MATH 54 (RIBET) HOMEWORK #6: PROOFS

These proof problems are more difficult than what we've done before, for two reasons. First, the definition of a vector space is rather long and complex. Second, and more problematic, is that mathematicians have a nasty habit of asking students to write proofs without explaining what makes a good proof.

General advice:

1. It's easier to understand the definition of a vector space if you divide the axioms into three categories. You can find an explanation at the end of this handout. (It's saved until then so you don't stop reading.)

2. When you are proving basic facts about vector space operations, the key to writing a good proof is to avoid circular reasoning. The best way to do this is to "play dumb" whenever you see *vector* addition or *scalar-times-vector* multiplication. In other words, write out the definitions given in the problem immediately, simplifying only after you have an expression in terms of numbers.

To see what I mean, look at the example solutions to the proof problems! They all break down into three steps: expand the definition of the left-hand side of an equation, then rewrite it using properties of numbers, and use the definitions in reverse to get the right-hand side of the equation.

PROOFS IN 3.2

Problem 36: *Prove* $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

An acceptable answer would be the equations in the following derivation. (The labels on the right are to show you how this matches up with the general advice.)

$$\begin{aligned}
&\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) && \\
=&\langle u_1, \dots, u_n\rangle \cdot (\langle v_1, \dots, v_n\rangle + \langle w_1, \dots, w_n\rangle) && \text{Expanding coordinates} \\
=&\langle u_1, \dots, u_n\rangle \cdot \langle v_1 + w_1, \dots, v_n + w_n\rangle && \text{Definition: vector addition} \\
=&u_1(v_1 + w_1) + \cdots + u_n(v_n + w_n) && \text{Definition: dot-product} \\
=&(u_1v_1 + \cdots + u_1v_n) + (u_1w_1 + \cdots + u_nw_n) && \text{Algebra \textit{with numbers}} \\
=&\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} && \text{Definition: dot-product}
\end{aligned}$$

Problem 37: *Prove* $r(\mathbf{u} \cdot \mathbf{v}) = (r\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (r\mathbf{v})$.

$$\begin{aligned}
&r(\mathbf{u} \cdot \mathbf{v}) && \\
=&r(\langle u_1, \dots, u_n\rangle \cdot \langle v_1, \dots, v_n\rangle) && \text{Expanding coordinates} \\
=&r(u_1v_1 + \cdots + u_nv_n) && \text{Definition: dot-product}
\end{aligned}$$

From here you can continue in two different ways, one to obtain $(r\mathbf{u}) \cdot \mathbf{v}$ and another (not shown, but almost identical) to obtain $\mathbf{u} \cdot (r\mathbf{v})$.

$$\begin{aligned}
=&(ru_1)v_1 + \cdots + (ru_n)v_n && \text{Algebra \textit{with numbers}} \\
=&(ru_1)v_1 + \cdots + (ru_n)v_n && \text{Algebra \textit{with numbers}} \\
=&\langle ru_1, \dots, ru_n\rangle \cdot \langle v_1, \dots, v_n\rangle && \text{Definition: dot-product} \\
=&(r\langle u_1, \dots, u_n\rangle) \cdot \langle v_1, \dots, v_n\rangle && \text{Definition: scalar multiplication} \\
=&(r\mathbf{u}) \cdot \mathbf{v} && \text{Definition: dot-product}
\end{aligned}$$

Proofs in 3.3

Problem 2: *Explain why ℓ [a line that does* not *go through the origin] is the set of all $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$, where t is any real number and $\mathbf{x}_0 \neq \mathbf{0}$ and $\mathbf{v}$ is not a multiple of $\mathbf{x}_0$.*

It's well-known that *every* line can be described as $\ell = \{\mathbf{x}_0 + t\mathbf{v} : t \in \mathbb{R}\}$ for a base-point $\mathbf{x}_0$ and a nonzero "velocity" $\mathbf{v}$. The condition that ℓ doesn't pass through the origin means that $\mathbf{0} \neq \mathbf{x}_0 + t\mathbf{v}$ for any t: in other words, there's no t satisfying $\mathbf{x}_0 = (-t)\mathbf{v}$. So $\mathbf{x}_0$ is not a multiple of $\mathbf{v}$.

Because $\mathbf{0}$ is a multiple of anything, the condition "$\mathbf{v} \neq \mathbf{0}$ and $\mathbf{x}_0$ not a multiple of $\mathbf{v}$" is equivalent to the condition "$\mathbf{x}_0 \neq \mathbf{0}$ and $\mathbf{v}$ not a multiple of $\mathbf{x}_0$". So, by rephrasing our conditions, we describe ℓ exactly like this idiot author did.

Problem 3: *Use exercise 2 to show that ℓ is not closed under addition.*

As above, we know $\mathbf{x}_0$ is not a multiple of $\mathbf{v}$, so in particular the sum $\mathbf{x}_0 + \mathbf{x}_0$ cannot be written in the form $\mathbf{x}_0 + t\mathbf{v}$. (In fact, if you pick *any* two points $\mathbf{x}_0 + t_1\mathbf{v}$ and $\mathbf{x}_0 + t_2\mathbf{v}$, their sum can't be written in the form $\mathbf{x}_0 + t\mathbf{v}$. For, if it could, then $\mathbf{x}_0$ would be the multiple $(t - t_1 - t_2)\mathbf{v}$. So ℓ is "the exact opposite" of closed under addition!)

Problem 11: *Show that axioms (3.19a, e, g) [commutative law of addition, distributive law (scalar by vector-sum), associative law] are satisfied for P_n.*

Addition and scalar multiplication are defined in example 8, top of page 160.

(a)

$$\begin{aligned}
& p+q \\
=& (a_0 + \cdots + a_n x^n) + (b_0 + \cdots + b_n x^n) && \text{Expanding coordinates} \\
=& (a_0 + b_0) + \cdots + (a_n + b_n)x^n && \text{Definition: vector addition} \\
=& (b_0 + a_0) + \cdots + (b_n + a_n)x^n && \text{Algebra with numbers} \\
=& (b_0 + \cdots + b_n x^n) + (a_0 + \cdots + a_n x^n) && \text{Definition: vector addition} \\
=& q + p && \text{Expanding coordinates}
\end{aligned}$$

(e)

$$\begin{aligned}
& c(p+q) \\
=& c[(a_0 + \cdots + a_n x^n) + (b_0 + \cdots + b_n x^n)] && \text{Expanding coordinates} \\
=& c[(a_0 + b_0) + \cdots + (a_n + b_n)x^n] && \text{Definition: vector addition} \\
=& [c(a_0 + b_0)] + \cdots + [c(a_n + b_n)]x^n && \text{Definition: scalar multiplication} \\
=& (ca_0 + cb_0) + \cdots + (ca_n + cb_n)x^n && \text{Algebra with numbers} \\
=& [(ca_0) + \cdots + (ca_n)x^n] + [(cb_0) + \cdots + (cb_n)x^n] && \text{Definition: vector addition} \\
=& c[a_0 + \cdots + a_n x^n] + c[b_0 + \cdots + b_n x^n] && \text{Definition: scalar multiplication} \\
=& cp + cq && \text{Expanding coordinates}
\end{aligned}$$

(g)

$$\begin{aligned}
& c(dp) \\
=& c[d(a_0 + \cdots + a_n x^n)] && \text{Expanding coordinates} \\
=& c[(da_0) + \cdots + (da_n)x^n] && \text{Definition: scalar multiplication} \\
=& [c(da_0)] + \cdots + [c(da_n)]x^n && \text{Definition: scalar multiplication} \\
=& [(cd)a_0] + \cdots + [(cd)a_n]x^n && \text{Algebra with numbers} \\
=& (cd)(a_0 + \cdots + a_n x^n) && \text{Definition: scalar multiplication} \\
=& (cd)p && \text{Expanding coordinates}
\end{aligned}$$

Problem 15: *Show that axioms (3.19b, f, h) [Associative law of addition, distributive law (scalar sum by vector], multiplicative identity] are satisfied for $C[a, b]$.*

Remember: f is an element of $C[a, b]$, but $f(x)$ is a number.

(b)

$$\begin{aligned}
&[(f+g)+h](x) && \\
=&(f+g)(x)+h(x) && \text{Definition: vector addition}\\
=&[f(x)+g(x)]+h(x) && \text{Definition: vector addition}\\
=&f(x)+[g(x)+h(x)] && \text{Algebra with numbers}\\
=&f(x)+(g+h)(x) && \text{Definition: vector addition}\\
=&[f+(g+h)](x) && \text{Definition: vector addition}
\end{aligned}$$

(d)

$$\begin{aligned}
&[(r+s)f](x) && \\
=&(r+s)f(x) && \text{Definition: scalar multiplication}\\
=&rf(x)+sf(x) && \text{Algebra with numbers}\\
=&(rf)(x)+(sf)(x) && \text{Definition: scalar multiplication}\\
=&[(rf)+(sf)](x) && \text{Definition: vector addition}
\end{aligned}$$

(h)

$$\begin{aligned}
&(1f)(x) && \\
=&1f(x) && \text{Definition: scalar multiplication}\\
=&f(x) && \text{Algebra with numbers}
\end{aligned}$$

Problem 27: It isn't. The book's answer isn't the only correct answer. Properties in Hill's definition of a vector space which aren't satisfied: additive closure, scalar-multiplicative closure, additive identity (zero), additive inverse (negatives).

Problem 34: $f \cdot g = \int_0^1 (x)(x^2)dx = x^4/4 \,|_0^1 = 1/4$.

Problem 35:

$$\|f\|^2 = f \cdot f = \int_0^1 (x)(x)dx = x^3/3 \,|_0^1 = 1/3 \therefore \|f\| = 1/\sqrt{3}.$$

$$\|g\|^2 = f \cdot g = \int_0^1 (x^2)(x^2)dx = x^5/5 \,|_0^1 = 1/5 \therefore \|f\| = 1/\sqrt{5}.$$

Problem 38:

$$\|g\|^2 = g \cdot g = \int_0^{2\pi} \cos(x)\cos(x)dx = \int_0^{2\pi} \frac{1+\cos(2x)}{2} = \pi \therefore \|f\| = \sqrt{\pi}.$$

$$\|h\|^2 = h \cdot h = \int_0^{2\pi} (x)(x)dx = x^3/3 \,|_0^{2\pi} = (2\pi)^3/3 \therefore \|f\| = (2\pi)^{3/2}/\sqrt{3}.$$

What the axioms are about

Category 1: Properties of Vectors (labelled as in Hill's definition)

(a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
(b) $(\mathbf{u}+\mathbf{v})+w=\mathbf{u}+(\mathbf{v}+w)$.
(e) $r(\mathbf{u}+\mathbf{v})=r\mathbf{u}+r\mathbf{v}$.
(f) $(r+s)\mathbf{u}=r\mathbf{u}+s\mathbf{u}$.
(g) $(rs)\mathbf{u}=r(s\mathbf{u})$.
(h) $1\mathbf{u}=\mathbf{u}$.

Category 2: Properties of Spaces (labelled as in Hill's definition)

(!) There is at least one element of V: that is, $V \neq \emptyset$.
(!) If $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u}+\mathbf{v}$ is in V.
(!) If $\mathbf{u}$ is in V and r is any number, then $r\mathbf{u}$ is in V.

Category 3: Convenient Facts (labelled as in Hill's definition)

(c) V contains an element $\mathbf{0}$, obeying the identity $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
(d) Any element of V has a negative, obeying the identity $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

The properties of vectors are just the facts which are true of all vectors. It is because of these properties that linear algebra works at all. Conveniently, if you know that V is a set of vectors (e.g., V is a subset of $\mathbb{R}^n$ and you wish to check whether it's a subspace), *you don't need to check these properties.*

The properties of spaces are not statements about vectors; rather, they are statements about the *set* V. These are what you check when deciding whether a sub*set* is a sub*space*.

The third category consists of things that you never have to check, and are only part of Hill's definition because he hates you. (They only become important as axioms in abstract algebra.)