NCG Sem 9/17/2007

<u>Recall</u> G cpt. Lie gp., $K \subseteq G$ closed subgroup
Want vector bundles over G/K

<u>Inducing construction</u>

$\pi : K \to GL(V)$ f.d. rep of K,
$\Xi_\pi := \{\xi : G \xrightarrow{cts} V \mid \xi(xs) = \pi_s^{-1}\xi(x)\ \forall x \in G, s \in K\}$
with (left) G-action $(\lambda_x \xi)(y) = \xi(x^{-1}y)$ $\quad =: A$
Ξ_π is a (right) fg proj module over $C(G/K) \subset C(G)$.

K cpt $\Rightarrow \exists$ K-invariant inner product on V.
$\rightsquigarrow$ (Riemannian, Hermitian) metric on Ξ_π.
Call this $\langle \cdot, \cdot \rangle_A : \Xi_\pi \times \Xi_\pi \to A$, $\langle \xi, \eta \rangle_A(x) = \langle \xi(x), \eta(x) \rangle_V$.

Main example: tangent bundle, given by adjoint rep.

Interlude (Prerequisite for construction)
Adjoint rep. of a Lie group
<u>Def</u> For $g \in G$, $C_g : G \to G$, $h \mapsto ghg^{-1}$ aut;
$(C_g)_* : \mathfrak{g} \to \mathfrak{g}$ map on tangent plane, gives $Ad : G \to GL(\mathfrak{g})$.

$\mathfrak{k} = T_e(K) \subseteq T_e(G) = \mathfrak{g}$. Give $\mathfrak{g}$ an Ad-invariant inner product, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $\mathfrak{m} = \mathfrak{k}^\perp$. Take restriction $Ad : K \to GL(\mathfrak{m})$ ($\mathfrak{k}$, $\langle \cdot, \cdot \rangle$ Ad-inv $\Rightarrow$ $\mathfrak{m}$ Ad-inv.) So $\mathcal{J}(G/K) = \Xi_{(Ad|_K, \mathfrak{m})} = \{w : G \to \mathfrak{m} \mid w(xs) = Ad_s^{-1} w(x)\}$.

$\mathcal{J}(G/K)$ acts on $C^\infty(G/K)$ by $(\delta_w f)(x) = D_0^t f(x \cdot \exp(t w(x)))$.
(Derivation :))

eg $G = SU(2)$, $\mathfrak{g} = su(2) = \text{span}_{\mathbb{R}}\{\underset{\sigma_1}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}, \underset{\sigma_2}{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}, \underset{\sigma_3}{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}\}$
$K = \{\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\} \Rightarrow \mathfrak{k} = \text{span}_{\mathbb{R}} \sigma_3$, $\mathfrak{m} = \text{span}_{\mathbb{R}}\{\sigma_1, \sigma_2\}$.
$\mathcal{J}(S^2) = \{w : SU(2) \to \mathfrak{m} : w(x\Theta) = Ad_\Theta^{-1} w(x)\}$ $\quad \Theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$

Write $w(x) = u(x)\sigma_1 + v(x)\sigma_2$: condition is

$$\begin{pmatrix} u(x\Theta) \\ v(x\Theta) \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

Convenient thing to do: identify ${}_{\mathbb{R}}m = {}_{\mathbb{R}}\mathbb{C}$: $\sigma_1 \leftrightarrow 1$, $\sigma_2 \leftrightarrow i$.

$W = \begin{pmatrix} 0 & u+iv \\ -u+iv & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{pmatrix}$ and $\alpha : SU(2) \to \mathbb{C}$ satisfies $\alpha(x\theta) = e^{-2i\theta}\alpha(x)$.

$\rightarrow$ Looks like (is) induced rep of a 1-D complex rep.

Can use this description to (relatively easily) prove statements about the tangent bundle of S^2.

e.g. nontrivial: we show $\nexists$ nonvanishing section.

If so, it corresponds to $\alpha : SU(2) \to \mathbb{C} \setminus \{0\}$, $\alpha(x\theta) = e^{-2i\theta}\alpha(x)$.

$\Rightarrow \alpha_* : \pi_1(SU(2)) \to \pi_1(\mathbb{C}\setminus\{0\})$ $\quad$ $(\because 0 \to \mathbb{Z})$ satisfies $\alpha_*([v(1)]) = 2$ — bullshit. $\square$ But complexified tangent bundle to S^2 is trivial; can exhibit 2 lin. ind. sections. Etc.

Clifford Algebras / Bundles

(V, g) $\mathbb{R}$-vector space, g inner product.

$\mathbb{C}l(V)$ is the $\mathbb{C}$ assoc., unital alg. gen. by V with relation $V^2 = g(V,V)\cdot 1 \Longleftrightarrow UV + VU = -2g(U,V)\cdot 1$. (Deformation of exterior algebra in direction of inner product, Rieffel said.)

- $\mathbb{C}l(\mathbb{R}^{2m}) \cong M_{2^m}(\mathbb{C})$
- $\mathbb{C}l(\mathbb{R}^{2m+1}) \cong M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$

<u>Universal Property</u>

If A complex, assoc., unital alg., & $f : V \to A$ is $\mathbb{R}$-linear s.t. $f(V^2) = -g(V,V)\cdot 1_A$, $\exists!$ alg. hom $\tilde{f} : \mathbb{C}l(V) \to A$ extending f.

e.g. If $T : V \to V$, $T \in O(V)$, $f_T : V \to \mathbb{C}l(V)$, $v \mapsto Tv \rightsquigarrow \tilde{f}_T : \mathbb{C}l(V) \circlearrowleft$.

$\tilde{f}_T \in Aut(\mathbb{C}l(V))$.

In our case, $K \to O(m)$ (by Ad-invariance of inner product) $\leftarrow$ compose w/ $O(m) \to Aut(\mathbb{C}l(m))$ to give a rep. of K on $\mathbb{C}l(m)$ — still denote this by Ad, but it's really an extension of Ad.

Not the classical definition.

Example; $\mathfrak{m} = \mathrm{span}\{\sigma_1, \sigma_2\} = \mathrm{span}_{\mathbb{R}}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\right\}$.

Claim: $\mathbb{C}l(\mathfrak{m}) \cong M_2(\mathbb{C})$. Any ideas of a map $\mathfrak{m} \to M_2(\mathbb{C})$? ....

Image of $\mathfrak{m}$ generates $M_2(\mathbb{C})$, so... ✓

$\mathrm{Ad}: U(1) \to GL(\mathbb{C}l(\mathfrak{m}))$ is still conjugation :)

Def Clifford bundle over G/K is $\Xi(\mathrm{Ad}, \mathbb{C}l(\mathfrak{m})) = \{e : G \to \mathbb{C}l(\mathfrak{m}) :$
$e(xs) = \mathrm{Ad}_s^{-1} e(x)\}$.
$\mathbb{C}l(G/K)$

A spin^c-structure on G/K is a $\mathbb{C}l(G/K) - C(G/K)$ bimodule, $\mathcal{S}$, s.t. $\mathrm{End}_{C(G/K)} \mathcal{S} \cong \mathbb{C}l(G/K)$. (Means $\mathbb{C}l(G/K)$ & $C(G/K)$ are strongly Morita equivalent.) [Only if G/K even-dim'l... ignore this lie.]

$\mathcal{S}$ then becomes a "spinor bundle"

How to build this on S^2?

Need reps of $K = U(1)$. Take $\pi_n : U(1) \to GL(\mathbb{C})$, $e^{i\theta} \mapsto (z \mapsto e^{-in\theta} z)$.

$\Xi_n = \{\xi : SU(2) \to \mathbb{C} : \xi(x\theta) = \pi_n(\theta)^{-1}\xi(x) = e^{in\theta}\xi(x)\}$.

These give all complex line bundles over S^2: clearly any over $S^2 \setminus \{N \text{ pole}\}$ or $S^2 \setminus \{S \text{ pole}\}$ trivial; care only about transition map on equator, $\therefore$ $\{\text{line bundles}\} \leftrightarrow \pi_1(S^1)$.

$\mathcal{S} = \Xi_{-1} \oplus \Xi_1 = \left\{\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right\} \ni \psi$ (WAG for Patrick & Tucker & me, but correct. Facts: bundles / S^2 split into line bundles, need self-duality...)

Module action is $W = \begin{pmatrix} 0 & \bar{\alpha} \\ -\alpha & 0\end{pmatrix} \in \mathfrak{I}(S^2)$, $(W \cdot \psi) = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}$
$= \begin{pmatrix}\alpha\eta \\ -\alpha\xi\end{pmatrix} \in \Xi_{-1} \oplus \Xi_1$: because $\xi(x\theta) = e^{-i\theta}\xi(x)$ ditto α

$\eta(x\theta) = e^{+i\theta}\eta(x)$ $\quad (\alpha\eta)(x\theta) = \alpha(x\theta)\eta(x\theta) = e^{-2i\theta}\alpha(x)e^{i\theta}\eta(x) = e^{-i\theta}(\alpha\eta)(x)$

$\Rightarrow \mathcal{S}$ is a spin^c-structure on S^2.

But we want a spin-structure on S^2.

Def A spin structure on G/K is a spin^c-structure $\mathcal{S}$, together with a conjugate-linear map $C: \mathcal{S} \to \mathcal{S}$ satisfying for $\psi, \psi' \in \mathcal{S}$, $\varphi \in \mathbb{C}l(G/K)$, $f \in C(G/K)$

(i) $C(\psi f) = C(\psi)\bar{f}$

(ii) $C(\varphi\psi) = \chi(\bar{\varphi}) \cdot C(\psi)$

(iii) $\langle C(\psi), C(\psi')\rangle_A = \langle \psi', \psi\rangle_A$.

$\chi \in \mathrm{Aut}(\mathbb{C}l(m))$ is induced by $m \to m$, $v \mapsto -v$, so that $\chi^2 = id$.
Grading operator; even part of $\mathbb{C}l$ unaffected; odd part hit by $-id$.
(So $\mathbb{Z}/2\mathbb{Z}$-grades $\mathbb{C}l(m)$.)
$\bar{\varphi}$: $\overline{\sum \alpha_I v_I} = \sum \bar{\alpha}_I v_I$,
$v_I = e_{i_1} \cdots e_{i_k}$, where $\{e_1, \dots e_n\}$ is o-n basis for m.
($\{\sigma_1, \sigma_2\}$ for $\mathbb{C}S$)

"Build a C, and a χ, and a bar, and we'll call it a day."

For S^2, $m = \mathrm{span}_{\mathbb{R}}\{\sigma_1, \sigma_2\}$, $\mathbb{C}l(m) \cong M_2(\mathbb{C})$

$\overline{\alpha \cdot I + \beta\sigma_1 + \gamma\sigma_2} = \bar{\alpha} I + \bar{\beta}\sigma_1 + \bar{\gamma}\sigma_2$ — or

$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}$.

$\chi(\alpha I + \beta\sigma_1 + \gamma\sigma_2 + \delta\sigma_1\sigma_2) = (\alpha \ -\beta \ -\gamma \ \delta)$

$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$.

$C: \boxed{-}_{-1} \oplus \boxed{-}_{1}$ is $C\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \bar{\eta} \\ \bar{\xi} \end{pmatrix} \in \mathcal{S}$

(i), (iii) clear. (ii): $C(\varphi\psi) = \chi(\bar{\varphi})C(\psi)$: $\chi \circ \mathrm{bar}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix}$

So LHS $= C\left(\begin{pmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = C\begin{pmatrix} \alpha\eta \\ -\bar{\alpha}\xi \end{pmatrix} = \begin{pmatrix} -\alpha\bar{\xi} \\ \bar{\alpha}\bar{\eta} \end{pmatrix}$

RHS $= \begin{pmatrix} 0 & -\bar{\alpha} \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} \bar{\eta} \\ \bar{\xi} \end{pmatrix} = \begin{pmatrix} -\bar{\alpha}\bar{\xi} \\ \bar{\alpha}\bar{\eta} \end{pmatrix} \Longrightarrow S^2$ is a spin manifold.