MATH 54 (RIBET) HOMEWORKS #16–17: 5.3.31–33, 5.4.{7,13,21–25}

Exercises from 5.3

Problem 31: Prove: If A is square, S is invertible, Λ is diagonal, and $A = S\Lambda S^{-1}$, then A has n linearly independent eigenvectors. [**Don't stop reading here. The other solutions are much shorter.**]

You had better expect the columns of S to be linearly independent eigenvectors!

The columns of S are linearly independent just because S is invertible. (It's a theorem that a square matrix is invertible if and only if its columns are independent. A good way to remember this is to consider the *rank* of S: to be invertible, S must have rank n, so it needs to have an n-dimensional column space.)

More specifically, if $S = [s_1 \dots s_n]$ and Λ is the diagonal matrix consisting of $\lambda_1, \dots, \lambda_n$, you want to show that $As_k = \lambda_k s_k$, $1 \le k \le n$. Many arguments are possible. You will probably like the first the most, but the second two can be stated much more quickly.

Argument 1: Calculate $As_k = S\Lambda S^{-1}s_k$ directly, except for the $S^{-1}s_k$ part. (It's too hard to do that directly.) If $\mathbf{x} = S^{-1}s_k$, then $\mathbf{x}$ is the solution to the equation $S\mathbf{x} = s_k$. But since $S\mathbf{x} = [s_1|\dots|s_k] \begin{bmatrix} x_1 \\ \dots \\ x_k \end{bmatrix} = x_1s_1 + \dots + x_ks_k + \dots + x_ns_n$, you can easily see that $\mathbf{x} = e_k$ (meaning $x_k = 1$ and the other coordinates of $\mathbf{x}$ are zero) is a solution. So $S^{-1}s_k = e_k$. Then by writing the matrices down you see $S\Lambda e_k = S(\lambda_k e_k) = \lambda_k s_k$.

Argument 2: Think in terms of two bases: the standard basis $B = \{e_1, \dots, e_n\}$, and the eigen basis $C = \{s_1, \dots, s_n\}$. Since $[C|B] = [S|I]$, S is the *transition matrix* from B to C. Since $[C|B] = [I|S]$ is row-equivalent to $[S^{-1}|I]$, S^{-1} is the transition matrix from C to B. Finally, *every vector in* $\mathbb{R}^n$ *is already equal to its coordinate vector with respect to the standard basis*: $v = [v]_B$. (That's what the standard basis is.) So $S\Lambda S^{-1}s_k = S\Lambda S^{-1}[s_k]_B = S\Lambda[s_k]_C = S\Lambda e_k = \lambda_k S[s_k]_C = \lambda_k[s_k]_B = \lambda_k s_k$.

Argument 3: If you form the $n \times n$ matrix $[As_1|\dots|As_n]$, then it is equal to $A[s_1|\dots|s_n]$. So, $[As_1|\dots|As_n] = (S\Lambda S^{-1})S = S\Lambda$. Multiplying by Λ on the right causes a column operation, just as multiplying by Λ on the left causes a row operation. Result: $[As_1|\dots|As_n] = [\lambda_1 s_1|\dots|\lambda_n s_n]$.

Problem 32: Suppose that A and B are square, diagonalizable, and have the same eigenvectors (but not necessarily the same eigenvalues). Show that A and B commute, $AB = BA$.

Let S be the $n \times n$ matrix consisting of n linearly independent eigenvectors (common to both matrices). This is possible because A and B are diagonalizable. Then there are diagonal matrices Λ_A and Λ_B (whose entries are just the eigenvalues associated to the columns of S) such that $A = S\Lambda_A S^{-1}$ and $B = S\Lambda_B S^{-1}$. Then $AB = S\Lambda_A \cancel{S^{-1}S}\Lambda_B S^{-1}$ and $BA = S\Lambda_B\cancel{S^{-1}S}\Lambda_A S^{-1}$. Clearly these are equal if $\Lambda_A\Lambda_B = \Lambda_B\Lambda_A$. And diagonal matrices *do* commute with each other; to multiply two diagonal matrices you just multiply the corresponding diagonal entries together.

Problem 33 Let T be a 3×3 upper triangular matrix with diagonal entries 1, 3, -6. Can T be diagonalized? Why or why not? If so, what is Λ?

Under these assumptions, the "variable matrix" $\lambda I - T$ is upper triangular with diagonal entries $\lambda - 1$, $\lambda - 3$, $\lambda + 6$. So $\det(\lambda I - T) = (\lambda - 1)(\lambda - 3)(\lambda + 6)$, so the eigenvalues of T are 1, 3, -6. Since these are *distinct*, it follows that T *is* diagonalizable, and Λ will just be the diagonal matrix consisting of 1, 3, and -6.

Problems 7 and 13: Find n linearly independent eigenvectors and verify the ones with distinct eigenvalues are orthogonal. Then find an orthogonal matrix Q and a diagonal matrix Λ such that $Q^{-1}AQ = \Lambda$. (In #7, A is the 3×3 matrix of ones. In #13, A is the 4×4 matrix of ones.)

Let's do it with a matrix of n ones. A previous homework asked you to prove that for $\lambda = n$, $[1, \ldots, 1]^T$ is an eigenvector, and for $\lambda = 0$, $e_k - e_{k+1}$ for $1 \leq k \leq n-1$ will give you $(n-1)$ independent eigenvectors. So quote that result. To finish part (a), note that the dot-product of $[1, \ldots, 1]^T$ and $e_k - e_{k+1}$ is $(1)(1) + (1)(-1) = 0$.

For part (b), you have to make these eigenvectors *orthonormal.* In other words, we must apply Gram-Schmidt to the one n-eigenvector and the $(n-1)$ 0-eigenvectors. Gram-Schmidt starting with one vector is easy: just normalize it. So we get $\frac{1}{\sqrt{n}}[1, \ldots, 1]^T$ for $\lambda = n$.

Starting with $(n-1)$ eigenvectors is not easy. Just do G-S by hand separately for $n = 3$ and $n = 4$. If you follow the procedure in the back of section 4.4, a pattern will emerge!

$$\text{The vectors } \begin{bmatrix} 1/1 \\ -1 \\ 0 \\ 0 \\ \cdots \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \\ \cdots \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \\ \cdots \end{bmatrix}, \ldots \text{ are mutually orthogonal.}$$

Why? If you take the dot-product $u_i \cdot u_j$, with $i < j$, then it *factors* as $\left(\frac{1}{i} + \cdots + \frac{1}{i} - 1\right)\frac{1}{j} = 0$! So then you normalize: $w_k = \sqrt{k/(k+1)}u_k$. End up with:

$$\Lambda_3 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 3 \end{bmatrix}, Q_3 = \begin{bmatrix} \frac{1}{\sqrt{1\cdot 2}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{1\cdot 2}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Lambda_4 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 4 \end{bmatrix}, Q_4 = \begin{bmatrix} \frac{1}{\sqrt{1\cdot 2}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{3\cdot 4}} & \frac{1}{\sqrt{4}} \\ \frac{-1}{\sqrt{1\cdot 2}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{3\cdot 4}} & \frac{1}{\sqrt{4}} \\ 0 & \frac{-2}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{3\cdot 4}} & \frac{1}{\sqrt{4}} \\ 0 & 0 & \frac{-3}{\sqrt{3\cdot 4}} & \frac{1}{\sqrt{4}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \\ 0 & 0 & \frac{-3}{2\sqrt{3}} & \frac{1}{2} \end{bmatrix}$$

Problem 21: Show that if there is an orthogonal matrix Q that diagonalizes A, then A is symmetric. (*HINT* If $Q^{-1}AQ = \Lambda$, then $A = Q\Lambda Q^{-1}$. Now find A^T and use the facts that $\Lambda^T = \Lambda$ and $Q^{-1} = Q^T$.)

$A = Q\Lambda Q^T$, so $A^T = (Q^T)^T(\Lambda)^T(Q)^T$. (Remember: $(XY)^T = Y^T X^T$. It's a lot like how inverses work.) $(Q^T)^T = Q$ and $\Lambda^T = \Lambda$, so it follows that $A^T = Q\Lambda Q^T$. So $A^T = A$.

Problem 22: Prove (5.40): A complex number $z = a + bi$ is real if and only if $z = \bar{z}$.

$a + bi = a - bi \iff a = a, b = -b \iff b = -b \iff b = 0.$

Problem 23: For a complex number $z = a + bi$, the *length* or *absolute value* of z is $|z| = \sqrt{a^2 + b^2}$. Show that $|z| = \sqrt{z\bar{z}}$.

Expand the right-hand side: $\sqrt{z\bar{z}} = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 - (bi)^2} = \sqrt{a^2 - (-1)b^2} = \sqrt{a^2 + b^2}$; this is by definition equal to $|z|$.

Problem 24: For $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, define the length or norm of $\mathbf{z}$ by $\|\mathbf{z}\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ and the *inner* (or *dot*) *product* by $\mathbf{z} \cdot \mathbf{w} = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$. Use exercise 23 to show that $\|\mathbf{z}\| = \sqrt{\mathbf{z} \cdot \mathbf{z}}$ (and the conjugates over the second factor in $\mathbf{z} \cdot \mathbf{w}$ are necessary for this to be true).

Exercise 23 shows that $|z|^2 = z\bar{z}$ for every complex number z. $\sqrt{\mathbf{z} \cdot \mathbf{z}} = \sqrt{z_1\bar{z}_1 + \cdots + z_n\bar{z}_n} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. This is by definition equal to $\|\mathbf{z}\|$. To see the absolute values and conjugates in the definitions are necessary, consider $\mathbf{z} = \mathbf{w} = \langle i \rangle$.

Problem 25: Use exercise 24 to show (5.41): For $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{C}^n$ and a complex number λ, $(\lambda\mathbf{z}) \cdot \mathbf{w} = \lambda(\mathbf{z} \cdot \mathbf{w})$ and $\mathbf{z} \cdot (\lambda\mathbf{w}) = \bar{\lambda}(\mathbf{z} \cdot \mathbf{w})$.

Remember that $\overline{zw} = \bar{z} \cdot \bar{w}$. So $(\lambda\mathbf{z}) \cdot \mathbf{w} = \sum_{i=1}^n (\lambda z_i)\bar{w}_i = \lambda \sum_{i=1}^n z_i\bar{w}_i = \lambda(\mathbf{z} \cdot \mathbf{w})$ and $\mathbf{z} \cdot (\lambda\mathbf{w}) = \sum_{i=1}^n z_i\overline{(\lambda w_i)} = \sum_{i=1}^n z_i\bar{\lambda}\bar{w}_i = \bar{\lambda}\sum_{i=1}^n z_i\bar{w}_i = \bar{\lambda}(\mathbf{z} \cdot \mathbf{w})$.