Math 252 9/19/2007

$|G|<\infty$ and char $k \nmid |G| \overset{}{\underset{\text{Maschke}}{\Leftrightarrow}} \boxed{\begin{matrix} kG \\ \text{s.s.}\end{matrix}}$ (This includes char $k=0$.

$\Rightarrow$ done last time (by averaging trick).

$\Leftarrow$: Assume kG semisimple (G possibly infinite).
Consider augmentation map $\varepsilon: kG \to k$, $\varepsilon(\sum a_g g) = \sum a_g \in k$.
This is the unique morphism s.t. $G \to 1$.
[Universal property of group algebra kG, $k \in$ (Ring):
$\forall k$-algebra A, group hom $e: \to U(A)$ [group of units],
$\exists!$ k-alg hom $\bar{e}: kG \to A$ extending e.]

$$\begin{tikzcd} G \arrow[r, "e"] \arrow[d, phantom, "\cap" description] & U(A) \arrow[d, phantom, "\cap" description] \\ kG \arrow[r, dashed, "\exists!\bar{e}"] & A \end{tikzcd}$$

$\ker\varepsilon$ is the so-called augmentation ideal, codimension 1. By semisimplicity, $\exists J$: ${}_{kG}kG = {}_{kG}(\ker\varepsilon)\oplus J$. Fix $0\neq\alpha\in J$. $\forall g\in G$, $(g-1)\alpha \in (\ker\varepsilon)\cap J = 0$, as $\ker\varepsilon$ is 2-sided. Thus $\alpha = g\alpha$, $\forall g\in G$. $\Rightarrow G$ finite, $\alpha = a(\sum g)$.
$\alpha \notin \ker\alpha \Rightarrow 0 \neq \varepsilon(\alpha) = a(|G|\cdot 1) \Rightarrow$ char $k \nmid |G|$. $\square$

Comments about Maschke's original approach:

Thm $|G|<\infty$, V f.d. $\mathbb{C}G$-module. Then $\exists$ positive-definite Hermitian form $\langle\cdot,\cdot\rangle$ on V s.t. every $g\in G$ acts unitarily on $(V, \langle\cdot,\cdot\rangle)$.

Pf Start with any pos-def Hermitian form $\langle\cdot,\cdot\rangle_0$ on $\mathbb{C}V$. Then define
$\langle u,v\rangle := \sum \langle gu, gv\rangle_0 \in \mathbb{C}$. Can then check $\langle\cdot,\cdot\rangle$ is sesquilinear, $\langle gu, gv\rangle \equiv \langle u,v\rangle$,
$\forall v\neq 0$, $\langle v,v\rangle_0 \in \dot{\mathbb{R}}^+ \Rightarrow \sum\langle gv, gv\rangle \in \dot{\mathbb{R}}^+$ $\square$

$\exists$ also a real analogue; clearly same proof works.

Cor 1 Any representation $G \xrightarrow{\rho} \begin{Bmatrix} GL_n(\mathbb{C}) \\ GL_n(\mathbb{R})\end{Bmatrix} \sim$ a $\begin{Bmatrix}\text{unitary} \\ \text{orthogonal}\end{Bmatrix}$ rep,
$G \xrightarrow{\rho} \begin{Bmatrix} U(n) \\ O(n)\end{Bmatrix}$.

Cor 2. $W \hookrightarrow V$, $\begin{Bmatrix}\mathbb{C}G \\ \mathbb{R}G\end{Bmatrix}$-module inclusion, always splits.

Pf Fix $\langle\cdot,\cdot\rangle$ as above. Take $W^\perp$, check it's a $\begin{Bmatrix}\mathbb{C}G \\ \mathbb{R}G\end{Bmatrix}$-complement.

From now on, we assume characteristic is good unless otherwise stated.

$R = kG$

$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$

Simple modules are $M_i = D_i^{n_i}$, $D_i = \mathrm{End}_R(M_i)$.

Let $m_i = \dim_k M_i$, $d_i = \dim_k D_i$.

1. $m_i = n_i d_i$
2. W-components are $\mathrm{End}((M_i)_{D_i})$.
3. ${}_R R \cong n_1 M_1 \oplus \cdots \oplus n_r M_r$
4. General magic equation: $|G| = \sum n_i m_i = \sum n_i^2 d_i \leq \sum m_i^2$.

Master Theorem (w/o "divisibility" part)

If $k = \bar{k}$ (good char.), then $|G| = \sum m_i^2$, $r = \#$ conjugacy classes of G.

Pf Here, $d_i = 1$: Suppose $D \supset k$. Let $a \in D$; form $k(a)$. $k(a)$ is a finite field extension (a commutes w/ k) $\therefore k(a) = k \therefore a \in k \therefore D \subset k$. ✓

For last part, compute centers out of $kG \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k)$.

$\dim_k Z(kG) = \sum \dim Z(M_{n_i}(k)) = r$. But consider $\alpha_i := \sum (i^{th}$ conj. class$) \in kG$.

Clearly α_i conj-invariant $\therefore \alpha_i \in Z(kG)$; clearly $\{\alpha_i\}$ k-linearly independent.

Conversely, if $\sum a_g g$ is conj-invariant, then $\forall g, g' \in i^{th}$ conj class, $a_g = a_{g'}$.

Thus $\{\alpha_i\}$ spans $Z(kG)$. $\square$

Example: G abelian.

1. Each $n_i = 1$, each D_i a field. $\therefore kG = D_1 \times \cdots \times D_r$ a product of fields.
   Each D_i supports a simple kG-module.
2. Let $e = \exp(G)$. Assume k has a primitive e^{th} root of unity. Then $kG \cong \underbrace{k \times \cdots \times k}_{|G|}$.
   To see this, consider $kG \to D_i$: $g \mapsto e^{th}$ root of $1 \ni k \therefore$ image of $kG = k \therefore D_i = k$. $\square$

Thus, if G, H abelian, $\mathbb{C}G \cong \mathbb{C}H$ as $\mathbb{C}$-algebras!