MATH 54 S215/S205: REVIEW FOR HILL §3.7–END

Note: It may be a good idea to look back at my review problems for the MT2 review for coordinate vectors and transition matrix material!

(1) The continuous 2π-periodic functions form a vector space. (Why?) Show that $A_0(f) = a_0$, $A_n(f) = a_n$, and $B_n(f) = b_n$ (a_0, a_n, and b_n being the usual Fourier coefficients for f) are linear transformations. Can any of them be represented by a matrix? Why or why not?

Solution: They form a vector space because the continuous functions form a vector space, and the subset of those which are 2π-periodic is closed under addition and scalar multiplication.

A_0, A_n, and B_n as defined are linear transformations because they preserve addition and scalar multiplication. For example:

$$A_n(f+g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f+g)(x)\sin(nx)dx = A_n(f+g).$$

The other calculations are very similar.

(2) Explain why $f \cdot g = \int_0^1 f(x)g(x)dx$ is an inner product on $C[0,1]$ but not on $C[-1,1]$.

Solution: The first is the usual inner product on $C[0,1]$, and an explanation can be found in Hill. The second, however, violates the rule that $f \cdot f$ should be positive (not zero) when $f \neq 0$. The function $f(x) = \begin{cases} x & -1 \leq x < 0 \\ 0 & 0 \leq x \leq 1 \end{cases}$ is not equal to the zero function, but this definition would make $f \cdot f = \int_0^1 (0)^2 dx = 0$.

(3) In a vector space with inner product, prove that the vectors orthogonal to a given vector x form a subspace. In the case of $\mathbb{R}^n$, prove this subspace is $NS(x^T)$.

Solution: It's the "nullspace" of the linear transformation $L(v) = x \cdot v$. More directly, you can argue that $x \cdot 0 = 0$, $x \cdot v = x \cdot w = 0 \implies x \cdot (v + w) = x \cdot (rv) = 0$: in other words, that the collection of vectors orthogonal to x (1) contains zero, (2) is closed under addition, and (3) is closed under scalar multiplication.

Since $x \cdot y = x^T y$, $y \perp x$ if and only if $x^T y = 0$ iff $y \in NS(x^T)$.

(4) If Q is an orthogonal matrix with entries $\pm 1/4$, what are the dimsensions of Q?

Solution: The columns of Q must be ortho*normal*, so $n(1/4)^2 = 1$. Same for the rows; alternatively, an orthogonal matrix must be square. So Q is a 16×16 matrix.

(5) A is a 2×2 matrix with $\mathrm{Tr}(A) = 3$ and $\det(A) = 2$. What are the eigenvalues of A?

Solution: $\lambda_1 + \lambda_2 = 3$, $\lambda_1 \lambda_2 = 2$. This makes λ_1, λ_2 1 and 2. (In fact, the characteristic polynomial fr a 2×2 matrix is $\lambda^2 - (\mathrm{Tr}\, A)\lambda + (\det A)$.)

(6) A is a 3×3 triangular matrix with diagonal entries 1,2,2. Can you find the eigenvalues? Do you know whether A is diagonalizable?

Solution: Yes. Notice that $\lambda I - A$ is also triangular! So its determinant $\det(\lambda I - A)$ is $(\lambda - 1)(\lambda - 2)(\lambda - 2)$, the product of the diagonal entries. This makes the eigenvalues 1,2,2. Based on this information there's no way to tell whether A is diagonalizable: it has repeated eigenvalues, but nothing is known about the eigenvectors.

(7) A is a 2×2 matrix with eigenvalues 0,1 and corresponding eigenvectors $\langle 1, 3\rangle$ and $\langle 3, -1\rangle$. Is A symmetric? What are $\mathrm{Tr}(A)$ and $\det(A)$? How many such A exist? What's $A^{1000} - A$?

Solution: A is orthogonally diagonalizable, hence symmetric, because it has real eigenvalues and n $(= 2)$ mutually orthogonal (independent) eigenvectors. $\mathrm{Tr}(A) = 0 + 1 = 1$, $\det(A) = 0 \cdot 1 = 0$. There is exactly one A, equal to $Q\Lambda Q^T$ (or $S\Lambda S^{-1}$) where these matrices are defined as usual. (Or, you can note that A is uniquely defined in the basis consisting of the eigenvectors of A.) Finally, $A^{1000} - A = Q(\Lambda^{1000} - \Lambda)Q^T$; but since $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Lambda^{1000} = \begin{bmatrix} 1^{1000} & 0 \\ 0 & 0^{1000} \end{bmatrix} = \Lambda$. So the answer to the last question is 0.

(8) If A is an $m \times n$ matrix with orthonormal columns, prove that the orthogonal projection onto $CS(A)$ is AA^T. Why is it "easy" to find the least-squares solution to $Ax = b$?

Solution: This boils down to realizing $A^T A = I_n$.