(\Sigma)} &\leq C\big(\|\mathrm{d}^*\beta\|_{L^p(\Sigma)} + \|\mathrm{d}\beta\|_{L^p(\Sigma)} + \|\pi(\beta)\|_{W^{1,p}(\Sigma)}\big) \\ &\leq C\big(\|\mathrm{d}^*\beta\|_{L^p(\Sigma)} + \|\mathrm{d}\beta\|_{L^p(\Sigma)} + \|\beta\|_{L^p(\Sigma)}\big). \end{aligned} \tag{9}$$

Here and in the following C denotes any finite constant that is uniform for all metrics g_x on Σ in a family of metrics that lies in a sufficiently small $\mathcal{C}^k$-neighbourhood of a fixed family of metrics. To prove (9) we use the Hodge decomposition $\beta = \mathrm{d}\xi + *\mathrm{d}\zeta + \pi(\beta)$. (See e.g. [Wa, Theorem 6.8] and recall that one can identify 2-forms on Σ with functions via the Hodge $*$ operator.) Here one chooses $\xi, \zeta \in W^{2,p}_z(\Sigma)$ such that they solve $\Delta\xi = \mathrm{d}^*\beta$ and $\Delta\zeta = *\mathrm{d}\beta$ respectively and concludes from proposition A.1 for some uniform constant C

$$\begin{aligned} \|\mathrm{d}\xi\|_{W^{1,p}(\Sigma)} &\leq \|\xi\|_{W^{2,p}(\Sigma)} \leq C\|\mathrm{d}^*\beta\|_{L^p(\Sigma)}, \\ \|*\mathrm{d}\zeta\|_{W^{1,p}(\Sigma)} &\leq \|\zeta\|_{W^{2,p}(\Sigma)} \leq C\|\mathrm{d}\beta\|_{L^p(\Sigma)}. \end{aligned}$$

The second step in (9) moreover uses the fact that the projection to the harmonic part is bounded as map $\pi : L^p(\Sigma, \mathrm{T}^*\Sigma) \to W^{1,p}(\Sigma, \mathrm{T}^*\Sigma)$.

Now consider the 1-forms $\alpha - \alpha^\nu \in L^p(\Omega \times \Sigma, \mathrm{T}^*\Sigma)$. For almost all $x \in \Omega$ we have $\alpha(x,\cdot) - \alpha^\nu(x,\cdot) \in L^p(\Sigma, \mathrm{T}^*\Sigma)$ as well as $*\mathrm{d}_\Sigma(\alpha(x,\cdot) - \alpha^\nu(x,\cdot)) \in L^p(\Sigma)$ and $\mathrm{d}^*_\Sigma(\alpha(x,\cdot) - \alpha^\nu(x,\cdot)) \in L^p(\Sigma)$. Then for these $x \in \Omega$ one concludes from the Hodge decomposition that in fact $\alpha(x,\cdot) - \alpha^\nu(x,\cdot) \in W^{1,p}(\Sigma, \mathrm{T}^*\Sigma)$. So we can apply (9) and integrate over $x \in \Omega$ to obtain for all $\nu \in \mathbb{N}$

$$\begin{aligned} &\|\alpha - \alpha^\nu\|^p_{L^p(\Omega\times\Sigma)} \\ &\leq \int_\Omega \|\alpha(x,\cdot) - \alpha^\nu(x,\cdot)\|^p_{L^p(\Sigma, g_x)} \\ &\leq C \int_\Omega \big(\|\mathrm{d}^*_\Sigma(\alpha - \alpha^\nu)\|^p_{L^p(\Sigma)} + \|\mathrm{d}_\Sigma(\alpha - \alpha^\nu)\|^p_{L^p(\Sigma)} + \|\pi(\alpha - \tilde{\alpha}^\nu)\|^p_{W^{1,p}(\Sigma)}\big) \\ &\leq C\big(\|f - f^\nu\|^p_{L^p(\Omega\times\Sigma)} + \|g - g^\nu\|^p_{L^p(\Omega\times\Sigma)} + \|\alpha - \tilde{\alpha}^\nu\|^p_{L^p(\Omega\times\Sigma)}\big). \end{aligned}$$

In the last step we again used the continuity of π. This proves the convergence $\alpha^\nu \to \alpha$ in the L^p-norm, and hence $\nabla_\Sigma \alpha^\nu \to \nabla_\Sigma \alpha$ in the distributional sense. Next, we use (9) to estimate for all $\nu \in \mathbb{N}$

$$\begin{aligned}\|\nabla_\Sigma \alpha^\nu\|^p_{L^p(\Omega\times\Sigma)} &= \int_\Omega \|\nabla_\Sigma \alpha^\nu(x,\cdot)\|^p_{L^p(\Sigma,g_x)} \\ &\leq C\int_\Omega \big(\|\mathrm{d}^*_\Sigma\alpha^\nu\|_{L^p(\Sigma)} + \|\mathrm{d}_\Sigma\alpha^\nu\|_{L^p(\Sigma)} + \|\alpha^\nu\|_{L^p(\Sigma)}\big)^p \\ &\leq C\big(\|\mathrm{d}^*_\Sigma\alpha^\nu\|^p_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}_\Sigma\alpha^\nu\|^p_{L^p(\Omega\times\Sigma)} + \|\alpha^\nu\|^p_{L^p(\Omega\times\Sigma)}\big).\end{aligned}$$

Here one deals with L^p-convergent sequences $\mathrm{d}^*_\Sigma\alpha^\nu = \Delta_\Sigma\xi^\nu = f^\nu \to f = \mathrm{d}^*_\Sigma\alpha$, $-*\mathrm{d}_\Sigma\alpha^\nu = \Delta_\Sigma\zeta^\nu = g^\nu \to g = -*\mathrm{d}_\Sigma\alpha$, and $\alpha^\nu \to \alpha$. So $(\nabla_\Sigma\alpha^\nu)_{\nu\in\mathbb{N}}$ is uniformly bounded in $L^p(\Omega\times\Sigma)$ and hence contains a weakly L^p-convergent subsequence. The limit is $\nabla_\Sigma\alpha$ since this already is the limit in the distributional sense. Thus we have proven the L^p-regularity of $\nabla_\Sigma\alpha$ on $\Omega\times\Sigma$, and moreover above estimate is preserved under the limit, which proves the lemma in the case $k=0$,

$$\begin{aligned}&\|\nabla_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} \\ &\leq \liminf_{\nu\to\infty} \|\nabla_\Sigma\alpha^\nu\|_{L^p(\Omega\times\Sigma)} \\ &\leq \liminf_{\nu\to\infty} C\big(\|\mathrm{d}^*_\Sigma\alpha^\nu\|_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}_\Sigma\alpha^\nu\|_{L^p(\Omega\times\Sigma)} + \|\alpha^\nu\|_{L^p(\Omega\times\Sigma)}\big) \\ &= C\big(\|\mathrm{d}^*_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} + \|\alpha\|_{L^p(\Omega\times\Sigma)}\big).\end{aligned}$$

In the case $k\geq 1$ one can now use the previous result to prove the lemma. Let $\alpha\in W^{k,p}(\Omega\times\Sigma,\mathrm{T}^*\Sigma)$ and suppose that $\mathrm{d}_\Sigma\alpha, \mathrm{d}^*_\Sigma\alpha$ are of class $W^{k,p}$. We denote by ∇ the covariant derivative on $\Omega\times\Sigma$. Then we have to show that $\nabla^k\nabla_\Sigma\alpha$ is of class L^p. So let $X_1,\dots,X_k$ be smooth vector fields on $\Omega\times\Sigma$ and introduce

$$\tilde\alpha := \nabla_{X_1}\dots\nabla_{X_k}\alpha \;\in\; L^p(\Omega\times\Sigma,\mathrm{T}^*\Sigma).$$

Both $\mathrm{d}_\Sigma\tilde\alpha$ and $\mathrm{d}^*_\Sigma\tilde\alpha$ are of class L^p since

$$\begin{aligned}\mathrm{d}_\Sigma\tilde\alpha &= [\mathrm{d}_\Sigma,\nabla_{X_1}\dots\nabla_{X_k}]\alpha + \nabla_{X_1}\dots\nabla_{X_k}\mathrm{d}_\Sigma\alpha, \\ \mathrm{d}^*_\Sigma\tilde\alpha &= [\mathrm{d}^*_\Sigma,\nabla_{X_1}\dots\nabla_{X_k}]\alpha + \nabla_{X_1}\dots\nabla_{X_k}\mathrm{d}^*_\Sigma\alpha.\end{aligned}$$

So the result for $k=0$ implies that $\nabla_\Sigma\tilde\alpha$ is of class L^p, hence $\nabla^k\nabla_\Sigma\alpha$ also is of class L^p since for all smooth vector fields

$$\nabla_{X_1}\dots\nabla_{X_k}\nabla_\Sigma\alpha = [\nabla_\Sigma,\nabla_{X_1}\dots\nabla_{X_k}]\alpha + \nabla_\Sigma\tilde\alpha.$$

With the same argument – using coordinate vector fields X_i and cutting them off – one obtains the estimate

$$\begin{aligned}&\|\nabla^k\nabla_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} \\ &\leq C\big(\|\nabla^k\mathrm{d}^*_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} + \|\nabla^k\mathrm{d}_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} + \|\alpha\|_{W^{k,p}(\Omega\times\Sigma)}\big).\end{aligned}$$

Now this proves the lemma,

$$\begin{aligned}\|\nabla_\Sigma\alpha\|_{W^{k,p}(\Omega\times\Sigma)} &\leq \|\nabla_\Sigma\alpha\|_{W^{k-1,p}(\Omega\times\Sigma)} + \|\nabla^k\nabla_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} \\ &\leq C\big(\|\mathrm{d}^*_\Sigma\alpha\|_{W^{k,p}(\Omega\times\Sigma)} + \|\mathrm{d}_\Sigma\alpha\|_{W^{k,p}(\Omega\times\Sigma)} + \|\alpha\|_{W^{k,p}(\Omega\times\Sigma)}\big).\end{aligned}$$

□

Proof of theorem 2.6 :
Recall that a neighbourhood of the boundary $\partial X \subset X$ can be covered by embeddings $\bar{\tau}_{0,i} : \mathcal{U}_i \times \Sigma_i \hookrightarrow X$ such that $\bar{\tau}_{0,i}^* g_0 = \mathrm{d}s^2 + \mathrm{d}t^2 + g_{0;s,t}$. (In the case (i) we put $g_0 := g$.) Since $K \subset X$ is compact one can cover it by a compact subset $K_{\text{int}} \subset \operatorname{int} X$ and $K_{\text{bdy}} := \bigcup_{i=1}^n \bar{\tau}_{0,i}(I_{0,i} \times [0,\delta_0] \times \Sigma_i)$ for some $\delta_0 > 0$ and $I_{0,i} \subset \mathcal{S}_i$ that are either compact intervals in $\mathbb{R}$ or equal to S^1. Moreover, one can ensure that $K_{\text{bdy}} \subset \operatorname{int} \mathcal{V}$ lies in the interior of the fixed neighbourhood of $K \cap \partial X$. Since X is exhausted by the compact submanifolds X_k one then finds $M := X_k \subset X$ such that both K_{bdy} and K_{int} are contained in the interior of M (and thus also $K \subset M$). Now let $A \in \mathcal{A}^{1,p}(M)$ be a solution of the boundary value problem (4) with respect to a metric g that is compatible with τ. Then we will prove its regularity and the corresponding estimates in the interior case on K_{int} and in the boundary case on K_{bdy} separately.

Interior case :
Firstly, since $K_{\text{int}} \subset \operatorname{int} M$ and $K_{\text{int}} \subset \operatorname{int} X = X \setminus \partial X$ we find a sequence of compact submanifolds $M_k \subset \operatorname{int} X$ such that $K_{\text{int}} \subset M_{k+1} \subset \operatorname{int} M_k \subset M$ for all $k \in \mathbb{N}$. We will prove inductively $A|_{M_k} \in \mathcal{A}^{k,p}(M_k)$ for all $k \in \mathbb{N}$, in each step improving the differentiability of A at the expense of restriction to a smaller submanifold. This will imply that $A|_{K_{\text{int}}} \in \mathcal{A}(K_{\text{int}})$ is smooth. Moreover, we inductively find constants $C_k, \delta_k > 0$ such that the additional assumptions of (ii) in the theorem imply

$$\|A - A_0\|_{W^{k,p}(M_k)} \leq C_k. \tag{10}$$

Here we use the fixed smooth metric g_0 to define the Sobolev norms – for a sufficiently small $\mathcal{C}^k$-neighbourhood of metrics, the Sobolev norms are equivalent with a uniform constant independent of the metric. Moreover, recall that the reference connection A_0 is smooth.

To start the induction we observe that this regularity and estimate are satisfied for $k = 1$ by assumption. For the induction step assume this regularity and estimate to hold for some $k \in \mathbb{N}$. Then we will use proposition 2.7 on $A|_{M_k} \in \mathcal{A}^{k,p}(M_k)$ to deduce the regularity and estimate on M_{k+1}.

Every coordinate vector field on M_{k+1} can be extended to a vector field X on M_k that vanishes near the boundary ∂M_k. So it suffices to consider such vector fields, i.e. use $N = \emptyset$ in the proposition. Then $\alpha := A - A_0$ satisfies the assumption (7). For the weak equation (8) we calculate for all $\beta = \phi \cdot \iota_X g$ with $\phi \in \mathcal{T} = \mathcal{C}_\delta^\infty(M_k, \mathfrak{g})$

$$-\int_{M_k} \langle\, F_A \,,\, \mathrm{d}_A \beta \,\rangle \;=\; \int_{M_k} \langle\, \mathrm{d}_A(\phi \cdot \iota_X g) \wedge F_A \,\rangle \;=\; \int_{\partial M_k} \langle\, \phi \cdot \iota_X g \wedge F_A \,\rangle \;=\; 0.$$

We have used Stokes' theorem while approximating A by smooth connections $\tilde{A}$, for which the Bianchi identity $\mathrm{d}_{\tilde{A}} F_{\tilde{A}} = 0$ holds. Now proposition 2.7 and remark 2.8 imply that $A|_{M_{k+1}} \in \mathcal{A}^{k+1,p}(M_{k+1})$. In the case (ii) of the theorem the proposition moreover provides $\delta_{k+1} > 0$ and a uniform constant C for all metrics g with $\|g - g_0\|_{\mathcal{C}^{k+1}(M_k)} \leq \delta_{k+1}$ such that the following holds: If (10) holds for some constant C_k, then

$$\begin{aligned}
\|A - A_0\|_{W^{k+1,p}(M_{k+1})} &\leq C \left(1 + \|A - A_0\|_{W^{k,p}(M_k)} + \|A - A_0\|^3_{W^{k,p}(M_k)}\right) \\
&\leq C \left(1 + C_k + C_k^3\right) \;=:\; C_{k+1}.
\end{aligned}$$

Here we have used the fact that the Sobolev norm of a 1-form is equivalent to an expression in terms of the Sobolev norms of its components in the coordinate charts. In case $k = 1$ and $p \leq 4$, this uniform bound is not found directly but after finitely many iterations of proposition 2.7 that give estimates on manifolds $N_1 = M_1$ and $M_2 \subset N_{i+1} \subset \operatorname{int} N_i$. In each step one chooses a smaller $\delta_2 > 0$ and a bigger C_2. This iteration uses the same Sobolev embeddings as remark 2.8. This proves the induction step on the interior part K_{int}.

Boundary case :
It remains to prove the regularity and estimates on the part K_{bdy} near the boundary. So consider a single boundary component $K' := \bar\tau_0(I_0 \times [0,\delta_0] \times \Sigma)$. We identify $I_0 = S^1 \cong \mathbb{R}/\mathbb{Z}$ or shift the compact interval such that $I_0 = [-r_0, r_0]$ and hence $K' = \bar\tau_0([-r_0,r_0] \times [0,\delta_0] \times \Sigma)$ for some $r_0 > 0$. Since K_{bdy} (and thus also K') lies in the interior of M as well as $\mathcal{V}$, one then finds $R_0 > r_0$ and $\Delta_0 > \delta_0$ such that $\bar\tau_0([-R_0,R_0] \times [0,\Delta_0] \times \Sigma) \subset M \cap \mathcal{V}$. Here $\bar\tau_0$ is the embedding that brings the metric g_0 into the standard form $\mathrm{d}s^2 + \mathrm{d}t^2 + g_{0;s,t}$. A different metric g compatible with τ defines a different embedding $\bar\tau$ such that $\bar\tau^* g = \mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}$. However, if g is sufficiently $\mathcal{C}^1$-close to g_0, then the geodesics are $\mathcal{C}^0$-close and hence $\bar\tau$ is $\mathcal{C}^0$-close to $\bar\tau_0$. (These embeddings are fixed for $t = 0$, and for $t > 0$ given by the normal geodesics.) Thus for a sufficiently small choice of $\delta_2 > 0$ one finds $R > r > 0$ and $\Delta > \delta > 0$ such that for all τ-compatible metrics g in the δ_2-ball around g_0

$$K' \subset \bar\tau([-r,r] \times [0,\delta] \times \Sigma) \qquad \text{and} \qquad \bar\tau([-R,R] \times [0,\Delta] \times \Sigma) \subset M \cap \mathcal{V}.$$

(In the case (i) this holds with r_0, δ_0, R_0, and Δ_0 for the fixed metric $g = g_0$.) We will prove the regularity and estimates for $\bar\tau^* A$ on $[-r,r] \times [0,\delta] \times \Sigma$. This suffices because for $\mathcal{C}^{k+2}$-close metrics the embedding $\bar\tau$ will be $\mathcal{C}^{k+1}$-close to the fixed $\bar\tau_0$, so that one obtains uniform constants in the estimates between the $W^{k,p}$-norms of A and $\bar\tau^* A$. Furthermore, the families $g_{s,t}$ of metrics on Σ will be $\mathcal{C}^k$-close to $g_{0;s,t}$ for $(s,t) \in [-R,R] \times [0,\Delta]$ if δ_k is chosen sufficiently small. Now choose compact submanifolds $\Omega_k \subset \mathbb{H} := \{(s,t) \in \mathbb{R}^2 \mid t \geq 0\}$ such that for all $k \in \mathbb{N}$

$$[-r,r] \times [0,\delta] \;\subset\; \Omega_{k+1} \;\subset\; \operatorname{int} \Omega_k \;\subset\; [-R,R] \times [0,\Delta].$$

We will prove the theorem by establishing the regularity and estimates for $\bar\tau^* A$ on the $\Omega_k \times \Sigma$ in Sobolev spaces of increasing differentiability. We distinguish the cases $p > 4$ and $4 \geq p > 2$. In case $p > 4$ one uses the following induction.

I) *Let $p > 2$ and suppose that $A \in \mathcal{A}^{1,2p}(M)$ solves (4). Then we will prove inductively that $\bar\tau^* A|_{\Omega_k \times \Sigma} \in \mathcal{A}^{k,p}(\Omega_k \times \Sigma)$ for all $k \geq 2$. Moreover, we will find a constant $\delta > 0$ and constants $C_k, \delta_k > 0$ for all $k \geq 2$ such that the following holds:*
If in addition $\|g - g_0\|_{\mathcal{C}^{k+2}(M)} \leq \delta_k$ and

$$\|A - A_0\|_{W^{1,2p}(M)} \leq C_1,$$
$$\|\bar\tau_0^*(A - A_0)|_\Sigma\|_{L^\infty(\mathcal{U},\mathcal{A}^{0,p}(\Sigma))} \leq \delta,$$

then for all $k \geq 2$

$$\|\bar\tau^*(A - A_0)\|_{W^{k,p}(\Omega_k \times \Sigma)} \leq C_k.$$

This is sufficient to conclude the theorem in case $p > 4$ as follows. One uses I) with p replaced by $\frac{1}{2}p$ to obtain regularity and estimates of $A - A_0$ in $\mathcal{A}^{1,p}(\Omega_1 \times \Sigma)$,

$\mathcal{A}^{2,\frac{p}{2}}(\Omega_2 \times \Sigma)$, and $\mathcal{A}^{k,\frac{p}{2}}(\Omega_k \times \Sigma)$ for all $k \geq 3$. Recall that the component K' of K_{bdy} is contained in each $\bar{\tau}(\Omega_k \times \Sigma)$. In addition, one has the Sobolev embeddings $W^{k+1,\frac{p}{2}} \hookrightarrow W^{k,p} \hookrightarrow \mathcal{C}^{k-1}$ on the compact 4-manifolds $\Omega_{k+1} \times \Sigma$, c.f. [Ad, Theorem 5.4]. So this proves the regularity and estimates on K_{bdy}.

In the case $4 \geq p > 2$ a preliminary iteration is required to achieve the regularity and estimates that are assumed in I). In contrast to I) the iteration is in p instead of k.

II) *Let $4 \geq p > 2$ and suppose that $A \in \mathcal{A}^{1,p}(M)$ solves (4). Then we will prove inductively that $\bar{\tau}^* A|_{\Omega_j \times \Sigma} \in \mathcal{A}^{1,p_j}(\Omega_j \times \Sigma)$ for a sequence (p_j) with $p_1 = p$ and $p_{j+1} = \theta(p_j) \cdot p_j$, where $\theta : (2,4] \to (1, \frac{17}{16}]$ is monotonely increasing and thus the sequence terminates with $p_N > 4$ for some $N \in \mathbb{N}$.*

Moreover, we will find constants $\delta > 0$ and $C_{1,j}, \delta_{1,j} > 0$ for $j = 2, \dots, N$ such that the following holds:

If for some $j = 1, \dots, N$ with $\|g - g_0\|_{\mathcal{C}^3(M)} \leq \delta_{1,j}$ we have

$$\|A - A_0\|_{W^{1,p}(M)} \leq C_1,$$
$$\|\bar{\tau}_0^*(A - A_0)|_\Sigma\|_{L^\infty(\mathcal{U}, \mathcal{A}^{0,p}(\Sigma))} \leq \delta,$$

then

$$\|\bar{\tau}^*(A - A_0)\|_{W^{1,p_j}(\Omega_j \times \Sigma)} \leq C_{1,j}.$$

Assuming I) and II) we first prove the theorem for the case $4 \geq p > 2$. After finitely many steps the iteration of II) gives regularity and estimates in $\mathcal{A}^{1,p_N}(\Omega_N \times \Sigma)$ with $p_N > 4$ and under the assumption $\|g - g_0\|_{\mathcal{C}^3(M)} \leq \delta_{1,N}$ on the metric. Now if necessary decrease p_N slightly such that $2p \geq p_N > 4$, then one still has $\mathcal{A}^{1,p_N}$-regularity and estimates on all components of K_{bdy} as well as on K_{int} (from the previous argument on the interior). Thus the assumptions of I) are satisfied with p replaced by $\frac{1}{2} p_N$ and C_1 replaced by a combination of $C_{1,N}$ and a constant from the interior iteration (both of which only depend on C_1). One just has to choose $\delta_2 \leq \delta_{1,N}$ and choose the $\delta > 0$ in I) smaller than the $\delta > 0$ from II). Then the iteration in I) gives regularity and estimates of $A - A_0$ in $\mathcal{A}^{k,\frac{1}{2}p_N}(\Omega_k \times \Sigma)$ for all $k \geq 2$. This proves the theorem in case $2 < p \leq 4$ due to the Sobolev embeddings $W^{k+1,\frac{1}{2}p_N} \hookrightarrow W^{k,p} \hookrightarrow \mathcal{C}^{k-2}$. So it remains to establish I) and II).

Proof of I):

As start of the induction we will use $\bar{\tau}^* A|_{\Omega_1 \times \Sigma} \in \mathcal{A}^{1,q}(\Omega_1 \times \Sigma)$ and $\|\bar{\tau}^*(A - A_0)\|_{W^{1,q}(\Omega_1 \times \Sigma)} \leq C_1'$, which holds by assumption with $q = 2p$ after replacing C_1 by a larger constant C_1' to make up for the effect of $\bar{\tau}^*$. For the induction step assume that $W^{k,q}$-regularity and estimates hold for some $k \in \mathbb{N}$ with $q = p$ or $q = 2p$ depending on whether $k \geq 2$ or $k = 1$. Then we consider the following decomposition of the connection A and its curvature:

$$\begin{aligned}
\bar{\tau}^* A &= \Phi\,\mathrm{d}s + \Psi\,\mathrm{d}t + B, \\
\bar{\tau}^* F_A &= F_B + (\mathrm{d}_B \Phi - \partial_s B) \wedge \mathrm{d}s + (\mathrm{d}_B \Psi - \partial_t B) \wedge \mathrm{d}t \\
&\quad + (\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi])\,\mathrm{d}s \wedge \mathrm{d}t.
\end{aligned} \tag{11}$$

Here $\Phi, \Psi \in W^{k,q}(\Omega_k \times \Sigma, \mathfrak{g})$, and $B \in W^{k,q}(\Omega_k \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ is a 2-parameter family of 1-forms on Σ. Choose a further compact submanifold $\Omega \subset \operatorname{int} \Omega_k$ such that

$\Omega_{k+1} \subset \operatorname{int}\Omega$. Now we shall use proposition 2.7 to deduce the higher regularity of Φ and Ψ on $\Omega \times \Sigma$. For this purpose one has to extend the vector fields ∂_s and ∂_t on $\Omega \times \Sigma$ to different vector fields on $\Omega_k \times \Sigma$, both denoted by X, and verify the assumptions (7) and (8) of proposition 2.7. These extensions will be chosen such that they vanish in a neighbourhood of $(\partial\Omega_k \setminus \partial\mathbb{H}) \times \Sigma$. Then $\alpha := \bar\tau^*(A - A_0)$ satisfies (7) on $M = \bar\tau(\Omega_k \times \Sigma)$ with $N = \bar\tau((\partial\Omega_k \cap \partial\mathbb{H}) \times \Sigma)$.

Choose a cutoff function $h \in \mathcal{C}^\infty(\Omega_k, [0,1])$ that equals 1 on Ω and vanishes in a neighbourhood of $\partial\Omega_k \setminus \partial\mathbb{H}$. Then firstly, $X := h\partial_t$ is a vector field as required that is perpendicular to the boundary $\partial\Omega_k \times \Sigma$. For this type of vector field we have to check the assumption (8) for all $\beta = \phi h \cdot \mathrm{d}t$ with $\phi \in \mathcal{C}^\infty_\delta(\Omega_k \times \Sigma, \mathfrak{g})$. Note that $\bar\tau_*\beta = (\phi \cdot h) \circ \bar\tau^{-1} \cdot \iota_{(\bar\tau_*\partial_t)} g$ can be trivially extended to M and then vanishes when restricted to ∂M. So we can use partial integration as in the interior case to obtain

$$\int_{\Omega_k \times \Sigma} \langle\, F_{\bar\tau^* A} \,,\, \mathrm{d}_{\bar\tau^* A}\beta \,\rangle \;=\; \int_M \langle\, F_A \,,\, \mathrm{d}_A \bar\tau_*\beta \,\rangle \;=\; -\int_{\partial M} \langle\, \bar\tau_*\beta \wedge F_A \,\rangle \;=\; 0.$$

Secondly, $X := h\partial_s$ also vanishes in a neighbourhood of $(\partial\Omega_k \setminus \partial\mathbb{H}) \times \Sigma$ and is tangential to the boundary $\partial\Omega_k \times \Sigma$. So we have to verify (8) for all $\beta = \phi h \cdot \mathrm{d}s$ with $\phi \in \mathcal{T} = \mathcal{C}^\infty_\nu(\Omega_k \times \Sigma, \mathfrak{g})$. Again, $\bar\tau_*\beta$ extends trivially to M. Then the partial integration yields

$$\begin{aligned} \int_{\Omega_k \times \Sigma} \langle\, F_{\bar\tau^* A} \,,\, \mathrm{d}_{\bar\tau^* A}\beta \,\rangle &= -\int_{\bar\tau^{-1}(\partial M)} \langle\, \beta \wedge \bar\tau^* F_A \,\rangle \\ &= -\int_{(\Omega_k \cap \partial\mathbb{H}) \times \Sigma} \langle\, \phi h \cdot \mathrm{d}s \wedge F_B \,\rangle \;=\; 0. \end{aligned}$$

The last step uses the fact that $B(s,0) = \tau^* A|_{\{s\}\times\Sigma} \in \mathcal{L} \subset \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$, and hence F_B vanishes on $\partial\mathbb{H} \times \Sigma$. However, we have to approximate A by smooth connections in order that Stokes' theorem holds and F_B is well-defined. So this calculation crucially uses the fact that a $W^{1,p}$-connection with boundary values in the Lagrangian submanifold $\mathcal{L}$ can be $W^{1,p}$-approximated by smooth connections with boundary values in $\mathcal{L} \cap \mathcal{A}(\Sigma)$. This was proven in [W2, Corollary 4.5]. So we have verified the assumptions of proposition 2.7 for both $\Phi = \bar\tau^* A(\partial_s)$ and $\Psi = \bar\tau^* A(\partial_t)$ and thus can deduce $\Phi, \Psi \in W^{k+1,q}(\Omega \times \Sigma)$. Moreover, under the additional assumptions of (ii) in the theorem we have the estimates

$$\begin{aligned} \|\Phi - \Phi_0\|_{W^{k+1,q}(\Omega\times\Sigma)} &\le C_s\left(1 + C_k + C_k^3\right) \;=:\; C^s_{k+1}, \\ \|\Psi - \Psi_0\|_{W^{k+1,q}(\Omega\times\Sigma)} &\le C_t\left(1 + C_k + C_k^3\right) \;=:\; C^t_{k+1}. \end{aligned} \tag{12}$$

The constants C_s and C_t are uniform for all metrics in some small $\mathcal{C}^{k+1}$-neighbourhood of $g_{0;s,t}$, so by a possibly smaller choice of $\delta_{k+1} > 0$ they become independent of $g_{s,t}$. Note that in the above estimates we also have decomposed the reference connection in the tubular neighbourhood coordinates, $\bar\tau^* A_0 = \Phi_0\,\mathrm{d}s + \Psi_0\,\mathrm{d}t + B_0$.

It remains to consider the Σ-component B in the tubular neighbourhood. The boundary value problem (4) becomes in the coordinates (11)

$$\left\{ \begin{aligned} \mathrm{d}^*_{B_0}(B - B_0) &= \nabla_s(\Phi - \Phi_0) + \nabla_t(\Psi - \Psi_0), \\ *F_B &= \partial_t\Phi - \partial_s\Psi + [\Psi, \Phi], \\ \partial_s B + *\partial_t B &= \mathrm{d}_B\Phi + *\mathrm{d}_B\Psi, \\ \Psi(s,0) - \Psi_0(s,0) &= 0 \qquad \forall (s,0) \in \partial\Omega_k, \\ B(s,0) &\in \mathcal{L} \qquad \forall (s,0) \in \partial\Omega_k. \end{aligned} \right. \tag{13}$$

Here d_B is the exterior derivative on Σ that is associated with the connection B, similarly $\mathrm{d}^*_{B_0}$ is the coderivative associated with B_0. Moreover, $*$ is the Hodge operator on Σ with respect to the metric $g_{s,t}$, and $\nabla_s\Phi := \partial_s\Phi + [\Phi_0,\Phi]$, $\nabla_t\Phi := \partial_t\Phi + [\Psi_0,\Phi]$. We rewrite the first two equations in (13) as a system of differential equations for $\alpha := B - B_0$ on Σ. For each $(s,t)\in\Omega_k$

$$\mathrm{d}^*_\Sigma\alpha(s,t) = \xi(s,t), \qquad \mathrm{d}_\Sigma\alpha(s,t) = *\zeta(s,t). \tag{14}$$

Here we have abbreviated

$$\begin{aligned}\xi &= *[B_0\wedge *(B-B_0)] + \nabla_s(\Phi-\Phi_0) + \nabla_t(\Psi-\Psi_0),\\ \zeta &= -*\mathrm{d}_\Sigma B_0 - *\tfrac{1}{2}[B\wedge B] + \partial_t\Phi - \partial_s\Psi + [\Psi,\Phi].\end{aligned}$$

These are both functions in $W^{k,q}(\Omega\times\Sigma,\mathfrak{g})$ due to the smoothness of A_0 and the previously established regularity of Φ and Ψ. (This uses the fact that $W^{k,q}\cdot W^{k,q}$ embeds into $W^{k,q}$ due to $W^{k,q}\hookrightarrow L^\infty$.) So lemma 2.9 asserts that $\nabla_\Sigma(B-B_0)$ is of class $W^{k,q}$ on $\Omega\times\Sigma$, and under the assumptions of (ii) in the theorem we obtain the estimate

$$\begin{aligned}&\|\nabla_\Sigma(B-B_0)\|_{W^{k,q}(\Omega\times\Sigma)}\\ &\le C\big(\|\xi\|_{W^{k,q}} + \|\zeta\|_{W^{k,q}} + \|B-B_0\|_{W^{k,q}}\big)\\ &\le C\big(1 + \|B-B_0\|_{W^{k,q}} + \|\Phi-\Phi_0\|_{W^{k+1,q}} + \|\Psi-\Psi_0\|_{W^{k+1,q}}\\ &\qquad + \|B-B_0\|^2_{W^{k,q}} + \|\Phi-\Phi_0\|_{W^{k,q}}\|\Psi-\Psi_0\|_{W^{k,q}}\big)\\ &\le C\big(1 + C_k + C^s_{k+1} + C^t_{k+1} + C^2_k\big) \;=:\; C^\Sigma_{k+1}.\end{aligned} \tag{15}$$

Here C denotes any constant that is uniform for all metrics in a $\mathcal{C}^{k+1}$-neighbourhood of the fixed $g_{0;s,t}$, so this might again require a smaller choice of $\delta_{k+1}>0$ in order that the constant C^Σ_{k+1} becomes independent of the metric $g_{s,t}$.

Now we have established the regularity and estimate for all derivatives of B of order $k+1$ containing at least one derivative in Σ-direction. (Note that in the case $k=1$ we even have L^q-regularity with $q=2p$ where only L^p-regularity was claimed. This additional regularity will be essential for the following argument.) It remains to consider the pure s- and t- derivatives of B and establish the L^p-regularity and -estimate for $\nabla^{k+1}_{\mathbb{H}}B$ on $\Omega_{k+1}\times\Sigma$, where $\nabla_{\mathbb{H}}$ is the standard covariant derivative on $\mathbb{H}$ with respect to the metric $\mathrm{d}s^2+\mathrm{d}t^2$. The reason for this regularity, as we shall show, is the fact that $B\in W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))$ satisfies a Cauchy-Riemann equation with Lagrangian boundary conditions,

$$\begin{cases}\partial_s B + *\partial_t B = G,\\ B(s,0)\in\mathcal{L} \qquad \forall (s,0)\in\partial\Omega.\end{cases} \tag{16}$$

The inhomogeneous term is

$$G := \mathrm{d}_B\Phi + *\mathrm{d}_B\Psi \;\in\; W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma)).$$

Here one uses the fact that $W^{k,q}(\Omega\times\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})\subset W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))$ since the smooth 1-forms are dense in both spaces and the norm on the second space is weaker than the $W^{k,q}$-norm on $\Omega\times\Sigma$, c.f. [W2, Lemma 2.2].

Now one has to apply the regularity result [W2, Theorem 1.2] for the Cauchy-Riemann equation on the complex Banach space $\mathcal{A}^{0,p}(\Sigma)$. As reference complex structure J_0 we use the Hodge $*$ operator on Σ with respect to the fixed family of metrics

$g_{0;s,t}$ on Σ (that varies smoothly with $(s,t) \in \Omega$). The smooth family J of complex structures in the equation is given by the Hodge operators with respect to the metrics $g_{s,t}$. The Lagrangian submanifold $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ is totally real with respect to any Hodge operator, and it is modelled on a closed subspace of $L^p(\Sigma, \mathbb{R}^n)$ for some $n \in \mathbb{N}$ (see [W2, Lemma 4.2, Corollary 4.4]) . In the case (ii) of the theorem moreover a family of connections $B_0 = \bar{\tau}^* A_0|_\Sigma \in \mathcal{C}^\infty(\Omega, \mathcal{A}(\Sigma))$ is given such that $B_0(s,0) \in \mathcal{L}$ for all $(s,0) \in \partial\Omega$ and B satisfies

$$\begin{aligned}\|B - B_0\|_{L^\infty(\Omega,\mathcal{A}^{0,p}(\Sigma))} &= \|\bar{\tau}^*(A - A_0)|_\Sigma\|_{L^\infty(\Omega,\mathcal{A}^{0,p}(\Sigma))} \\ &\leq C\|\bar{\tau}_0^*(A - A_0)|_\Sigma\|_{L^\infty(\mathcal{U},\mathcal{A}^{0,p}(\Sigma))} \;\leq\; C\delta.\end{aligned}$$

Here one uses the fact that $\bar{\tau}(\Omega \times \Sigma) \subset \bar{\tau}_0(\mathcal{U} \times \Sigma)$ lies in a component of the fixed neighbourhood $\mathcal{V}$ of $K \cap \partial X$. The assumption of closeness to A_0 in $\mathcal{A}^{0,p}(\Sigma)$ was formulated for $\bar{\tau}_0^*(A - A_0)|_\Sigma$. However, for a metric g in a sufficiently small $\mathcal{C}^2$-neighbourhood of the fixed metric g_0 the extensions $\bar{\tau}$ and $\bar{\tau}_0$ are $\mathcal{C}^1$-close and one obtains the above estimate with a constant C independent of the metric. Moreover, by induction hypothesis we have

$$\|B - B_0\|_{W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))} \leq C\|\bar{\tau}^*(A - A_0)\|_{W^{k,q}(\Omega_k\times\Sigma)} \;\leq\; CC_k,$$

where the additional constant C comes from the continuous embedding $L^q(\Sigma) \to L^p(\Sigma)$, and is nontrivial in case $k = 1$ due to $q = 2p$. So the function $B \in W^{k,q}(\Omega, \mathcal{A}^{0,p}(\Sigma))$ satisfies the assumptions of [W2, Theorem 1.2] if $\delta > 0$ is chosen sufficiently small. (Note that this choice is independent of $k \in \mathbb{N}$.)

Now [W2, Theorem 1.2] gives $B \in W^{k+1,p}(\Omega_{k+1}, \mathcal{A}^{0,p}(\Sigma))$. By [W2, Lemma 2.2] this also proves $\nabla_{\mathbb{H}}^{k+1} B \in L^p(\Omega_{k+1}, \mathcal{A}^{0,p}(\Sigma)) = L^p(\Omega_{k+1} \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$, and this finishes the induction step $\bar{\tau}^* A|_{\Omega_{k+1}\times\Sigma} \in \mathcal{A}^{k+1,p}(\Omega_{k+1} \times \Sigma)$ for the regularity near the boundary. The induction step for the estimate in case (ii) of the theorem now follows from the estimate from [W2, Theorem 1.2],

$$\begin{aligned}&\|\nabla_{\mathbb{H}}^{k+1}(B - B_0)\|_{L^p(\Omega_{k+1}\times\Sigma)} \\ &\leq \|B - B_0\|_{W^{k+1,p}(\Omega_{k+1},\mathcal{A}^{0,p}(\Sigma))} \\ &\leq C\big(\|G\|_{W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))} + \|B - B_0\|_{W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))}\big) \\ &\leq C\big(C_k + C_k^2 + C_{k+1}^s + C_{k+1}^t\big) \;=:\; C_{k+1}^{\mathbb{H}}. \end{aligned} \tag{17}$$

Here the constant from [W2, Theorem 1.2] is uniform for a sufficiently small $\mathcal{C}^{k+1}$-neighbourhood of complex structures. In this case, these are the families of Hodge operators on Σ that depend on the metric $g_{s,t}$. Thus for sufficiently small $\delta_{k+1} > 0$ that constant (and also the further Sobolev constants that come into the estimate) becomes independent of the metric. The final constant C_{k+1} then results from all the separate estimates, see the decomposition (11) and the estimates in (12), (15), and (17),

$$\|\bar{\tau}^*(A - A_0)\|_{W^{k+1,p}(\Omega_{k+1}\times\Sigma)} \;\leq\; C_k + C_{k+1}^s + C_{k+1}^t + C_{k+1}^\Sigma + C_{k+1}^{\mathbb{H}}.$$

Proof of II):
Except for the higher differentiability of B in direction of $\mathbb{H}$ this iteration works by the same decomposition and equations as in I). The start of the induction $k = 1$

is given by assumption. For the induction step we assume that the claimed W^{1,p_k}-regularity and -estimates hold for some $k \in \mathbb{N}$ with $p_k \leq 4$. Then proposition 2.7 gives $\Phi, \Psi \in W^{2,q_k}(\Omega \times \Sigma)$ with corresponding estimates and

$$q_k = \begin{cases} \frac{4p_k}{8-p_k} & ; \text{if } p_k < 4, \\ 3 & ; \text{if } p_k = 4. \end{cases}$$

(In the case $p_k = 4$ one applies the proposition only assuming $W^{1,p_k'}$-regularity for $p_k' = \frac{24}{7} < 4$, then one obtains W^{2,q_k}-regularity with $q_k = 3$.) Now the right hand sides in (14) lie in $W^{1,q_k}(\Omega \times \Sigma)$, so lemma 2.9 gives W^{1,q_k}-regularity and -estimates for $\nabla_\Sigma B$ on $\Omega \times \Sigma$. Next, $B \in W^{1,p_k}(\Omega, \mathcal{A}^{0,p}(\Sigma))$ satisfies the Cauchy-Riemann equation (16) with the inhomogeneous term $G \in W^{1,q_k}(\Omega \times \Sigma, \mathrm{T}^*(\Omega \times \Sigma) \otimes \mathfrak{g})$. Now we shall use the Sobolev embedding $W^{1,q_k}(\Omega \times \Sigma) \hookrightarrow L^{r_k}(\Omega \times \Sigma)$ with

$$r_k = \frac{4q_k}{4 - q_k} = \begin{cases} \frac{2p_k}{4-p_k} & ; \text{if } p_k < 4, \\ 12 & ; \text{if } p_k = 4. \end{cases}$$

Note that $r_k > p_k \geq p$ due to $p_k > 2$, so that we have $G \in L^{r_k}(\Omega, \mathcal{A}^{0,p}(\Sigma))$. We cannot apply [W2, Theorem 1.2] directly because that would require the initial regularity $B \in W^{1,2p}(\Omega, \mathcal{A}^{0,p}(\Sigma))$ for some $p > 2$. However, we still proceed as in its proof and introduce the coordinates from [W2, Lemma 4.3] that straighten out the Lagrangian submanifold,

$$\Theta_{s,t} : \mathcal{W}_{s,t} \to \mathcal{A}^{0,p}(\Sigma).$$

Here $\mathcal{W}_{s,t} \subset Y \times Y$ is a neighbourhood of zero, Y is a closed subspace of $L^p(\Sigma, \mathbb{R}^m)$ for some $m \in \mathbb{N}$, Θ is in $\mathcal{C}^{k+1}$-dependence on (s,t) in a neighbourhood $U \subset \Omega$ of some $(s_0, 0) \in \Omega \cap \partial\mathbb{H}$ and it maps diffeomorphically to a neighbourhood of $B(s,t)$ or $B_0(s,t)$ in case (ii). Thus one can write

$$B(s,t) = \Theta_{s,t}(v(s,t)) \qquad \forall (s,t) \in U$$

with $v = (v_1, v_2) \in W^{1,p_k}(U, Y \times Y)$. Moreover, we already have the W^{1,q_k}-regularity of both B and $\nabla_\Sigma B$ on $U \times \Sigma$, so $B \in W^{1,q_k}(U, \mathcal{A}^{1,q_k}(\Sigma)) \subset W^{1,q_k}(U, \mathcal{A}^{0,s_k}(\Sigma))$ with corresponding estimates. Here we use the Sobolev embedding [Ad, Theorem 5.4] $W^{1,q_k}(\Sigma) \hookrightarrow L^{s_k}(\Sigma)$ with

$$s_k = \begin{cases} \frac{2q_k}{2-q_k} = \frac{4p_k}{8-3p_k} & ; \text{if } p_k < \frac{8}{3}, \\ \frac{44p_k - 80}{8-p_k} & ; \text{if } p_k \geq \frac{8}{3}, \\ \frac{31}{2} & ; \text{if } p_k = 4. \end{cases}$$

(Here we have chosen suitable values of s_k for later calculations in case $p_k \geq \frac{8}{3}$ and thus $q_k \geq 2$.) The special structure of the coordinate map Θ in [W2, Lemma 4.3] (it is a local diffeomorphism between $\mathcal{A}^{0,s_k}(\Sigma)$ and a closed subset of $L^{s_k}(\Sigma, \mathbb{R}^{2m})$ since $s_k > p_k > 2$) implies that $v \in W^{1,q_k}(U, L^{s_k}(\Sigma, \mathbb{R}^{2m}))$, which will be important later.

The Cauchy-Riemann equation (16) now becomes

$$\begin{cases} \partial_s v + I \partial_t v = f, \\ v_2(s,0) = 0 \qquad \forall s \in \mathbb{R}. \end{cases}$$

Here $I = (\mathrm{d}_v\Theta)^{-1} * (\mathrm{d}_v\Theta) \in W^{1,p_k}(U, \mathrm{End}(Y \times Y))$ and

$$f \;=\; (\mathrm{d}_v\Theta)^{-1}(G - \partial_s\Theta(v) - *\partial_t\Theta(v)) \;\in\; L^{r_k}(U, Y \times Y).$$

We now approximate f in $L^{r_k}(U, Y \times Y)$ by smooth functions that vanish on ∂U, then partial integration in [W2, (2.4) or (10)] yields for all $\phi \in \mathcal{C}^\infty(U, Y^* \times Y^*)$ and a smooth cutoff function $h \in \mathcal{C}^\infty(U, [0,1])$ with $\partial_t h|_{t=0} = 0$ as in the proof of [W2, Theorem 1.2]

$$\begin{aligned}\int_U \langle\, hv \,,\, \Delta\phi \,\rangle = \int_U \langle\, f \,,\, \partial_s(h\phi) - \partial_t(h \cdot I^*\phi) \,\rangle \;+ \int_U \langle\, \tilde{F} \,,\, \phi \,\rangle \\ + \int_{\partial U \cap \partial\mathbb{H}} \langle\, v_1 \,,\, \partial_t(h\phi_1) + \partial_s(h\phi_2) \,\rangle. \end{aligned} \tag{18}$$

Here $\tilde{F} = (\Delta h)v + 2(\partial_s h)\partial_s v + 2(\partial_t h)\partial_t v + h(\partial_t I)\partial_s v - h(\partial_s I)\partial_t v$ contains the crucial terms $(\partial_t I)(\partial_s v)$ and $(\partial_s I)(\partial_t v)$ and thus lies in $L^{\frac{1}{2}p_k}(U, Y \times Y)$. This is a weak Laplace equation with Dirichlet boundary conditions for hv_2, Neumann boundary conditions for hv_1, and with the inhomogeneous term in $W^{-1,r_k}(U, Y \times Y)$. The latter is the dual space of $W^{1,r'_k}(U, Y^* \times Y^*)$ with $\frac{1}{r_k} + \frac{1}{r'_k} = 1$. (The inclusion $L^{\frac{1}{2}p_k}(U) \hookrightarrow W^{-1,r_k}(U)$ is continuous as can be seen via the dual embedding that is due to $\frac{1}{2} - \frac{1}{r'_k} \geq -1 + \frac{1}{p_k/2}$.) Recall that $Y \subset L^p(\Sigma, \mathbb{R}^m)$ is a closed subspace. Since $r_k > p$ the special regularity result [W2, Lemma 2.1] for the Laplace equation with values in a Banach space cannot be applied to deduce $hv \in W^{1,r_k}(U, Y \times Y)$. However, the general regularity theory for the Laplace equation extends to functions with values in a Hilbert space. So we use the embedding $L^p(\Sigma) \hookrightarrow L^2(\Sigma)$. Then (18) is a weak Laplace equation with the inhomogeneous term in $W^{-1,r_k}(U, L^2(\Sigma, \mathbb{R}^{2m}))$ and enables us to deduce $hv \in W^{1,r_k}(U, L^2(\Sigma, \mathbb{R}^{2m}))$ and thus $v \in W^{1,r_k}(\tilde{U}, L^2(\Sigma, \mathbb{R}^{2m}))$ with the corresponding estimates for some smaller domain $\tilde{U}$ (where $h|_{\tilde{U}} \equiv 1$; a finite union of such domains still covers a neighbourhood of $\Omega \cap \partial\mathbb{H}$). Furthermore, recall that $v \in W^{1,q_k}(U, L^{s_k}(\Sigma, \mathbb{R}^{2m}))$. Now we claim that the following inclusion with the corresponding estimates holds for some suitable p_{k+1}

$$W^{1,r_k}(\tilde{U}, L^2(\Sigma)) \;\cap\; W^{1,q_k}(\tilde{U}, L^{s_k}(\Sigma)) \;\subset\; W^{1,p_{k+1}}(\tilde{U}, L^{p_{k+1}}(\Sigma)). \tag{19}$$

To show (19) it suffices to estimate the $L^{p_{k+1}}(\tilde{U} \times \Sigma)$-norm of a smooth function by its $L^{r_k}(\tilde{U}, L^2(\Sigma))$- and $L^{q_k}(\tilde{U}, L^{s_k}(\Sigma))$-norms. Let $\alpha > 2$ and $t \in [1,2)$, then the Hölder inequality gives for all $f \in \mathcal{C}^\infty(\tilde{U} \times \Sigma, \mathbb{R}^{2m})$

$$\begin{aligned}\|f\|^\alpha_{L^\alpha(\tilde{U}\times\Sigma)} &= \int_{\tilde{U}} \int_\Sigma |f|^t |f|^{\alpha - t} \\ &\leq \int_{\tilde{U}} \|f\|^t_{L^2(\Sigma)} \|f\|^{\alpha-t}_{L^{2\frac{\alpha-t}{2-t}}(\Sigma)} \\ &\leq \|f\|^t_{L^r(\tilde{U}, L^2(\Sigma))} \|f\|^{\alpha-t}_{L^{r\frac{\alpha-t}{r-t}}(\tilde{U}, L^{2\frac{\alpha-t}{2-t}}(\Sigma))} \\ &\leq \|f\|^\alpha_{L^r(\tilde{U}, L^2(\Sigma))} + \|f\|^\alpha_{L^{r\frac{\alpha-t}{r-t}}(\tilde{U}, L^{2\frac{\alpha-t}{2-t}}(\Sigma))}.\end{aligned}$$

Here we abbreviated $r := r_k > p_k > 2$. Now we want

$$q_k = \frac{r_k(\alpha - t)}{r_k - t} \qquad \text{and} \qquad s_k = \frac{2(\alpha - t)}{2 - t}. \tag{20}$$

Indeed, in the case $p_k = 4$ our choices $q_k = 3$, $r_k = 12$, and $s_k = \frac{31}{2}$ together with $t := \frac{5}{3}$ and $\alpha := \frac{17}{4}$ solve these equations. So we obtain $p_{k+1} = \alpha = \frac{17}{16}p_k$. In case $p_k < 4$ the first equation gives

$$\alpha = \frac{4+t}{8-p_k}p_k. \tag{21}$$

If moreover $p_k \geq \frac{8}{3}$, then we choose $t := \frac{5}{3}$ to obtain $\alpha = \frac{17}{24-3p_k}p_k \geq \frac{17}{16}p_k$. This also solves (20) with our choice $s_k = \frac{44p_k-80}{8-p_k}$, so we obtain $p_{k+1} = \frac{17}{16}p_k$. Finally, in case $\frac{8}{3} > p_k > 2$ one obtains from (20)

$$t = \frac{p_k^2}{-p_k^2+7p_k-8} \in [1,2).$$

Inserting this in (21) yields $\alpha = \theta(p_k)\cdot p_k$ with

$$\theta(p_k) = \frac{3p_k-4}{-p_k^2+7p_k-8}.$$

One then checks that $\theta(2) = 1$ and $\theta'(p) > 0$ for $p > 2$, thus $\theta(p) > 1$ for $p > 2$. Moreover, $\theta(\frac{8}{3}) = \frac{9}{8}$, so $\theta(p') = \frac{17}{16}$ for some $p' \in (2, \frac{8}{3})$. Now for $p \geq p'$ we extend the function constantly to obtain a monotonely increasing function $\theta : (2,4] \to (1, \frac{17}{16}]$. With this modified function we finally choose $p_{k+1} = \theta(p_k)\cdot p_k$ for all $2 < p_k \leq 4$. This finishes the proof of (19) and thus shows that $v \in W^{1,p_{k+1}}(\tilde{U}, L^{p_{k+1}}(\Sigma))$.

In addition, note that our choice of $p_{k+1} \leq \alpha$ will always satisfy $p_{k+1} \leq r_k$. In case $p_k = 4$ see the actual numbers, in case $p_k < 4$ this is due to (21), $t \leq 2$, and $p_k > 2$,

$$\alpha \leq \frac{6}{8-p_k}p_k \leq \frac{2}{4-p_k}p_k = r_k.$$

Now we again use the special structure of the coordinates Θ in [W2, Lemma 4.3] to deduce that $B = \Theta \circ v \in W^{1,p_{k+1}}(\tilde{U}, \mathcal{A}^{0,p_{k+1}}(\Sigma))$ with the corresponding estimates. Above, we already established the W^{1,r_k}- and thus $W^{1,p_{k+1}}$-regularity and -estimates for Φ and Ψ as well as $B \in L^{p_{k+1}}(\tilde{U}, \mathcal{A}^{1,p_{k+1}}(\Sigma))$. (Recall the Sobolev embedding $W^{1,q_k} \hookrightarrow L^{r_k}$, and that $p_k \geq q_k$ and $r_k \geq p_{k+1}$, so we have $L^{r_k}(\tilde{U}, L^{r_k}(\Sigma))$-regularity of B as well as $\nabla_\Sigma B$.) Putting all this together we have established the $W^{1,p_{k+1}}$-regularity and -estimates for $\bar{\tau}^*A$ over $\tilde{U}_i \times \Sigma$, where the $\tilde{U}_i$ cover a neighbourhood of $\Omega_{k+1} \cap \partial\mathbb{H}$. The interior regularity again follows directly from proposition 2.7.

This iteration gives a sequence (p_k) with $p_{k+1} = \theta(p_k)\cdot p_k \geq \theta(p)\cdot p_k$. So this sequence grows at a rate greater or equal to $\theta(p) > \theta(2) = 1$ and hence reaches $p_N > 4$ after finitely many steps. This finishes the proof of II) and the theorem. □

Proof of theorem A :
Fix a solution $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$ of (2) with $p > 2$. We have to find a gauge transformation $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(X)$ such that $u^*A \in \mathcal{A}(X)$ is smooth. Recall that the manifold $X = \bigcup_{k\in\mathbb{N}} X_k$ is exhausted by compact submanifolds X_k meeting the assumptions of proposition 2.1. So it suffices to prove for every $k \in \mathbb{N}$ that there exists a gauge transformation $u \in \mathcal{G}^{2,p}(X_k)$ such that $u^*A|_{X_k}$ is smooth.

For that purpose fix $k \in \mathbb{N}$ and choose a compact submanifold $M \subset X$ that is large enough such that theorem 2.6 applies to the compact subset $K := X_k \subset M$.

Next, choose $A_0 \in \mathcal{A}(M)$ such that $\|A - A_0\|_{W^{1,p}(M)}$ and $\|A - A_0\|_{L^q(M)}$ are sufficiently small for the local slice theorem, proposition 2.2, to apply to A_0 with the reference connection $\hat{A} = A$. Here due to $p > 2$ one can choose $q > 4$ in the local slice theorem such that the Sobolev embedding $W^{1,p}(M) \hookrightarrow L^q(M)$ holds. Then by proposition 2.2 and remark 2.3 (i) one obtains a gauge transformation $u \in \mathcal{G}^{2,p}(M)$ such that u^*A is in relative Coulomb gauge with respect to A_0. Moreover, u^*A also solves (2) since both the anti-self-duality equation and the Lagrangian submanifolds $\mathcal{L}_i$ are gauge invariant. The latter is due to the fact that u restricts to a gauge transformation in $\mathcal{G}^{1,p}(\Sigma_i)$ on each boundary slice $\tau_i(\{s\} \times \Sigma_i)$ due to the Sobolev embedding $\mathcal{G}^{2,p}(\mathcal{U}_i \times \Sigma) \subset W^{1,p}(\mathcal{U}_i, \mathcal{G}^{1,p}(\Sigma_i)) \hookrightarrow \mathcal{C}^0(\mathcal{U}_i, \mathcal{G}^{1,p}(\Sigma_i))$. So $u^*A \in \mathcal{A}^{1,p}(M)$ is a solution of (4) and theorem 2.6 (i) asserts that $u^*A|_{X_k} \in \mathcal{A}(X_k)$ is indeed smooth.

Such a gauge transformation $u \in \mathcal{G}^{2,p}(X_k)$ can be found for every $k \in \mathbb{N}$, hence proposition 2.1 (i) asserts that there exists a gauge transformation $u \in \mathcal{G}^{2,p}_{\text{loc}}(X)$ on the full noncompact manifold such that $u^*A \in \mathcal{A}(X)$ is smooth as claimed. □

Proof of theorem B :
Fix a smoothly convergent sequence of metrics $g^\nu \to g$ that are compatible to τ and let $A^\nu \in \mathcal{A}^{1,p}_{\text{loc}}(X)$ be a sequence of solutions of (2) with respect to the metrics g^ν. Recall that the manifold $X = \bigcup_{k\in\mathbb{N}} X_k$ is exhausted by compact submanifolds X_k meeting the assumptions of proposition 2.1. We will find a subsequence (again denoted A^ν) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{\text{loc}}(X)$ such that the sequence $u^{\nu\,*}A^\nu$ is bounded in the $W^{\ell,p}$-norm on X_k for all $\ell \in \mathbb{N}$ and $k \in \mathbb{N}$. Then due to the compact Sobolev embeddings $W^{\ell,p}(X_k) \hookrightarrow \mathcal{C}^{\ell-2}(X_k)$ one finds a further diagonal subsequence that converges uniformly with all derivatives on every compact subset of X.

By proposition 2.1 (ii) it suffices to construct the gauge transformations and establish the claimed uniform bounds over X_k for all $k \in \mathbb{N}$ and for any subsequence of the connections (again denoted A^ν). So fix $k \in \mathbb{N}$ and choose a compact submanifold $M \subset X$ such that theorem 2.6 holds with $K = X_k \subset M$. Choose a further compact submanifold $M' \subset X$ such that theorem 2.6 holds with $K = M \subset M'$. Then by assumption of the theorem $\|F_{A^\nu}\|_{L^p(M')}$ is uniformly bounded. So the weak Uhlenbeck compactness, proposition 2.4, provides a subsequence (still denoted A^ν), a limit connection $A_0 \in \mathcal{A}^{1,p}(M')$, and gauge transformations $u^\nu \in \mathcal{G}^{2,p}(M')$ such that $u^{\nu\,*}A^\nu \to A_0$ in the weak $W^{1,p}$-topology. The limit A_0 then satisfies the boundary value problem (2) with respect to the limit metric g. (For the boundary conditions this follows from the compact embedding in lemma 2.5 and the fact that every $\mathcal{L}_i \subset \mathcal{A}^{0,p}(\Sigma_i)$ is a Banach submanifold and hence L^p-closed.) So as in the proof of theorem A one finds a gauge transformation $u_0 \in \mathcal{G}^{2,p}(M)$ such that $u_0^*A_0 \in \mathcal{A}(M)$ is smooth. (Note however that we do not have sufficient boundary conditions on $\partial M' \setminus \partial X$ to obtain smoothness on M'. Thus we had to start out from the larger submanifold $M' \neq M$.) Now replace A_0 by $u_0^*A_0$ and u^ν by $u^\nu u_0 \in \mathcal{G}^{2,p}(M)$, then still $u^{\nu\,*}A^\nu \to A_0$ (since $(u^\nu u_0)^*A^\nu - u_0^*A_0 = u_0^{-1}(u^{\nu\,*}A^\nu - A_0)u_0$ and conjugation by u_0 is continuous in the weak $W^{1,p}$-topology). Thus one has a $W^{1,p}$-bound, $\|u^{\nu\,*}A^\nu - A_0\|_{W^{1,p}(M)} \leq c_0$ for some constant c_0.

Due to $p > 2$ one can now choose $q > 4$ in the local slice theorem such that the Sobolev embedding $W^{1,p}(M) \hookrightarrow L^q(M)$ is compact. Hence for a further subsequence of the connections $u^{\nu\,*}A^\nu \to A_0$ in the L^q-norm. Let $\varepsilon > 0$ be the constant from proposition 2.2 for the reference connection $\hat{A} = A_0$, then one finds a further subsequence such that $\|u^{\nu\,*}A^\nu - A_0\|_{L^q(M)} \leq \varepsilon$ for all $\nu \in \mathbb{N}$. So the local slice theorem provides further gauge transformations $\tilde{u}^\nu \in \mathcal{G}^{2,p}(M)$ such that the $\tilde{u}^{\nu\,*}A^\nu$ are in relative Coulomb gauge with respect to A_0. The gauge transformed connections still solve (2), hence the $\tilde{u}^{\nu\,*}A^\nu$ are solutions of (4). Moreover, we have $\|\tilde{u}^{\nu\,*}A^\nu - A_0\|_q \leq C_{CG}\|u^{\nu\,*}A^\nu - A_0\|_q$, hence $\tilde{u}^{\nu\,*}A^\nu \to A_0$ in the L^q-norm, and

$$\left\|\tilde{u}^{\nu\,*}A^\nu - A_0\right\|_{W^{1,p}(M)} \leq C_{CG}c_0.$$

The higher $W^{k,p}$-bounds will now follow from theorem 2.6 after we verify its assumptions. Fix the metric $g_0 := g$ and a compact neighbourhood $\mathcal{V} = \bigcup_{i=1}^n \bar{\tau}_{0,i}(\mathcal{U}_i \times \Sigma_i)$ of $K \cap \partial X$. Then the $\bar{\tau}_{0,i}^*(\tilde{u}^{\nu\,*}A^\nu - A_0)|_{\Sigma_i}$ are uniformly $W^{1,p}$-bounded and converge to zero in the L^q-norm on $\mathcal{U}_i \times \mathcal{S}_i$ as seen above. Now the embedding

$$W^{1,p}(\mathcal{U}_i \times \Sigma_i, \mathrm{T}^*\Sigma_i \otimes \mathfrak{g}) \hookrightarrow L^\infty(\mathcal{U}_i, \mathcal{A}^{0,p}(\Sigma_i))$$

is compact by lemma 2.5. Thus one finds a subsequence such that the $\bar{\tau}_{0,i}^*(\tilde{u}^{\nu\,*}A^\nu)|_{\Sigma_i}$ converge in $L^\infty(\mathcal{U}_i, \mathcal{A}^{0,p}(\Sigma_i))$. The limit can only be $\bar{\tau}_{0,i}^*A_0|_{\Sigma_i}$ since this already is the L^q-limit. Now in theorem 2.6 (ii) choose the constant $C_1 = C_{CG}c_0$ and let $\delta > 0$ be the constant determined from C_1. Then one can take a subsequence such that

$$\|\bar{\tau}_{0,i}^*(\tilde{u}^{\nu\,*}A^\nu - A_0)_{\Sigma_i}\|_{L^\infty(\mathcal{U}_i, \mathcal{A}^{0,p}(\Sigma_i))} \leq \delta \qquad \forall i = 1, \ldots, n, \forall \nu.$$

Now theorem 2.6 (ii) provides the claimed uniform bounds as follows. Fix $\ell \in \mathbb{N}$, then $\|g^\nu - g\|_{\mathcal{C}^{\ell+2}(M)} \leq \delta_\ell$ for all $\nu \geq \nu_\ell$ with some $\nu_\ell \in \mathbb{N}$, and thus

$$\left\|\tilde{u}^{\nu\,*}A^\nu - A_0\right\|_{W^{\ell,p}(X_k)} \leq C_\ell \qquad \forall \nu \geq \nu_\ell.$$

This finally implies the uniform bound for this subsequence,

$$\sup_{\nu \in \mathbb{N}} \left\|\tilde{u}^{\nu\,*}A^\nu\right\|_{W^{\ell,p}(X_k)} < \infty.$$

Here the gauge transformations $\tilde{u}^\nu \in \mathcal{G}^{2,p}(X_k)$ still depend on $k \in \mathbb{N}$ and are only defined on X_k. But now proposition 2.1 (ii) provides a subsequence of (A^ν) and gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{\mathrm{loc}}(X)$ defined on the full noncompact manifold such that $u^{\nu\,*}A^\nu$ is uniformly bounded in every $W^{\ell,p}$-norm on every compact submanifold X_k. Now one can iteratively use the compact Sobolev embeddings $W^{\ell+2,p}(X_\ell) \hookrightarrow \mathcal{C}^\ell(X_\ell)$ for each $\ell \in \mathbb{N}$ to find a further subsequence of the connections that converges in $\mathcal{C}^\ell(X_\ell)$. If in each step one fixes one further element of the sequence, then this iteration finally yields a sequence of connections that converges uniformly with all derivatives on every compact subset of X to a smooth connection $A \in \mathcal{A}(X)$. $\Box$

3. Fredholm theory

This section concerns the linearization of the boundary value problem (2) in the special case of a compact 4-manifold of the form $X = S^1 \times Y$, where Y is a compact orientable 3-manifold whose boundary $\partial Y = \Sigma$ is a disjoint union of connected Riemann surfaces. The aim of this section is to prove theorem C. (The actual Fredholm property in part (i) will be proven last, building on (ii) and (iii), which are stated separately for future reference.)

So we equip $S^1 \times Y$ with a product metric $\tilde{g} = \mathrm{d}s^2 + g_s$ (where g_s is an S^1-family of metrics on Y) and assume that this is compatible with the natural space-time splitting of the boundary $\partial X = S^1 \times \Sigma$. This means that for some $\Delta > 0$ there exists an embedding

$$\tau : S^1 \times [0,\Delta) \times \Sigma \hookrightarrow S^1 \times Y$$

preserving the boundary, $\tau(s,0,z) = (s,z)$ for all $s \in S^1$ and $z \in \Sigma$, such that

$$\tau^* \tilde{g} = \mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}.$$

Here $g_{s,t}$ is a smooth family of metrics on Σ. This assumption on the metric implies that the normal geodesics are independent of $s \in S^1$ in a neighbourhood of the boundary. So in fact, the embedding is given by $\tau(s,t,z) = (s, \gamma_z(t))$, where γ is the normal geodesic starting at $z \in \Sigma$. This seems like a very restrictive assumption, but it suffices for our application to Riemannian 4-manifolds with a boundary space-time splitting. Indeed, the neighbourhoods of the compact boundary components are isometric to $S^1 \times Y$ with $Y = [0,\Delta] \times \Sigma$ and a metric $\mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}$.

Now fix $p > 2$ and let $\mathcal{L} \subset \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ be a gauge invariant Lagrangian submanifold of $\mathcal{A}^{0,p}(\Sigma)$ as in the introduction. Then for $\tilde{A} \in \mathcal{A}^{1,p}(S^1 \times Y)$ we consider the nonlinear boundary value problem

$$\begin{cases} *F_{\tilde{A}} + F_{\tilde{A}} = 0, \\ \tilde{A}|_{\{s\}\times\partial Y} \in \mathcal{L} \quad \forall s \in S^1. \end{cases} \tag{22}$$

Fix a smooth connection $\tilde{A} \in \mathcal{A}(S^1 \times Y)$ with Lagrangian boundary values (but not necessarily a solution of this boundary value problem). It can be decomposed as $\tilde{A} = A + \Phi \mathrm{d}s$ with $\Phi \in \mathcal{C}^\infty(S^1 \times Y, \mathfrak{g})$ and with $A \in \mathcal{C}^\infty(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ satisfying $A_s := A(s)|_{\partial Y} \in \mathcal{L}$ for all $s \in S^1$. Similarly, a tangent vector $\tilde{\alpha}$ to $\mathcal{A}^{1,p}(S^1 \times Y)$ decomposes as $\tilde{\alpha} = \alpha + \varphi \mathrm{d}s$ with $\varphi \in W^{1,p}(S^1 \times Y, \mathfrak{g})$ and $\alpha \in W^{1,p}(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$. Now let $E^{1,p}_A \subset W^{1,p}(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ be the subspace of S^1-families of 1-forms α that satisfy the boundary conditions from the linearization of (22) and the Coulomb gauge,

$$*\alpha(s)|_{\partial Y} = 0 \qquad \text{and} \qquad \alpha(s)|_{\partial Y} \in \mathrm{T}_{A_s}\mathcal{L} \qquad \text{for all } s \in S^1.$$

Then the linearized operator for the study of the moduli space of gauge equivalence classes of solutions of (22) is as in the introduction

$$D_{(A,\Phi)} : E^{1,p}_A \times W^{1,p}(S^1 \times Y, \mathfrak{g}) \longrightarrow L^p(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g}) \times L^p(S^1 \times Y, \mathfrak{g})$$

given by

$$D_{(A,\Phi)}(\alpha,\varphi) = \big(\nabla_s \alpha - \mathrm{d}_A \varphi + *\mathrm{d}_A \alpha \,,\, \nabla_s \varphi - \mathrm{d}_A^* \alpha\big).$$

Here d_A denotes the exterior derivative corresponding to the connection $A(s)$ on Y for all $s \in S^1$, $*$ denotes the Hodge operator on Y with respect to the s-dependent metric g_s on Y, and we use the notation $\nabla_s \alpha := \partial_s \alpha + [\Phi, \alpha]$. Our main result, theorem C (i), is the Fredholm property of $D_{(A,\Phi)}$. We now give an outline of its proof.

The first crucial point is the estimate in theorem C (ii), which ensures that $D_{(A,\Phi)}$ has a closed image and a finite dimensional kernel. It can be rephrased as follows due to the identities

$$\begin{aligned} \mathrm{d}^+_{\tilde{A}} \tilde{\alpha} &= \tfrac{1}{2} * \big(\nabla_s \alpha - \mathrm{d}_A \varphi + *\mathrm{d}_A \alpha\big) - \tfrac{1}{2}\big(\nabla_s \alpha - \mathrm{d}_A \varphi + *\mathrm{d}_A \alpha\big) \wedge \mathrm{d}s, \\ \mathrm{d}^*_{\tilde{A}} \tilde{\alpha} &= -\nabla_s \varphi + \mathrm{d}^*_A \alpha. \end{aligned} \tag{23}$$

Lemma 3.1. *There is a constant C such that for all $\tilde\alpha \in W^{1,p}(X, \mathrm{T}^*X \otimes \mathfrak{g})$ satisfying*

$$*\tilde\alpha|_{\partial X} = 0 \qquad \textit{and} \qquad \tilde\alpha|_{\{s\}\times\partial Y} \in \mathrm{T}_{A_s}\mathcal{L} \quad \forall s \in S^1$$

one has the estimate

$$\|\tilde\alpha\|_{W^{1,p}} \leq C\bigl(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_p + \|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_p + \|\tilde\alpha\|_p\bigr).$$

The second part of the Fredholm theory for $D_{(A,\Phi)}$ is the identification of the cokernel with the kernel of a slightly modified linearized operator, which will be used to prove that the cokernel is finite dimensional. To be more precise let $\sigma : S^1 \times Y \to S^1 \times Y$ denote the reflection given by $\sigma(s, y) := (-s, y)$, where $S^1 \cong \mathbb{R}/\mathbb{Z}$. Then we will establish the following duality:

$$(\beta, \zeta) \in (\operatorname{im} D_{(A,\Phi)})^\perp \iff (\beta \circ \sigma, \zeta \circ \sigma) \in \ker D_{\sigma^*(A,\Phi)},$$

where $D_{\sigma^*(A,\Phi)}$ is the linearized operator at the connection $\sigma^*\tilde A = A\circ\sigma - \Phi\circ\sigma\mathrm{d}s$ with respect to the metric $\sigma^*\tilde g$ on $S^1 \times Y$. Once we know that $\operatorname{im} D_{(A,\Phi)}$ is closed, this gives an isomorphism between $(\operatorname{coker} D_{(A,\Phi)})^* \cong (\operatorname{im} D_{(A,\Phi)})^\perp$ and $\ker D_{\sigma^*(A,\Phi)}$. Here Z^* denotes the dual space of a Banach space Z, and for a subspace $Y \subset Z$ we denote by $Y^\perp \subset Z^*$ the space of linear functionals that vanish on Y. Now the estimate in theorem C (ii) will also apply to $D_{\sigma^*(A,\Phi)}$, and this implies that its kernel – and hence the cokernel of $D_{(A,\Phi)}$ – is of finite dimension. The main difficulty in establishing the above duality is the regularity result theorem C (iii).

This regularity as well as the estimate in theorem C (ii) or lemma 3.1 will be proven analogously to the nonlinear regularity and estimates in section 2. Again, the interior regularity and estimate is standard elliptic theory, and one has to use a splitting near the boundary. We shall show that the S^1- and the normal component both satisfy a Laplace equation with Neumann and Dirichlet boundary conditions respectively. The Σ-component will again gives rise to a (weak) Cauchy-Riemann equation in a Banach space, only this time the boundary values will lie in the tangent space of the Lagrangian. In contrast to the required L^p-estimates we shall first show that the L^2-estimate for L^p-regular 1-forms can be obtained by more elementary methods. These were already outlined in [S] as indication for the Fredholm property of the boundary value problem (22).

Let $\tilde\alpha \in W^{1,p}(X, \mathrm{T}^*X \otimes \mathfrak{g})$ be as in lemma 3.1 for some $p > 2$. From the first boundary condition $*\tilde\alpha|_{\partial X} = 0$ one obtains

$$\|\nabla\tilde\alpha\|_2^2 = \|\mathrm{d}\tilde\alpha\|_2^2 + \|\mathrm{d}^*\tilde\alpha\|_2^2 - \int_{\partial X} \tilde g(Y_{\tilde\alpha}, \nabla_{Y_{\tilde\alpha}}\nu).$$

Here the vector field $Y_{\tilde\alpha}$ is given by $\iota_{Y_{\tilde\alpha}}\tilde g = \tilde\alpha$, so $\int_{\partial X} \tilde g(Y_{\tilde\alpha}, \nabla_{Y_{\tilde\alpha}}\nu) \geq -C\|\tilde\alpha\|^2_{L^2(\partial X)}$. For this last term one then uses the following special version of the Sobolev trace theorem for general $1 < q < \infty$.

Let $\tau : [0,\Delta) \times \partial X \to X$ be a diffeomorphism to a tubular neighbourhood of ∂X in X. Then for all $\delta > 0$ one finds a constant C_δ such that for all $f \in W^{1,q}(X)$

$$\begin{aligned}
&\|f\|^q_{L^q(\partial X)} \\
&= \int_{\partial X}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\Big((s-1)|f(\tau(s,z))|^q\Big)\,\mathrm{d}s\,\mathrm{d}^3z \\
&\le \int_{\partial X}\int_0^1 |f(\tau(s,z))|^q\,\mathrm{d}s\,\mathrm{d}^3z + \int_{\partial X}\int_0^1 q|f(\tau(s,z))|^{q-1}|\partial_s f(\tau(s,z))|\,\mathrm{d}s\,\mathrm{d}^3z \\
&\le C\big(\|f\|^q_{L^q(X)} + \|f\|^{q-1}_{L^q(X)}\|\nabla f\|_{L^q(X)}\big) \\
&\le \big(\delta\|f\|_{W^{1,q}(X)} + C_\delta\|f\|_{L^q(X)}\big)^q. \qquad (24)
\end{aligned}$$

This uses the fact that for all $x, y \ge 0$ and $\delta > 0$

$$x^{q-1}y \;\le\; \begin{Bmatrix} \delta^q y^q & ;\text{if } x \le \delta^{\frac{q}{q-1}} y \\ \delta^{-\frac{q}{q-1}} x^q & ;\text{if } x \ge \delta^{\frac{q}{q-1}} y \end{Bmatrix} \;\le\; \big(\delta y + \delta^{-\frac{1}{q-1}} x\big)^q.$$

So we obtain

$$\|\tilde\alpha\|_{W^{1,2}} \le C\big(\|\mathrm{d}_{\tilde A}\tilde\alpha\|_2 + \|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_2 + \|\tilde\alpha\|_2\big). \qquad (25)$$

In fact, the analogous $W^{1,p}$-estimates hold true for general p, as is proven e.g. in [W1, Theorem 5.1]. However, in the case $p = 2$ one can calculate further for all $\delta > 0$

$$\begin{aligned}
\|\mathrm{d}_{\tilde A}\tilde\alpha\|_2^2 &= \int_X \langle\, \mathrm{d}_{\tilde A}\tilde\alpha\,,\, 2\mathrm{d}^+_{\tilde A}\tilde\alpha\,\rangle - \int_X \langle\, \mathrm{d}_{\tilde A}\tilde\alpha \wedge \mathrm{d}_{\tilde A}\tilde\alpha\,\rangle \\
&= 2\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_2^2 - \int_X \langle\, \tilde\alpha \wedge [F_{\tilde A}\wedge\tilde\alpha]\,\rangle - \int_{\partial X}\langle\, \tilde\alpha\wedge \mathrm{d}_{\tilde A}\tilde\alpha\,\rangle \\
&\le 2\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_2^2 + C_\delta\|\tilde\alpha\|_2^2 + \delta\|\tilde\alpha\|^2_{W^{1,2}}. \qquad (26)
\end{aligned}$$

Here the boundary term above is estimated as follows. We use the universal covering of $S^1 = \mathbb{R}/\mathbb{Z}$ to integrate over $[0,1]\times\partial Y$ instead of $\partial X = S^1 \times \partial Y$. Introduce $A := (A_s)_{s\in S^1}$, which is a smooth path in $\mathcal{L}$. Then using the splitting $\tilde\alpha|_{\partial X} = \alpha + \varphi\mathrm{d}s$ with $\alpha : S^1\times\Sigma \to \mathrm{T}^*\Sigma\otimes\mathfrak{g}$ and $\varphi : S^1\times\Sigma\to\mathfrak{g}$ one obtains

$$\begin{aligned}
&-\int_{\partial X}\langle\, \tilde\alpha\wedge\mathrm{d}_{\tilde A}\tilde\alpha\,\rangle \\
&= -\int_0^1\int_\Sigma \langle\, \varphi\,,\,\mathrm{d}_A\alpha\,\rangle\,\mathrm{dvol}_\Sigma\wedge\mathrm{d}s \;-\int_0^1\int_\Sigma\langle\,\alpha\wedge(\mathrm{d}_A\varphi - \nabla_s\alpha)\,\rangle\wedge\mathrm{d}s \\
&= \int_0^1\int_\Sigma \langle\,\alpha\wedge\nabla_s\alpha\,\rangle\wedge\mathrm{d}s \\
&\le \delta\|\tilde\alpha\|^2_{W^{1,2}(X)} + C'_\delta\|\tilde\alpha\|^2_{L^2(X)}.
\end{aligned}$$

Firstly, we have used the fact that $\mathrm{d}_A\alpha|_\Sigma = 0$ since $\alpha(s) \in \mathrm{T}_{A_s}\mathcal{L} \subset \ker\mathrm{d}_{A_s}$ for all $s\in S^1$. Secondly, we have also used that both α and $\mathrm{d}_A\varphi$ lie in $\mathrm{T}_A\mathcal{L}$, hence the symplectic form $\int_\Sigma\langle\,\alpha\wedge\mathrm{d}_A\varphi\,\rangle$ vanishes for all $s\in S^1$. This is not strictly true since $\tilde\alpha$ only restricts to an L^p-regular 1-form on ∂X. However, as 1-form on $[0,1]\times Y$ it can be approximated as follows by smooth 1-forms that meet the Lagrangian boundary condition on $[0,1]\times\Sigma$.

We use the linearization of the coordinates in [W2, Lemma 4.3] at A_s for every $s\in[0,1]$. Since the path $s\mapsto A_s\in\mathcal{L}\cap\mathcal{A}(\Sigma)$ is smooth, this gives a smooth path of diffeomorphisms Θ_s for any $q>2$,

$$\Theta_s:\begin{array}{rcl} Z\times Z & \longrightarrow & L^q(\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g}) \\ (\xi,v,\zeta,w) & \longmapsto & \mathrm{d}_{A_s}\xi+\sum_{i=1}^m v^i\gamma_i(s)+*\mathrm{d}_{A_s}\zeta+\sum_{i=1}^m w^i*\gamma_i(s), \end{array}$$

with $Z:=W^{1,q}_z(\Sigma,\mathfrak{g})\times\mathbb{R}^m$ and where the $\gamma_i\in\mathcal{C}^\infty([0,1]\times\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})$ satisfy $\gamma_i(s)\in\mathrm{T}_{A_s}\mathcal{L}$ for all $s\in[0,1]$. We perform the above estimate on $[0,1]\times Y$ since we can not necessarily achieve $\gamma_i(0)=\gamma_i(1)$. In these coordinates, we mollify to obtain the required smooth approximations of $\tilde\alpha$ near the boundary. Furthermore, we use these coordinates for $q=3$ to write the smooth approximations on the boundary as $\alpha(s)=\mathrm{d}_{A_s}\xi(s)+\sum_{i=1}^m v^i(s)\gamma_i(s)$ with $\|\xi(s)\|_{W^{1,3}(\Sigma)}+|v(s)|\leq C\|\alpha(s)\|_{L^3(\Sigma)}$. Then for all $s\in[0,1]$

$$\begin{aligned}\int_\Sigma\langle\,\alpha(s)\wedge\nabla_s\alpha(s)\,\rangle &= \int_\Sigma\langle\,\alpha\wedge\bigl(\mathrm{d}_{A_s}\partial_s\xi+\textstyle\sum_{i=1}^m\partial_s v^i\cdot\gamma_i\bigr)\,\rangle \\ &\quad+\int_\Sigma\langle\,\alpha\wedge\bigl([\Phi,\alpha]+[\partial_s A,\xi]+\textstyle\sum_{i=1}^m v^i\cdot\partial_s\gamma_i\bigr)\,\rangle \\ &\leq C\|\alpha(s)\|_{L^2(\Sigma)}\|\alpha(s)\|_{L^3(\Sigma)}.\end{aligned}$$

Here the crucial point is that $\mathrm{d}_A\partial_s\xi$ and $\partial_s v^i\cdot\gamma_i$ are tangent to the Lagrangian, hence the first term vanishes. Now one uses (24) for $q=2$ and the Sobolev trace theorem (the restriction $W^{1,2}(X)\to L^3(\partial X)$ is continuous by e.g. [Ad, Theorem 6.2]) to obtain the estimate,

$$\begin{aligned}\int_0^1\int_\Sigma\langle\,\alpha\wedge\nabla_s\alpha\,\rangle\wedge\mathrm{d}s &\leq C\|\tilde\alpha\|_{L^2(\partial X)}\|\tilde\alpha\|_{L^3(\partial X)} \\ &\leq\tfrac{\delta}{2}\|\tilde\alpha\|^2_{W^{1,2}(X)}+C_\delta\|\tilde\alpha\|_{L^2(X)}\|\tilde\alpha\|_{W^{1,2}(X)} \\ &\leq\delta\|\tilde\alpha\|^2_{W^{1,2}(X)}+C'_\delta\|\tilde\alpha\|^2_{L^2(X)}.\end{aligned}$$

This proves (26). Now $\delta>0$ can be chosen arbitrarily small, so the term $\|\tilde\alpha\|_{W^{1,2}}$ can be absorbed into the left hand side of (25), and thus one obtains the claimed estimate

$$\|\tilde\alpha\|_{W^{1,2}}\;\leq\;C\bigl(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_2+\|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_2+\|\tilde\alpha\|_2\bigr).$$

Proof of theorem C (ii) or lemma 3.1 :
We will use lemma A.2 for the manifold $M:=S^1\times Y$ in several different cases to obtain the estimate for different components of $\tilde\alpha$. The first weak equation in lemma A.2 is the same in all cases. For all $\eta\in\mathcal{C}^\infty(M;\mathfrak{g})$

$$\begin{aligned}\int_M\langle\,\tilde\alpha\,,\,\mathrm{d}\eta\,\rangle &= \int_M\langle\,\mathrm{d}^*\tilde\alpha\,,\,\eta\,\rangle+\int_{\partial M}\langle\,\eta\,,\,*\tilde\alpha\,\rangle \\ &= \int_M\langle\,\mathrm{d}^*_{\tilde A}\tilde\alpha+*[\tilde A\wedge*\tilde\alpha]\,,\,\eta\,\rangle\quad=\int_M\langle\,f\,,\,\eta\,\rangle.\end{aligned}$$

Here one uses the fact that $*\tilde\alpha|_{\partial M}=0$. Then $f\in L^p(M,\mathfrak{g})$ and

$$\|f\|_p\leq\|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_p+2\|\tilde A\|_\infty\|\tilde\alpha\|_p. \tag{27}$$

To obtain the second weak equation in lemma A.2 we calculate for all $\lambda \in \Omega^1(M;\mathfrak{g})$

$$
\begin{aligned}
\int_M \langle\, \tilde\alpha\,,\, \mathrm{d}^*\mathrm{d}\lambda \,\rangle &= \int_M \langle\, \tilde\alpha\,,\, \mathrm{d}^*\mathrm{d}\lambda + \mathrm{d}^* * \mathrm{d}\lambda \,\rangle \\
&= \int_M \langle\, \gamma\,,\, \mathrm{d}\lambda \,\rangle - \int_{S^1\times\partial Y} \langle\, \tilde\alpha \wedge *\mathrm{d}\lambda \,\rangle - \int_{S^1\times\partial Y} \langle\, \tilde\alpha \wedge \mathrm{d}\lambda \,\rangle,
\end{aligned} \tag{28}
$$

where $\gamma \,=\, \mathrm{d}\tilde\alpha + *\mathrm{d}\tilde\alpha \,=\, 2\mathrm{d}^+_{\tilde A}\tilde\alpha - 2[\tilde A \wedge \tilde\alpha]^+ \,\in L^p(M, \Lambda^2\mathrm{T}^*M \otimes \mathfrak{g})$ and

$$
\|\gamma\|_p \le 2\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_p + 4\|\tilde A\|_\infty \|\tilde\alpha\|_p. \tag{29}
$$

Now recall that there is an embedding $\tau : S^1 \times [0,\Delta) \times \Sigma \hookrightarrow S^1 \times Y$ to a tubular neighbourhood of $S^1 \times \partial Y$ such that $\tau^*\tilde g = \mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}$ for a family $g_{s,t}$ of metrics on Σ. One can then cover $M = S^1 \times Y$ with $\tau(S^1 \times [0, \frac{\Delta}{2}] \times \Sigma)$ and a compact subset $V \subset M \setminus \partial M$.

For the claimed estimate of $\tilde\alpha$ over V it suffices to use lemma A.2 for vector fields $X \in \Gamma(\mathrm{T}M)$ that are equal to coordinate vector fields on V and vanish on ∂M. So one has to consider (28) for $\lambda = \phi \cdot \iota_X \tilde g$ with $\phi \in \mathcal{C}^\infty_\delta(M, \mathfrak{g})$. Then both boundary terms vanish and hence lemma A.2 directly asserts, with some constants C and C_V, that

$$
\begin{aligned}
\|\tilde\alpha\|_{W^{1,p}(V)} &\le C\bigl(\|f\|_{L^p(M)} + \|\gamma\|_{L^p(M)} + \|\tilde\alpha\|_{L^p(M)}\bigr) \\
&\le C_V\bigl(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\tilde\alpha\|_{L^p(M)}\bigr).
\end{aligned}
$$

So it remains to establish the estimate for $\tilde\alpha$ near the boundary $\partial M \,=\, S^1 \times \Sigma$. For that purpose we introduce the decomposition $\tau^*\tilde\alpha \,=\, \varphi\mathrm{d}s + \psi\mathrm{d}t + \alpha$, where $\varphi, \psi \in W^{1,p}(S^1 \times [0,\Delta) \times \Sigma, \mathfrak{g})$ and $\alpha \in W^{1,p}(S^1 \times [0,\Delta) \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$. Let $\Omega := S^1 \times [0, \frac{3}{4}\Delta]$ and let $K := S^1 \times [0, \frac{\Delta}{2}]$. Then we will prove the estimate for φ and ψ on $\Omega \times \Sigma$ and for α on $K \times \Sigma$.

Firstly, note that $\psi = \tilde\alpha(\tau_*\partial_t) \circ \tau$, where $-\tau_*\partial_t|_{\partial M} = \nu$ is the outer unit normal to ∂M. So one can cut off $\tau_*\partial_t$ outside of $\tau(\Omega \times \Sigma)$ to obtain a vector field $X \in \Gamma(\mathrm{T}M)$ that satisfies the assumption of lemma A.2, that is $X|_{\partial M} = -\nu$ is perpendicular to the boundary. Then one has to test (28) with $\lambda = \phi \cdot \iota_X \tilde g$ for all $\phi \in \mathcal{C}^\infty_\delta(M, \mathfrak{g})$. Again both boundary terms vanish. Indeed, on $S^1 \times \partial Y$ we have $\phi \equiv 0$ and $\iota_X \tilde g \,=\, \tau_*\mathrm{d}t$, hence $\mathrm{d}\lambda|_{\mathbb{R}\times\partial Y} \,=\, 0$ and $*\mathrm{d}\lambda|_{\mathbb{R}\times\partial Y} \,=\, -\frac{\partial\phi}{\partial\nu} * \tau_*(\mathrm{d}t \wedge \mathrm{d}t) \,=\, 0$. Thus lemma A.2 yields the following estimate.

$$
\begin{aligned}
\|\psi\|_{W^{1,p}(\Omega\times\Sigma)} &\le C\|\tilde\alpha(X)\|_{W^{1,p}(M)} \\
&\le C\bigl(\|f\|_{L^p(M)} + \|\gamma\|_{L^p(M)} + \|\tilde\alpha\|_{L^p(M)}\bigr) \\
&\le C_t\bigl(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\tilde\alpha\|_{L^p(M)}\bigr).
\end{aligned}
$$

Here C denotes any finite constant and the bounds on the derivatives of τ enter into the constant C_t.

Next, for the regularity of $\varphi = \tilde\alpha(\partial_s) \circ \tau$ one can apply lemma A.2 with the tangential vector field $X = \partial_s$. Recall that τ preserves the S^1-coordinate. One has to verify the second weak equation for all $\phi \in \mathcal{C}^\infty_\nu(M, \mathfrak{g})$, i.e. consider (28) for $\lambda = \phi \cdot \iota_X \tilde g = \phi \cdot \mathrm{d}s$. The first boundary term vanishes since one has $*\mathrm{d}\lambda|_{S^1\times\partial Y} \,=\, -\frac{\partial\phi}{\partial\nu}\,\mathrm{dvol}_{\partial Y} \,=\, 0$. For the second term one can choose any $\delta > 0$ and then finds a constant C_δ such that for all $\phi \in \mathcal{C}^\infty_\nu(M,\mathfrak{g})$

$$\begin{aligned}\left|\int_{S^1\times\partial Y}\langle\,\tilde\alpha\wedge\mathrm{d}\lambda\rangle\right| &= \left|\int_{S^1}\int_\Sigma\langle\,\alpha(s,0)\wedge\mathrm{d}_\Sigma(\phi\circ\tau)(s,0)\,\rangle\wedge\mathrm{d}s\right| \\ &= \left|\int_{S^1\times\partial Y}\langle\,\tilde\alpha\wedge[\tilde A,\phi]\,\rangle\wedge\mathrm{d}s\right| \\ &\le \|\tilde\alpha\|_{L^p(\partial M)}\|\tilde A\|_\infty\|\phi\|_{L^{p^*}(\partial M)} \\ &\le \bigl(\delta\|\tilde\alpha\|_{W^{1,p}(M)} + C_\delta\|\tilde\alpha\|_{L^p(M)}\bigr)\|\phi\|_{W^{1,p^*}(M)}.\end{aligned}$$

This uses the fact that $\alpha(s,0)$ and $\mathrm{d}_{A_s}(\phi\circ\tau)|_{(s,0)\times\Sigma}$ both lie in the tangent space $\mathrm{T}_{A_s}\mathcal{L}$ to the Lagrangian, on which the symplectic form vanishes. So we have the identity $\int_\Sigma\langle\,\alpha\wedge\mathrm{d}_A(\phi\circ\tau)\,\rangle = 0$. Moreover, we have used the trace theorem for Sobolev spaces, in particular (24) with $q = p$. Now lemma A.2 and remark A.3 yield with $c_1 = \|f\|_p$, $c_2 = \|\gamma\|_{L^p(M)} + \delta\|\tilde\alpha\|_{W^{1,p}(M)} + C_\delta\|\tilde\alpha\|_{L^p(M)}$, and using (27), (29)

$$\begin{aligned}&\|\varphi\|_{W^{1,p}(\Omega\times\Sigma)} \\ &\le C\bigl(\|f\|_{L^p(M)} + c_2 + \|\tilde\alpha\|_{L^p(M)}\bigr) \\ &\le \delta\|\tilde\alpha\|_{W^{1,p}(M)} + C_s(\delta)\bigl(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\tilde\alpha\|_{L^p(M)}\bigr).\end{aligned}$$

Here again $\delta > 0$ can be chosen arbitrarily small and the constant $C_s(\delta)$ depends on this choice.

It remains to establish the estimate for the Σ-component α near the boundary. In the coordinates τ on $\Omega\times\Sigma$, the forms $\mathrm{d}^*_{\tilde A}\tilde\alpha$ and $\mathrm{d}^+_{\tilde A}\tilde\alpha$ become

$$\begin{aligned}\tau^*\mathrm{d}^*_{\tilde A}\tilde\alpha &= -\partial_s\varphi - \partial_t\psi + \mathrm{d}^*_\Sigma\alpha - \tau^*(*[\tilde A\wedge *\tilde\alpha]), \\ \tau^*\mathrm{d}^+_{\tilde A}\tilde\alpha &= \tfrac12\bigl(-(\partial_s\alpha + *_\Sigma\partial_t\alpha)\wedge\mathrm{d}s + *_\Sigma(\partial_s\alpha + *_\Sigma\partial_t\alpha)\wedge\mathrm{d}t\bigr) \\ &\quad + \tfrac12\bigl(\mathrm{d}_\Sigma\alpha + (*_\Sigma\mathrm{d}_\Sigma\alpha)\mathrm{d}s\wedge\mathrm{d}t\bigr) + \tau^*([\tilde A\wedge\tilde\alpha]^+).\end{aligned}$$

So one obtains the following bounds: The components in the mixed direction of Ω and Σ of the second equation yields for some constant C_1

$$\begin{aligned}\|\partial_s\alpha + *_\Sigma\partial_t\alpha\|_{L^p(\Omega\times\Sigma)} &\le \bigl\|\tau^*\mathrm{d}^+_{\tilde A}\tilde\alpha\bigr\|_{L^p(\Omega\times\Sigma)} + \bigl\|\tau^*([\tilde A\wedge\tilde\alpha]^+)\bigr\|_{L^p(\Omega\times\Sigma)} \\ &\le C_1\bigl(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\tilde\alpha\|_{L^p(M)}\bigr).\end{aligned}$$

Similarly, a combination of the first equation and the Σ-component of the second equation can be used for every $\delta > 0$ to find a constant $C_2(\delta)$ such that

$$\begin{aligned}&\|\mathrm{d}_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}^*_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} \\ &\le C\bigl(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\tilde\alpha\|_{L^p(M)} + \|\varphi\|_{W^{1,p}(\Omega\times\Sigma)} + \|\psi\|_{W^{1,p}}\bigr) \\ &\le \delta\|\tilde\alpha\|_{W^{1,p}(M)} + C_2(\delta)\bigl(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\tilde\alpha\|_M)\bigr).\end{aligned}$$

Now lemma 2.9 provides an L^p-estimate for the derivatives of α in Σ-direction,

$$\begin{aligned}&\|\nabla_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} \\ &\le C\bigl(\|\mathrm{d}_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}^*_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} + \|\alpha\|_{L^p(\Omega\times\Sigma)}\bigr) \\ &\le \delta\|\tilde\alpha\|_{W^{1,p}(M)} + C_\Sigma(\delta)\bigl(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_{L^p(M)} + \|\tilde\alpha\|_{L^p(M)}\bigr),\end{aligned}$$

where again $C_\Sigma(\delta)$ depends on the choice of $\delta > 0$. For the derivatives in s- and t-direction, we will apply [W2, Theorem 1.3] on the Banach space $X = L^p(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ with the complex structure $*_\Sigma$ determined by the metric $g_{s,t}$ on Σ and hence depending smoothly on $(s,t) \in \Omega$. The Lagrangian submanifold $\mathcal{L} \subset X$ is totally real with respect to all Hodge operators and it is modelled on a closed subspace of $L^p(\Sigma, \mathbb{R}^n)$ as seen in [W2, Lemma 4.2, Corollary 4.4]. Now $\alpha \in W^{1,p}(\Omega, X)$ satisfies the Lagrangian boundary condition $\alpha(s,0) \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s \in S^1$, where $s \mapsto A_s$ is a smooth loop in $\mathcal{L}$. Thus [W2, Corollary 1.4] yields a constant C such that the following estimate holds:

$$\begin{aligned}
\|\nabla_\Omega \alpha\|_{L^p(K\times\Sigma)} &\leq \|\alpha\|_{W^{1,p}(K,X)} \\
&\leq C\big(\|\partial_s\alpha + *_\Sigma \partial_t \alpha\|_{L^p(\Omega,X)} + \|\alpha\|_{L^p(\Omega,X)}\big) \\
&\leq C_K\big(\|\mathrm{d}^+_{\tilde{A}}\tilde{\alpha}\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}\big).
\end{aligned}$$

Here C_K also includes the above constant C_1. Now adding up all the estimates for the different components of $\tilde{\alpha}$ gives for all $\delta > 0$

$$\begin{aligned}
\|\tilde{\alpha}\|_{W^{1,p}} \leq{}& (C_V + C_t + C_s(\delta) + C_\Sigma(\delta) + C_K)\big(\|\mathrm{d}^+_{\tilde{A}}\tilde{\alpha}\|_p + \|\mathrm{d}^*_{\tilde{A}}\tilde{\alpha}\|_p + \|\tilde{\alpha}\|_p\big) \\
&+ 2\delta\|\tilde{\alpha}\|_{W^{1,p}}.
\end{aligned}$$

With $\delta = \frac{1}{4}$ the term $\|\tilde{\alpha}\|_{W^{1,p}}$ can be absorbed into the left hand side, which finishes the proof of the lemma. $\square$

Proof of theorem C (iii) :
Let $\beta \in L^q(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ and $\zeta \in L^q(S^1 \times Y, \mathfrak{g})$ be as supposed in theorem C. Then there exists a constant c such that for all $\alpha \in \mathcal{C}^\infty(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ satisfying (3) and for all $\varphi \in \mathcal{C}^\infty(S^1 \times Y, \mathfrak{g})$

$$\begin{aligned}
&\left| \int_{S^1}\int_Y \langle\, \nabla_s\alpha - \mathrm{d}_A\varphi + *\mathrm{d}_A\alpha \,,\, \beta \,\rangle \;+ \int_{S^1}\int_Y \langle\, \nabla_s\varphi - \mathrm{d}_A^*\alpha \,,\, \zeta \,\rangle \right| \\
&= \left| \int_{S^1\times Y} \langle\, D_{(A,\Phi)}(\alpha,\varphi) \,,\, (\beta,\zeta) \,\rangle \right| \\
&\leq c\|(\alpha,\varphi)\|_{q^*}. \qquad (30)
\end{aligned}$$

The higher regularity of ζ is most easily seen if we go back to the notation $\tilde{\alpha} = \alpha + \varphi\mathrm{d}s$. With this we can write $D_{(A,\Phi)}(\alpha,\varphi) = (2\gamma\,,\, -\mathrm{d}^*_{\tilde{A}}\tilde{\alpha})$, where $\mathrm{d}^+_{\tilde{A}}\tilde{\alpha} = *\gamma - \gamma \wedge \mathrm{d}s$. We abbreviate $M := S^1 \times Y$, then we have for all $\tilde{\alpha} \in \mathcal{C}^\infty(M, \mathrm{T}^*M \otimes \mathfrak{g})$ with $*\tilde{\alpha}|_{\partial M} = 0$ and $\tilde{\alpha}|_{\{s\}\times\partial Y} \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s \in S^1$

$$\left| \int_M \langle\, 2\mathrm{d}^+_{\tilde{A}}\tilde{\alpha} \,,\, \beta \wedge \mathrm{d}s \,\rangle + \int_M \langle\, \mathrm{d}^*_{\tilde{A}}\tilde{\alpha} \,,\, \zeta \,\rangle \right| \leq c\|\tilde{\alpha}\|_{q^*}.$$

Now use the embedding $\tau : S^1 \times [0,\Delta) \times \Sigma \hookrightarrow M$ to construct a connection $\hat{A} \in \mathcal{A}(M)$ such that $\tau^*\hat{A}(s,t,z) = A_s(z)$ near the boundary (this can be cut off and then extends trivially to all of M). Then $\tilde{\alpha} := \mathrm{d}_{\hat{A}}\phi$ satisfies the above boundary conditions for all

$\phi \in \mathcal{C}^\infty_\nu(M,\mathfrak{g})$ since $\mathrm{d}_{\hat{A}}\phi(\nu) = \frac{\partial\phi}{\partial\nu} + [\hat{A}(\nu),\phi] = 0$ and $\mathrm{d}_{\hat{A}}\phi|_{\{s\}\times\partial Y} = \mathrm{d}_{A_s}\phi \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s\in S^1$. Thus we obtain for all $\phi \in \mathcal{C}^\infty_\nu(M,\mathfrak{g})$ in view of $\Delta\phi = \mathrm{d}^*(\tilde{\alpha} - [\hat{A},\phi])$

$$
\begin{aligned}
&\left|\int_M \langle\, \Delta\phi\,,\,\zeta\,\rangle\right| \\
&= \left|\int_M \langle\, \mathrm{d}^*_{\tilde{A}}\tilde{\alpha} + *[\tilde{A}\wedge *\tilde{\alpha}] - \mathrm{d}^*[\hat{A},\phi]\,,\,\zeta\,\rangle\right| \\
&\le c\|\tilde{\alpha}\|_{q^*} + \left|\int_M \langle\, -2\mathrm{d}^+_{\tilde{A}}\mathrm{d}_{\hat{A}}\phi\,,\,\beta\wedge\mathrm{d}s\,\rangle\right| + \left|\int_M \langle\, *[\tilde{A}\wedge *\mathrm{d}_{\hat{A}}\phi] - \mathrm{d}^*[\hat{A},\phi]\,,\,\zeta\,\rangle\right| \\
&\le C\big(c + \|\beta\|_q + \|\zeta\|_q\big)\|\phi\|_{W^{1,q^*}}.
\end{aligned}
$$

(Here and in the following we denote by C any constant $C = C(q,\tilde{A},\hat{A})$ that is independent of (β,ζ).) The regularity theory for the Neumann problem, e.g. proposition A.1, then asserts that $\zeta\in W^{1,q}(M)$ with

$$
\|\zeta\|_{W^{1,q}} \le C(c + \|(\beta,\zeta)\|_q). \tag{31}
$$

To deduce the higher regularity of β we will mainly use lemma A.2. The first weak equation in the lemma is given by choosing $\alpha = 0$ in (30). For all $\varphi\in\mathcal{C}^\infty(M,\mathfrak{g})$

$$
\begin{aligned}
\left|\int_M \langle\,\beta\,,\,\mathrm{d}\varphi\,\rangle\right| &= \left|\int_{S^1}\int_Y \langle\,\beta\,,\,\mathrm{d}_A\varphi - [A,\varphi]\,\rangle\right| \\
&\le c\|\varphi\|_{q^*} + \left|\int_{S^1}\int_Y \langle\,\nabla_s\zeta\,,\,\varphi\,\rangle\right| + \left|\int_{S^1}\int_Y \langle\,[\beta\wedge *A]\,,\,\varphi\,\rangle\right| \\
&\le \big(c + C\big(\|\zeta\|_{W^{1,q}} + \|\beta\|_q\big)\big)\|\varphi\|_{q^*}.
\end{aligned}
$$

For the second weak equation let $\varphi = 0$ and $\alpha = *\mathrm{d}\lambda - \partial_s\lambda$ for $\lambda = \phi\cdot\iota_X\tilde{g}$ with ϕ in $\mathcal{C}^\infty_\delta(M,\mathfrak{g})$ or $\mathcal{C}^\infty_\nu(M,\mathfrak{g})$ corresponding to the vector field $X\in\mathcal{C}^\infty(M,\mathrm{T}Y)$. If the boundary conditions for $\alpha\in E^{1,p}_A$ are satisfied, then we obtain with $\mathrm{d} = \mathrm{d}_Y$

$$
\begin{aligned}
&\left|\int_M \langle\,\beta\,,\,\mathrm{d}^*_M\mathrm{d}_M(\phi\cdot\iota_X g)\,\rangle\right| \\
&= \left|\int_{S^1}\int_Y \langle\,\beta\,,\,*\mathrm{d}*\mathrm{d}\lambda - \partial_s^2\lambda - *(\partial_s *)\partial_s\lambda\,\rangle\right| \\
&= \left|\int_{S^1}\int_Y \langle\,\beta\,,\,*\mathrm{d}_A\alpha - *[A\wedge *\mathrm{d}\lambda] + *\mathrm{d}_A\partial_s\lambda\right. \\
&\qquad\qquad\left. + \nabla_s\alpha - [\Phi,\partial_s\lambda] - \nabla_s *\mathrm{d}\lambda - *(\partial_s *)\partial_s\lambda\,\rangle\right| \\
&\le c\|\alpha\|_{q^*} + \left|\int_{S^1}\int_Y \langle\,\zeta\,,\,\mathrm{d}^*_A\alpha\,\rangle\right| + C\|\beta\|_q\|\lambda\|_{W^{1,q^*}} \\
&\le C\big(c + \|\zeta\|_{W^{1,q}} + \|\beta\|_q\big)\|\phi\|_{W^{1,q^*}}.
\end{aligned}
$$

Here we have used the identity

$$
*\mathrm{d}_A\partial_s\lambda - \nabla_s *\mathrm{d}\lambda = *[A\wedge\partial_s\lambda] - [\Phi, *\mathrm{d}\lambda] - (\partial_s *)\mathrm{d}\lambda.
$$

Moreover, partial integration with vanishing boundary term $*\alpha|_{\partial Y} = 0$ gives

$$\int_{S^1}\int_Y \langle\, \zeta\,,\, \mathrm{d}_A^*\alpha \,\rangle = \int_{S^1}\int_Y \langle\, \mathrm{d}_A\zeta\,,\, *\mathrm{d}\lambda - \partial_s\lambda \,\rangle.$$

Now let $X \in \mathcal{C}^\infty(M, \mathrm{T}Y)$ be perpendicular to the boundary $\partial M = S^1 \times \partial Y$, then $\alpha = *\mathrm{d}\lambda - \partial_s\lambda$ satisfies the boundary conditions (3) for every $\phi \in \mathcal{C}^\infty_\delta(M)$. Indeed, on the boundary $\partial M = S^1 \times \partial Y$ the 1-form $\lambda = \phi\cdot\iota_X\tilde{g}$ vanishes, we have $\iota_X\tilde{g} = h\cdot\tau_*\mathrm{d}t$ for some smooth function h, and moreover $\mathrm{d}\phi = -\frac{\partial\phi}{\partial\nu}\cdot\tau_*\mathrm{d}t$. Hence

$$\begin{aligned} *\alpha|_{\partial Y} &= \mathrm{d}\lambda|_{\partial Y} - *\partial_s\lambda|_{\partial Y} = 0, \\ \alpha|_{\partial Y} &= *\mathrm{d}\lambda|_{\partial Y} - \partial_s\lambda|_{\partial Y} = -\tfrac{\partial\phi}{\partial\nu}h*(\tau_*\mathrm{d}t\wedge\tau_*\mathrm{d}t) = 0. \end{aligned}$$

Thus for all vector fields $X \in \mathcal{C}^\infty(M, \mathrm{T}Y)$ that are perpendicular to the boundary, lemma A.2 and remark A.3 assert that $\beta(X) \in W^{1,q}(M, \mathfrak{g})$ and

$$\|\beta(X)\|_{W^{1,q}} \;\leq\; C\bigl(c + \|\zeta\|_{W^{1,q}} + \|\beta\|_q\bigr) \;\leq\; C\bigl(c + \|(\zeta,\beta)\|_q\bigr).$$

Here we have also used the previously established regularity and estimate (31) for ζ. In particular, this implies $W^{1,q}$-regularity and -estimate for β on all compact subsets in the interior of $S^1 \times Y$. So it remains to prove the regularity on the neighbourhood $\tau(S^1 \times [0, \frac{\Delta}{2}] \times \Sigma)$ of the boundary. We pull back (β, ζ) from $\tau(S^1 \times [0, \Delta) \times \Sigma)$ and write

$$\tau^*\beta = \xi\mathrm{d}t + \hat{\beta}, \qquad \tau^*\zeta = \eta.$$

We have already established that $\eta = \zeta\circ\tau$ and $\xi = \beta(\tau_*\partial_t)\circ\tau$ lie in $W^{1,q}(\Omega\times\Sigma, \mathfrak{g})$ with the according estimate, where $\Omega := S^1 \times [0, \frac{3}{4}\Delta)$. Here a vector field X on Y that is perpendicular to ∂Y is constructed by cutting off $\tau_*\partial_t$ outside of $\tau(\Omega\times\Sigma)$. So it remains to consider $\hat{\beta} \in L^q(\Omega\times\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g})$ and establish its $W^{1,q}$-regularity and -estimate on $S^1 \times [0, \frac{\Delta}{2}] \times \Sigma$.

In order to derive a weak equation for $\hat{\beta}$ on $\Omega\times\Sigma$ from (30) we use the test 1-form $\tilde{\alpha} = \tau_*(\varphi\mathrm{d}s + \psi\mathrm{d}t + \hat{\alpha})$ with $\varphi, \psi \in \mathcal{C}^\infty_0(\Omega\times\Sigma, \mathfrak{g})$ (supported in $\mathrm{int}(\Omega)\times\Sigma$)) and $\hat{\alpha} \in \mathcal{C}^\infty(\Omega\times\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g})$ with compact support $\operatorname{supp}\hat{\alpha} \subset S^1 \times [0, \frac{3}{4}\Delta) \times \Sigma$ and $\hat{\alpha}(s, 0, \cdot) \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s \in S^1$. This $\tilde{\alpha}$ satisfies the boundary conditions (3) and it can be extended trivially to a smooth 1-form on all of $S^1 \times Y$. Thus we obtain

$$\begin{aligned} &\left|\int_{\Omega\times\Sigma} \langle\, \nabla_s\hat{\alpha} + *\nabla_t\hat{\alpha} - \mathrm{d}_A\varphi - *\mathrm{d}_A\psi\,,\, \hat{\beta} \,\rangle\right| \\ &\leq \left|\int_{\Omega\times\Sigma} \langle\, -\nabla_s\psi + \nabla_t\varphi - *\mathrm{d}_A\hat{\alpha}\,,\, \xi \,\rangle \;+ \int_{\Omega\times\Sigma} \langle\, \nabla_s\varphi + \nabla_t\psi - \mathrm{d}_A^*\hat{\alpha}\,,\, \eta \,\rangle\right| \\ &\quad + c\|\tau_*(\varphi\,\mathrm{d}s + \psi\,\mathrm{d}t + \hat{\alpha})\|_{L^{q^*}(S^1\times Y)}. \end{aligned}$$

Here we have decomposed $\tau^*\tilde{A} = \Phi\mathrm{d}s + \Psi\mathrm{d}t + A$ with $A \in \mathcal{C}^\infty(\Omega\times\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g})$ satisfying $A(s, 0) = A_s \in \mathcal{L}$ for all $s \in S^1$. We also use the notation $\nabla_t\varphi = \partial_t\varphi + [\Psi, \varphi]$ and denote by d_A and $*$ the differential and Hodge operator on Σ. Now if we put $\hat{\alpha} = 0$, then we obtain for all $\varphi, \psi \in \mathcal{C}^\infty_0(\Omega\times\Sigma, \mathfrak{g})$ by partial integration

$$\begin{aligned} \left|\int_{\Omega\times\Sigma} \langle\, \mathrm{d}_A\varphi\,,\, \hat{\beta} \,\rangle\right| &\leq \bigl(Cc + \|\nabla_t\xi - \nabla_s\eta\|_{L^q(\Omega\times\Sigma)}\bigr)\|\varphi\|_{L^{q^*}(\Omega\times\Sigma)}, \\ \left|\int_{\Omega\times\Sigma} \langle\, *\mathrm{d}_A\psi\,,\, \hat{\beta} \,\rangle\right| &\leq \bigl(Cc + \|\nabla_s\xi + \nabla_t\eta\|_{L^q(\Omega\times\Sigma)}\bigr)\|\psi\|_{L^{q^*}(\Omega\times\Sigma)}. \end{aligned}$$

This shows that the weak derivatives $\mathrm{d}_\Sigma^*\hat\beta$ and $\mathrm{d}_\Sigma\hat\beta$ are of class L^q on $\overline{\Omega}\times\Sigma$. Now lemma 2.9 implies that $\nabla_\Sigma\hat\beta$ is of class L^q on $\Omega\times\Sigma$ with

$$\begin{aligned}\|\nabla_\Sigma\hat\beta\|_{L^q(\Omega\times\Sigma)} &\leq C\bigl(c+\|\nabla_t\xi-\nabla_s\eta\|_q+\|\nabla_s\xi+\nabla_t\eta\|_q+\|\hat\beta\|_q\bigr)\\ &\leq C\bigl(c+\|(\beta,\zeta)\|_{L^q(S^1\times Y)}\bigr).\end{aligned}$$

So it remains to deduce the L^q-regularity of $\partial_s\hat\beta$ and $\partial_t\hat\beta$ on $S^1\times[0,\frac{\Delta}{2}]\times\Sigma$ from the above inequality for $\varphi=\psi=0$, namely from

$$\left|\int_{\Omega\times\Sigma}\langle\,\nabla_s\hat\alpha+*\nabla_t\hat\alpha\,,\,\hat\beta\,\rangle\right|\leq\bigl(Cc+\|\mathrm{d}_A\eta+*\mathrm{d}_A\xi\|_{L^q(\Omega\times\Sigma)}\bigr)\|\hat\alpha\|_{L^{q^*}(\Omega\times\Sigma)}. \quad (32)$$

This holds for all $\hat\alpha\in\mathcal{C}^\infty(\Omega\times\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})$ with compact support and $\hat\alpha(s,0,\cdot)\in\mathrm{T}_{A_s}\mathcal{L}$ for all $s\in S^1$. We now employ different arguments in the cases $q>2$ and $q<2$.

Case $\mathbf{q>2}$:

In this case the regularity of $\partial_s\hat\beta$ and $\partial_t\hat\beta$ will follow from [W2, Theorem 1.3] on the Banach space $X=L^q(\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})$ with the complex structure given by the Hodge operator on Σ with respect to the metric $g_{s,t}$. From (32) one obtains the following estimate for some constant C and all $\hat\alpha$ as above:

$$\begin{aligned}&\left|\int_\Omega\int_\Sigma\langle\,\hat\beta\,,\,\partial_s\hat\alpha+\partial_t(*\hat\alpha)\,\rangle\right|\\ &\leq C\bigl(c+\|\eta\|_{W^{1,q}(\Omega\times\Sigma)}+\|\xi\|_{W^{1,q}(\Omega\times\Sigma)}+\|\hat\beta\|_{L^q(\Omega\times\Sigma)}\bigr)\|\hat\alpha\|_{L^{q^*}(\Omega\times\Sigma)}\\ &\leq C\bigl(c+\|(\beta,\zeta)\|_{L^q(S^1\times Y)}\bigr)\|\hat\alpha\|_{L^{q^*}(\Omega,X^*)}.\end{aligned} \quad (33)$$

Note that this extends to the $W^{1,q^*}(\Omega,L^{q^*}(\Sigma))$-closure of the admissible $\hat\alpha$ in (32). In particular the estimate above holds for all $\hat\alpha\in W^{1,q}(\Omega,X)$ that are compactly supported and satisfy $\hat\alpha(s,0,\cdot)\in\mathrm{T}_{A_s}\mathcal{L}$ for all $s\in S^1$. To see that these can be approximated by smooth $\hat\alpha$ with Lagrangian boundary conditions one uses the Banach submanifold coordinates for $\mathcal{L}$ given by [W2, Lemma 4.3] as before. Here the Lagrangian submanifold $\mathcal{L}\subset X$ is totally real with respect to all Hodge operators as before, and it is the L^q-restriction or -completion of the original submanifold in $\mathcal{A}^{0,p}(\Sigma)$, hence it is modelled on $W^{1,q}_z(\Sigma,\mathfrak{g})\times\mathbb{R}^m$, a closed subspace of $L^q(\Sigma,\mathbb{R}^n)$ (see [W2, Lemma 4.2, 4.3]). However, in order to be able to apply [W2, Theorem 1.3], we need to extend this estimate to all $\hat\alpha\in W^{1,\infty}(\Omega,X^*)$ with compact support and $\hat\alpha(s,0)\in(*\mathrm{T}_{A_s}\mathcal{L})^\perp$ for all $s\in S^1$. This is possible since any such $\hat\alpha$ can be approximated in $W^{1,q^*}(\Omega,X^*)$ by $\hat\alpha_i\in\mathcal{C}^\infty(\Omega,X)$ that are compactly supported and satisfy the above stronger boundary condition $\hat\alpha_i(s,0)\in\mathrm{T}_{A_s}\mathcal{L}$ for all $s\in S^1$.

Indeed, [W2, Lemma 2.2] provides such an approximating sequence α_i without the Lagrangian boundary conditions. From the proof via mollifiers one sees that the approximating sequence can be chosen with compact support in Ω. Now for all $s\in S^1$ one has the topological splitting $X=\mathrm{T}_{A_s}\mathcal{L}\oplus*\mathrm{T}_{A_s}\mathcal{L}$ and thus $X^*=(\mathrm{T}_{A_s}\mathcal{L})^\perp\oplus(*\mathrm{T}_{A_s}\mathcal{L})^\perp$. Since $q>2$ the embedding $X\hookrightarrow X^*$ is continuous. This identification uses the L^2-inner product on X which equals the metric $\omega(\cdot,*\cdot)$ given by the symplectic form ω and the complex structure $*$. So due to the Lagrangian condition this embedding maps $\mathrm{T}_{A_s}\mathcal{L}\hookrightarrow(*\mathrm{T}_{A_s}\mathcal{L})^\perp$ and $*\mathrm{T}_{A_s}\mathcal{L}\hookrightarrow(\mathrm{T}_{A_s}\mathcal{L})^\perp$. We write $\hat\alpha=\gamma+\delta$ and $\alpha_i=\gamma_i+\delta_i$ according to these splittings to obtain $\gamma,\delta\in\mathcal{C}^\infty(\Omega,X^*)$ and $\gamma_i,\delta_i\in\mathcal{C}^\infty(\Omega,X)$ such that $*\mathrm{T}_A\mathcal{L} \ni \gamma_i \to \gamma \in (\mathrm{T}_A\mathcal{L})^\perp$ and $\mathrm{T}_A\mathcal{L} \ni \delta_i \to \delta \in (*\mathrm{T}_A\mathcal{L})^\perp$ with convergence in $W^{1,q^*}(\Omega, X^*)$. The boundary condition on $\hat\alpha$ gives $\gamma|_{t=0} \equiv 0$. Moreover, $\partial_t\gamma$ is uniformly bounded in X^*, so one can find a constant C such that $\|\gamma(s,t)\|_{X^*} \le Ct$ for all $t \in [0, \frac{3}{4}\Delta)$ and hence for sufficiently small $\varepsilon > 0$

$$\|\gamma\|_{L^{q^*}(S^1\times[0,\varepsilon],X^*)} \le \tfrac{C}{1+q^*}\varepsilon^{1+\frac{1}{q^*}}.$$

Now let $\delta > 0$ be given and choose $1 > \varepsilon > 0$ such that $\|\gamma\|_{L^{q^*}(S^1\times[0,\varepsilon],X^*)} \le \varepsilon\delta$ and $\|\gamma\|_{W^{1,q^*}(S^1\times[0,\varepsilon],X^*)} \le \delta$. Next, choose a sufficiently large $i \in \mathbb{N}$ such that $\|\gamma_i - \gamma\|_{W^{1,q^*}(\Omega,X^*)} \le \varepsilon\delta$, and let $h \in \mathcal{C}^\infty([0,\frac{3}{4}\Delta],[0,1])$ be a cutoff function with $h(0) = 0$, $h|_{t\ge\varepsilon} \equiv 0$, and $|h'| \le \frac{2}{\varepsilon}$. Then $\hat\alpha_i := h\gamma_i + \delta_i \in \mathcal{C}^\infty(\Omega, X)$ satisfies the Lagrangian boundary condition $\hat\alpha_i(s,0) \in \mathrm{T}_{A_s}\mathcal{L}$ and approximates $\hat\alpha$ in view of the following estimate:

$$\begin{aligned}\|\hat\alpha_i - \hat\alpha\|_{W^{1,q^*}(\Omega,X^*)} &\le \|h(\gamma_i-\gamma)\|_{W^{1,q^*}(\Omega,X^*)} + \|(1-h)\gamma\|_{W^{1,q^*}(\Omega,X^*)} \\ &\le \|\gamma_i-\gamma\|_{W^{1,q^*}(\Omega,X^*)} + \tfrac{2}{\varepsilon}\|\gamma_i-\gamma\|_{L^{q^*}(\Omega,X^*)} \\ &\quad + \|\gamma\|_{W^{1,q^*}(S^1\times[0,\varepsilon],X^*)} + \tfrac{2}{\varepsilon}\|\gamma\|_{L^{q^*}(S^1\times[0,\varepsilon],X^*)} \\ &\le 6\delta.\end{aligned}$$

This approximation shows that (33) holds indeed true for all $\hat\alpha \in W^{1,\infty}(\Omega, X^*)$ with compact support and $\alpha(s,0) \in (*\mathrm{T}_{A_s}\mathcal{L})^\perp$ for all $s \in S^1$. Thus [W2, Theorem 1.3] asserts that $\hat\beta \in W^{1,q}(S^1 \times [0,\frac{\Delta}{2}], X)$, and hence $\partial_s\hat\beta$ and $\partial_t\hat\beta$ are of class L^q on $K \times \Sigma$ wit $K := S^1 \times [0,\frac{\Delta}{2}]$ as claimed, with

$$\|\partial_s\hat\beta\|_{L^q(K\times\Sigma)} + \|\partial_t\hat\beta\|_{L^q(K\times\Sigma)} \le \|\hat\beta\|_{W^{1,q}(K,X)} \le C\big(c + \|(\beta,\zeta)\|_{L^q(S^1\times Y)}\big).$$

Case $\mathbf{q < 2}$:
In this case we cover S^1 by two intervals, $S^1 = I_1 \cup I_2$ such that there are isometric embeddings $(0,1) \hookrightarrow S^1$ identifying $[\frac14,\frac34]$ with I_1 and I_2 respectively. Abbreviate $K := [\frac14,\frac34] \times [0,\frac{\Delta}{2}]$ and let $\Omega' \subset (0,1)\times[0,\frac34\Delta]$ be a compact submanifold of the half space $\mathbb{H}$ such that $K \subset \operatorname{int}\Omega'$. Then for each of the above identifications $S^1 \setminus \{pt\} \cong (0,1)$ one has L^q-regularity of $\hat\beta$ on $\Omega' \times \Sigma$ by assumption and of $*\mathrm{d}_A\xi + \mathrm{d}_A\eta$ from above. Now the task is to establish in both cases the L^q-regularity of $\partial_s\hat\beta$ and $\partial_t\hat\beta$ on $K \times \Sigma$ using (32). For that purpose choose a cutoff function $h \in \mathcal{C}^\infty(\mathbb{H},[0,1])$ supported in Ω' such that $h|_K \equiv 1$. Then it suffices to show that for all $\gamma \in \mathcal{C}_0^\infty(\Omega'\times\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g})$ (compactly supported in $\operatorname{int}(\Omega')\times\Sigma$)

$$\left|\int_{\Omega'\times\Sigma} \langle\, \partial_s\gamma \,,\, h\hat\beta\,\rangle\right| \le C\big(c + \|(\beta,\zeta)\|_{L^q(S^1\times\Sigma)}\big)\|\gamma\|_{L^{q^*}(\Omega'\times\Sigma)}.$$

This gives the required L^q-regularity and -estimate for the weak derivative $\partial_s(h\hat\beta)$ and hence for $\partial_s\hat\beta$ on $K\times\Sigma$. The regularity and estimate for $\partial_t\hat\beta$ follows by the same argument with $\partial_s\gamma$ replaced by $\partial_t\gamma$.

We linearize the submanifold chart maps along $(A_s)_{s\in(0,1)} \in \mathcal{L}\cap\mathcal{A}(\Sigma)$ given by [W2, Lemma 4.3] for the Lagrangian $\mathcal{L} \subset \mathcal{A}^{0,q^*}(\Sigma)$. Note that this uses the L^{q^*}-completion of the actual Lagrangian in $\mathcal{A}^{0,p}(\Sigma)$. Abbreviate $Z := W^{1,q^*}_z(\Sigma,\mathfrak{g})\times\mathbb{R}^m$

and let $*_{s,t}$ denote the Hodge operator on Σ with respect to the metric $g_{s,t}$. Then one obtains a smooth family of bounded isomorphisms

$$\Theta_{s,t} : \ Z \times Z \xrightarrow{\sim} L^{q^*}(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g}) \ =: \ X$$

defined for all $(s,t) \in \Omega'$ by

$$\Theta_{s,t}(\xi, v, \zeta, w) \ = \ \mathrm{d}_{A_s}\xi + \textstyle\sum_{i=1}^m v^i \gamma_i(s) + *_{s,t}\mathrm{d}_{A_s}\zeta + \sum_{i=1}^m w^i *_{s,t} \gamma_i(s).$$

Here $\gamma_i \in \mathcal{C}^\infty((0,1) \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ with $\gamma_i(s) \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s \in (0,1)$. If we abbreviate $Z^\infty := \mathcal{C}^\infty_z(\Sigma, \mathfrak{g}) \times \mathbb{R}^m \subset Z$, then $\Theta_{s,t}$ maps $Z^\infty \times Z^\infty$ into the set of smooth 1-forms $\Omega^1(\Sigma, \mathfrak{g})$. So given any $\gamma \in \mathcal{C}^\infty_0(\Omega' \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ we have $f := \Theta^{-1} \circ \gamma \in \mathcal{C}^\infty_0(\Omega', Z^\infty \times Z^\infty)$ and for some constant C

$$\|f\|_{L^{q^*}(\Omega', Z\times Z)} \ \le \ C\|\gamma\|_{L^{q^*}(\Omega', X)} \ = \ C\|\gamma\|_{L^{q^*}(\Omega'\times\Sigma)}.$$

Write $f = (f_1, f_2)$ with $f_i \in \mathcal{C}^\infty_0(\Omega', Z^\infty)$ and note that $\int_{\Omega'} \partial_s f_1 = 0$ due to the compact support. So one can solve $\Delta_{\Omega'}\phi_1 = \partial_s f_1$ by $\phi_1 \in \mathcal{C}^\infty_\nu(\Omega', Z^\infty)$ with $\int_{\Omega'} \phi_1 = 0$ and $\Delta_{\Omega'}\phi_2 = \partial_s f_2$ by $\phi_2 \in \mathcal{C}^\infty_\delta(\Omega', Z^\infty)$. (For the $\mathcal{C}^\infty_z(\Sigma, \mathfrak{g})$-component of Z^∞ one has solutions of the Laplace equation on every $\Omega' \times \{x\}$ that depend smoothly on $x \in \Sigma$.) Now let $\Phi := (\phi_1, \phi_2) \in \mathcal{C}^\infty(\Omega', Z \times Z)$ and consider the 1-form

$$\hat\alpha_\gamma \ := \ h \cdot \Theta(-\partial_s\Phi + J_0\partial_t\Phi) \ \in \ \mathcal{C}^\infty(\Omega', X).$$

This extends to a 1-form on $\Omega \times \Sigma$ that is admissible in (32). Indeed, $\hat\alpha_\gamma$ vanishes for s close to 0 or 1 and thus trivially extends to $s \in S^1$. The Lagrangian boundary condition is met since for all $s \in S^1$

$$\hat\alpha_\gamma(s,0) = h(s,0) \cdot \Theta_{s,0}(-\partial_s\phi_1 - \partial_t\phi_2, -\partial_s\phi_2 + \partial_t\phi_1) \in \Theta_{s,0}(Z, 0) = \mathrm{T}_{A_s}\mathcal{L}.$$

So from (32) we obtain for all $\hat\alpha_\gamma$ of the above form

$$\left| \int_{\Omega'\times\Sigma} \langle\, \hat\beta \,,\, \partial_s\hat\alpha_\gamma + \partial_t(*\hat\alpha_\gamma)\,\rangle \right| \ \le \ C\big(c + \|(\beta,\zeta)\|_{L^q(S^1\times\Sigma)}\big)\|\hat\alpha_\gamma\|_{L^{q^*}(\Omega,X)}$$

Moreover, one has for all $\gamma \in \mathcal{C}^\infty_0(\Omega' \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ and the associated f, Φ and $\hat\alpha_\gamma$

$$\|\hat\alpha_\gamma\|_{L^{q^*}(\Omega,X)} \ \le \ C\|\Phi\|_{W^{1,q^*}(\Omega', Z\times Z)} \ \le \ C\|f\|_{L^{q^*}(\Omega', Z\times Z)} \ \le \ C\|\gamma\|_{L^{q^*}(\Omega'\times\Sigma)}.$$

Here the second inequality follows from $\Delta_{\Omega'}\Phi = \partial_s f$ and [W2, Lemma 2.1] as follows. In the $\mathbb{R}^m$-component of Z, it is the usual elliptic estimate for the Dirichlet or Neumann problem. For the components in the infinite dimensional part $Y := W^{1,q^*}_z(\Sigma, \mathfrak{g})$ of Z (still denoted by ϕ_i and f_i) this uses the following estimate. For all $\psi \in \mathcal{C}^\infty_\nu(\Omega', Y^*)$ in the case $i = 1$ and for all $\psi \in \mathcal{C}^\infty_\delta(\Omega', Y^*)$ in the case $i = 2$

$$\begin{aligned}
\left| \int_{\Omega'\times\Sigma} \langle\, \phi_i \,,\, \Delta_{\Omega'}\psi \,\rangle \right| \ &= \ \left| \int_{\Omega'\times\Sigma} \langle\, \Delta_{\Omega'}\phi_i \,,\, \psi \,\rangle \right| \\
&= \ \left| \int_{\Omega'\times\Sigma} \langle\, \partial_s f_i \,,\, \psi \,\rangle \right| \\
&= \ \left| \int_{\Omega'\times\Sigma} \langle\, f_i \,,\, \partial_s\psi \,\rangle \right| \\
&\le \ \|f_i\|_{L^{q^*}(\Omega',Y)} \|\psi\|_{W^{1,q}(\Omega',Y^*)}.
\end{aligned}$$

Now a calculation shows that

$$\partial_s\hat{\alpha}_\gamma + \partial_t(*\hat{\alpha}_\gamma) = h\cdot\Theta(\Delta\Phi) + \partial_s(h\cdot\Theta)(-\partial_s\Phi + J_0\partial_t\Phi) + \partial_t(h\cdot\Theta)(\partial_t\Phi - J_0\partial_s\Phi).$$

We then use $\Delta\Phi = \partial_s f$ to obtain

$$\begin{aligned}
&\left|\int_{\Omega'\times\Sigma}\langle\, h\cdot\hat{\beta}\,,\,\partial_s\gamma\,\rangle\right| \\
&= \left|\int_{\Omega'\times\Sigma}\langle\,\hat{\beta}\,,\,h\cdot\Theta(\Delta\Phi) + h\cdot\partial_s\Theta(f)\,\rangle\right| \\
&\leq \left|\int_{\Omega'\times\Sigma}\langle\,\hat{\beta}\,,\,\partial_s\hat{\alpha}_\gamma + \partial_t(*\hat{\alpha}_\gamma)\,\rangle\right| \\
&\quad + C\|\hat{\beta}\|_{L^q(\Omega',X^*)}\big(\|-\partial_s\Phi + J_0\partial_t\Phi)\|_{L^{q^*}(\Omega',Z\times Z)} + \|f\|_{L^{q^*}(\Omega',Z\times Z)}\big) \\
&\leq C\big(c + \|(\beta,\zeta)\|_{L^q(S^1\times\Sigma)}\big)\|\gamma\|_{L^{q^*}(\Omega'\times\Sigma)}.
\end{aligned}$$

This holds with uniform constants for all $\gamma\in\mathcal{C}_0^\infty(\Omega'\times\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})$ and thus implies the L^q-regularity of $\partial_s\hat{\beta}$ on $K\times\Sigma$ together with the estimate

$$\|\partial_s\hat{\beta}\|_{L^q(K\times\Sigma)} \leq C\big(c + \|(\beta,\zeta)\|_{L^q(S^1\times Y)}\big).$$

This establishes the L^q-regularity and -estimate for $\partial_s\hat{\beta}$ (and analogously of $\partial_t\hat{\beta}$) on $S^1\times[0,\frac{\Delta}{2}]$ and thus finishes the proof of theorem C (iii). □

Proof of theorem C (i) :
Lemma 3.1 and the identities (23) imply that for some constant C and for all (α,φ) in the domain of $D_{(A,\Phi)}$

$$\|(\alpha,\varphi)\|_{W^{1,p}} \leq C\big(\|D_{(A,\Phi)}(\alpha,\varphi)\|_p + \|(\alpha,\varphi)\|_p\big).$$

Note that the embedding $W^{1,p}(X)\hookrightarrow L^p(X)$ is compact, so this estimate already implies that $\ker D_{(A,\Phi)}$ is finite dimensional and $\operatorname{im} D_{(A,\Phi)}$ is closed (see e.g. [Z, 3.12]). So it remains to consider the cokernel of $D_{(A,\Phi)}$. For that purpose we abbreviate $Z := L^p(S^1\times Y,\mathrm{T}^*Y\otimes\mathfrak{g})\times L^p(S^1\times Y,\mathfrak{g})$, then $\operatorname{coker} D_{(A,\Phi)} = Z/\operatorname{im} D_{(A,\Phi)}$ is a Banach space since $\operatorname{im} D_{(A,\Phi)}$ is closed. So it has the same dimension as its dual space $(Z/\operatorname{im} D_{(A,\Phi)})^* \cong (\operatorname{im} D_{(A,\Phi)})^\perp$. Now let $\sigma : S^1\times Y\to S^1\times Y$ denote the reflection $\sigma(s,y) := (-s,y)$ on $S^1\cong\mathbb{R}/\mathbb{Z}$, then we claim that there is an isomorphism

$$\begin{array}{ccc} (\operatorname{im} D_{(A,\Phi)})^\perp & \xrightarrow{\sim} & \ker D_{\sigma^*(A,\Phi)} \\ (\beta,\zeta) & \longmapsto & (\beta\circ\sigma,\zeta\circ\sigma). \end{array} \tag{34}$$

Here $D_{\sigma^*(A,\Phi)} = D_{(A',\Phi')}$ is the linearized operator corresponding to the reflected connection $\sigma^*\tilde{A} = A' + \Phi'\mathrm{d}s$ with respect to the metric $\sigma^*\tilde{g}$ on X. Note that $\ker D_{\sigma^*(A,\Phi)}$ has finite dimension since the estimate in theorem C (ii) also holds for the operator $D_{\sigma^*(A,\Phi)}$. So this would indeed prove that $\operatorname{coker} D_{\tilde{A}}$ is of finite dimension and hence $D_{\tilde{A}}$ is a Fredholm operator.

To establish the above isomorphism consider any $(\beta,\zeta)\in(\operatorname{im}D_{(A,\Phi)})^{\perp}$, that is $\beta\in L^{p^*}(S^1\times Y,\mathrm{T}^*Y\otimes\mathfrak{g})$ and $\zeta\in L^{p^*}(S^1\times Y,\mathfrak{g})$ such that for all $\alpha\in E_A^{1,p}$ and $\varphi\in W^{1,p}(S^1\times Y,\mathfrak{g})$

$$\int_{S^1\times Y}\langle\, D_{(A,\Phi)}(\alpha,\varphi)\,,\,(\beta,\zeta)\,\rangle=0.$$

Iteration of theorem C (iii) implies that β and ζ are in fact $W^{1,p}$-regular: We start with $q=p^*<2$, then the lemma asserts W^{1,p^*}-regularity. Next, the Sobolev embedding theorem gives L^{q_1}-regularity for some $q_1\in(\frac{4}{3},2)$ with $q_1>p^*$. Indeed, the Sobolev embedding holds for any $q_1\leq\frac{4p^*}{4-p^*}$, and $\frac{4}{3}<\frac{4p^*}{4-p^*}$ as well as $p^*<\frac{4p^*}{4-p^*}$ holds due to $p^*>1$. So the lemma together with the Sobolev embeddings can be iterated to give $L^{q_{i+1}}$-regularity for $q_{i+1}=\frac{4q_i}{4-q_i}$ as long as $4>q_i>2$ or $2>q_i\geq p^*$. This iteration yields $q_2\in(2,4)$ and $q_3>4$. Thus another iteration of the lemma gives W^{1,q_3}- and thus also L^p-regularity of β and ζ. Finally, since $p>2$ the lemma applies again and asserts the claimed $W^{1,p}$-regularity of β and ζ. Now by partial integration

$$\begin{aligned}
0&=\int_{S^1\times Y}\langle\, D_{(A,\Phi)}(\alpha,\varphi)\,,\,(\beta,\zeta)\,\rangle\\
&=\int_{S^1}\int_Y\langle\,\nabla_s\alpha-\mathrm{d}_A\varphi+*\mathrm{d}_A\alpha\,,\,\beta\,\rangle\;+\int_{S^1}\int_Y\langle\,\nabla_s\varphi-\mathrm{d}_A^*\alpha\,,\,\zeta\,\rangle\\
&=\int_{S^1}\int_Y\langle\,\alpha\,,\,-\nabla_s\beta-\mathrm{d}_A\zeta+*\mathrm{d}_A\beta\,\rangle\;+\int_{S^1}\int_Y\langle\,\varphi\,,\,-\nabla_s\zeta-\mathrm{d}_A^*\beta\,\rangle\\
&\quad-\int_{S^1}\int_\Sigma\langle\,\alpha\wedge\beta\,\rangle\;-\int_{S^1}\int_\Sigma\langle\,\varphi\,,\,*\beta\,\rangle.
\end{aligned}\tag{35}$$

Testing this with all $\alpha\in\mathcal{C}_0^\infty(S^1\times Y,\mathrm{T}^*Y\otimes\mathfrak{g})\subset E_A^{1,p}$ and $\varphi\in\mathcal{C}_0^\infty(S^1\times Y,\mathfrak{g})$ implies $-\nabla_s\beta-\mathrm{d}_A\zeta+*\mathrm{d}_A\beta=0$ and $-\nabla_s\zeta-\mathrm{d}_A^*\beta=0$. Then furthermore we deduce $*\beta(s)|_{\partial Y}=0$ for all $s\in S^1$ from testing with φ that run through all of $\mathcal{C}^\infty(S^1\times\Sigma,\mathfrak{g})$ on the boundary. Finally, $\int_{S^1}\int_\Sigma\langle\,\alpha\wedge\beta\,\rangle=0$ remains from (35). Since both α and β restricted to $S^1\times\Sigma$ are continuous paths in $\mathcal{A}^{0,p}(\Sigma)$, this implies that for all $s\in S^1$ and every $\alpha\in\mathrm{T}_{A_s}\mathcal{L}$

$$0\;=\;\int_\Sigma\langle\,\alpha\wedge\beta(s)\,\rangle\;=\;\omega(\alpha,\beta(s)),$$

where ω is the symplectic structure on $\mathcal{A}^{0,p}(\Sigma)$. Since $\mathrm{T}_{A_s}\mathcal{L}$ is a Lagrangian subspace, this proves $\beta(s)|_{\partial Y}\in\mathrm{T}_{A_s}\mathcal{L}$ for all $s\in S^1$ and thus $\beta\in E_A^{1,p}$, or equivalently $\beta\circ\sigma\in E_{A\circ\sigma}$. So $(\beta\circ\sigma,\zeta\circ\sigma)$ lies in the domain of $D_{\sigma^*(A,\Phi)}$. Now note that $\sigma^*\tilde{A}=A\circ\sigma-(\Phi\circ\sigma)\mathrm{d}s$, thus one obtains $(\beta\circ\sigma,\zeta\circ\sigma)\in\ker D_{\sigma^*(A,\Phi)}$ since

$$D_{\sigma^*(A,\Phi)}(\beta\circ\sigma,\zeta\circ\sigma)\;=\;\big((-\nabla_s\beta-\mathrm{d}_A\zeta+*\mathrm{d}_A\beta)\circ\sigma\,,\,(-\nabla_s\zeta-\mathrm{d}_A^*\beta)\circ\sigma\big)\;=\;0.$$

This proves that the map in (34) indeed maps into $\ker D_{\sigma^*(A,\Phi)}$. To see the surjectivity of this map consider any $(\beta,\zeta)\in\ker D_{\sigma^*(A,\Phi)}$. Then the same partial integration as in (35) shows that $(\beta\circ\sigma,\zeta\circ\sigma)\in(\operatorname{im}D_{(A,\Phi)})^{\perp}$, and thus (β,ζ) is the image of this element under the map (34). So this establishes the isomorphism (34) and thus shows that $D_{(A,\Phi)}$ is Fredholm. □

A. Dirichlet and Neumann problem

Throughout this paper we use various regularity results for the Laplace operator. For convenience these are summarized in this appendix.

We deal with (homogeneous) Dirichlet boundary conditions and with possibly inhomogeneous Neumann boundary conditions. Often, the equations are formulated weakly with the help of the following test function spaces:

$$\mathcal{C}^\infty_\delta(M) = \bigl\{\phi \in \mathcal{C}^\infty(M) \bigm| \phi|_{\partial M} = 0\bigr\},$$
$$\mathcal{C}^\infty_\nu(M) = \bigl\{\phi \in \mathcal{C}^\infty(M) \bigm| \tfrac{\partial\phi}{\partial\nu}\big|_{\partial M} = 0\bigr\}.$$

Here and throughout this appendix let M be a manifold with boundary. We abbreviate $\Delta := \mathrm{d}^*\mathrm{d}$, and denote by $\frac{\partial\phi}{\partial\nu}$ the Lie derivative in the direction of the outer unit normal. Moreover, we use the notation $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, \dots\}$. The regularity theory for the Dirichlet and Neumann problem that is used in this paper can be summarized as follows. References are for example [GT] and [W1, Theorems 2.3',3.2,D.2].

Proposition A.1. *Let $1 < p < \infty$ and $k \in \mathbb{N}$, then there exists a constant C such that the following holds. Let $f \in W^{k-1,p}(M)$ and $G \in W^{k,p}(M)$ and suppose that $u \in W^{k,p}(M)$ is a weak solution of the inhomogeneous Neumann problem (or the Dirichlet problem, in which case one can drop G), that is for all $\psi \in \mathcal{C}^\infty_\nu(M)$ (or for all $\psi \in \mathcal{C}^\infty_\delta(M)$)*

$$\int_M u \cdot \Delta\psi \; = \int_M f \cdot \psi + \int_{\partial M} G \cdot \psi.$$

Then $u \in W^{k+1,p}(M)$ and

$$\|u\|_{W^{k+1,p}} \leq C\bigl(\|f\|_{W^{k-1,p}} + \|G\|_{W^{k,p}} + \|u\|_{W^{k,p}}\bigr).$$

In the special case $k = 0$ there exists a constant C such that the following holds: Suppose that $u \in L^p(M)$ and that there exists a constant c such that for all $\psi \in \mathcal{C}^\infty_\nu(M)$ (or for all $\psi \in \mathcal{C}^\infty_\delta(M)$)

$$\int_M u \cdot \Delta\psi \; \leq c\|\psi\|_{W^{1,p^*}}.$$

Then $u \in W^{1,p}(M)$ and $\|u\|_{W^{1,p}} \leq C\bigl(c + \|u\|_{L^p}\bigr)$.

We also frequently encounter Laplace equations for 1-forms, where the components satisfy different boundary conditions. In these cases the following lemma allows to obtain regularity results for the components separately. The proof relies on the above standard regularity theory for the Laplace operator.

Lemma A.2. *Let (M, g) be a compact Riemannian manifold (possibly with boundary), let $k \in \mathbb{N}_0$ and $1 < p < \infty$. Let $X \in \Gamma(\mathrm{T}M)$ be a smooth vector field that is either perpendicular to the boundary, i.e. $X|_{\partial M} = h \cdot \nu$ for some $h \in \mathcal{C}^\infty(\partial M)$, or tangential, i.e. $X|_{\partial M} \in \Gamma(\mathrm{T}\partial M)$. In the first case let $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$, in the latter case let $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$. Then there exists a constant C such that the following holds:*

Fix a function $f \in W^{k,p}(M)$ and a 2-form $\gamma \in W^{k,p}(M, \Lambda^2\mathrm{T}^*M)$ and suppose that the 1-form $\alpha \in W^{k,p}(M, \mathrm{T}^*M)$ satisfies

$$\int_M \langle\, \alpha\,,\, \mathrm{d}\eta \,\rangle = \int_M f \cdot \eta \qquad \forall \eta \in \mathcal{C}^\infty(M),$$
$$\int_M \langle\, \alpha\,,\, \mathrm{d}^*\omega \,\rangle = \int_M \langle\, \gamma\,,\, \omega \,\rangle \qquad \forall \omega = \mathrm{d}(\phi \cdot \iota_X g)\,,\; \phi \in \mathcal{T}.$$

Then $\alpha(X) \in W^{k+1,p}(M)$ *and*

$$\|\alpha(X)\|_{W^{k+1,p}} \leq C\big(\|f\|_{W^{k,p}} + \|\gamma\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}\big).$$

Remark A.3. In the case $k=0$ let $\frac{1}{p}+\frac{1}{p^*}=1$, then the weak equations for α can be replaced by the following: There exist constants c_1 and c_2 such that

$$\begin{aligned} \left|\int_M \langle\, \alpha\,,\, \mathrm{d}\eta\,\rangle\right| &\leq c_1\ \|\eta\|_{p^*} &\qquad \forall \eta \in \mathcal{C}^\infty(M), \\ \left|\int_M \langle\, \alpha\,,\, \mathrm{d}^*\mathrm{d}(\phi\cdot\iota_X g)\,\rangle\right| &\leq c_2\ \|\phi\|_{W^{1,p^*}} &\qquad \forall \phi \in \mathcal{T}. \end{aligned}$$

The estimate then becomes $\|\alpha(X)\|_{W^{1,p}} \leq C\big(c_1 + c_2 + \|\alpha\|_p\big)$.

Proof of lemma A.2 and remark A.3 :
Let $\alpha^\nu \in \mathcal{C}^\infty(M, \mathrm{T}^*M)$ be an L^p-approximating sequence for α such that $\alpha^\nu \equiv 0$ near ∂M. Then one obtains for all $\phi \in \mathcal{T}$

$$\begin{aligned}
\int_M \alpha(X)\cdot\Delta\phi &= \lim_{\nu\to\infty}\Bigl(\int_M \langle\, \mathcal{L}_X\alpha^\nu \,,\, \mathrm{d}\phi\,\rangle - \int_M \langle\, \iota_X \mathrm{d}\alpha^\nu \,,\, \mathrm{d}\phi\,\rangle\Bigr) \\
&= \lim_{\nu\to\infty}\Bigl(-\int_M \langle\, \alpha^\nu \,,\, \mathcal{L}_X\mathrm{d}\phi\,\rangle - \int_M \langle\, \alpha^\nu \,,\, \mathrm{div}X\cdot\mathrm{d}\phi\,\rangle \\
&\qquad - \int_M \langle\, \alpha^\nu \,,\, \iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g\,\rangle - \int_M \langle\, \mathrm{d}\alpha^\nu \,,\, \iota_X g\wedge\mathrm{d}\phi\,\rangle\Bigr) \\
&= \int_M \langle\, \alpha \,,\, \mathrm{d}(-\mathcal{L}_X\phi - \mathrm{div}X\cdot\phi)\,\rangle - \int_M \langle\, \alpha \,,\, \mathrm{d}^*(\iota_X g\wedge\mathrm{d}\phi)\,\rangle \\
&\quad + \int_M \langle\, \alpha \,,\, \phi\cdot\mathrm{d}(\mathrm{div}X) - \iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g\,\rangle \\
&= \int_M \langle\, f \,,\, -\mathcal{L}_X\phi - \mathrm{div}X\cdot\phi\,\rangle + \int_M \langle\, \gamma \,,\, \mathrm{d}(\phi\cdot\iota_X g)\,\rangle \\
&\quad - \int_M \langle\, \alpha \,,\, \mathrm{d}^*(\phi\cdot\mathrm{d}\iota_X g)\,\rangle + \int_M \langle\, \alpha \,,\, \phi\cdot\mathrm{d}(\mathrm{div}X) - \iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g\,\rangle.
\end{aligned}$$

Here the vector field $Y_{\mathrm{d}\phi}$ is given by $\iota_{Y_{\mathrm{d}\phi}} g = \mathrm{d}\phi$. In the case $k \geq 1$ further partial integration yields for all $\phi \in \mathcal{T}$

$$\int_M \alpha(X)\cdot\Delta\phi = \int_M F\cdot\phi + \int_{\partial M} G\cdot\phi,$$

where $F \in W^{k-1,p}(M)$, $G \in W^{k,p}(M)$, and for some constant C

$$\|F\|_{W^{k-1,p}} + \|G\|_{W^{k,p}} \leq C\bigl(\|f\|_{W^{k,p}} + \|\gamma\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}\bigr).$$

So the regularity proposition A.1 for the weak Laplace equation with either Neumann (i.e. $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$) or Dirichlet (i.e. $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$) boundary conditions proves that $\alpha(X) \in W^{k+1,p}(M)$ with the according estimate.

In the case $k = 0$ one works with the following inequality: Let $\frac{1}{p^*} + \frac{1}{p} = 1$, then there is a constant C such that for all $\phi \in \mathcal{T}$

$$\Bigl|\int_M \alpha(X)\cdot\Delta\phi\Bigr| \leq C\bigl(\|f\|_p + \|\gamma\|_p + \|\alpha\|_p\bigr)\|\phi\|_{W^{1,p^*}}.$$

(Under the assumptions of remark A.3, one simply replaces $\|f\|_p$ and $\|\gamma\|_p$ by c_1 and c_2 respectively.) The regularity and estimate for $\alpha(X)$ then follow from proposition A.1. □

Acknowledgement. I would like to thank Dietmar Salamon for his constant help and encouragement in pursuing this project. Part of this research was supported by the Swiss National Science Foundation.

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