

MATH 54 S215/S205: REVIEW FOR HILL §1.1–3.6

- (1) Let A' be an $n \times (n + 1)$ augmented matrix representing a linear system. Show that the n -vector x is a solution to the system if and only if $A' \begin{bmatrix} x \\ -1 \end{bmatrix} = 0$.

Solution: The augmented matrix representing the linear system $Ax = b$ is just $[A|b]$; A is an $n \times n$ block and b is an $n \times 1$ block. So the matrix product $A' \begin{bmatrix} A & b \end{bmatrix} \begin{bmatrix} x \\ -1 \end{bmatrix}$ is $[Ax - b(1)]$. This is of course zero exactly when $Ax = b$.

- (2) If A is an $n \times n$ matrix in row echelon form, is $A + I$ in row echelon form? In this case, where are its pivots, and is $A + I$ invertible?

Solution: No; coming up with a counterexample is as easy as taking a matrix which is square but not in REF, and subtracting I . On the other hand, if none of the entries of A are negative, then $A + I$ will be a matrix in REF with pivots on the diagonal. Having n pivots, it will be invertible.

- (3) A spreadsheet contains two matrices, A and B , with row and column headings.

A holds prices per kg.: its rows are labeled “Walmart” and “Petco”, and its columns are labeled “cat food” and “duck chow”.

B holds kg. per month quantities: its rows are labeled “cat food” and “duck chow”, and its columns are labeled Jan.–Mar.

Interpret the matrix product $AB \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: The column labels of A match the row labels of B ; these indicate that the entries of AB will represent a total for both pets. Specifically, it will have rows representing stores, columns representing months, and entries representing the price per month of shopping there. Multiplying AB by $[1, 1, 1]^T$ will simply total the costs over the three months.

- (4) Are elementary matrices always square? Always invertible? If E is an elementary matrix and A is a matrix of the same shape, which of A , AE , EA are row-equivalent?

Solution: Yes, yes, and in this list A and EA are row-equivalent. (Make sure you know what this means.)

- (5) If A and B are invertible matrices, which of the following are invertible?

$$A^2, ABA, A + B, A + I, \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

What are $NS(A)$, $RS(A)$, and $CS(A)$?

Show that B is invertible if and only if B^2 is invertible.

Solution: A^2 and ABA are invertible; their inverses are $(A^{-1})^2$ and $A^{-1}B^{-1}A^{-1}$. $A + B$ and $A + I$ need not be; just consider $A = -I$, $B = I$. And the block matrix is invertible, with inverse given by $\begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix}$. The correct way to justify this is to note that $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Watch out: if the A^{-1} and B^{-1} switch places, the answer is wrong. If you couldn't spot the inverse, you could still solve the problem by showing the block matrix has a rank of $2n$ or a nullity of zero, or showing that it's row-equivalent to a matrix whose inverse you can find correctly.

If B^{-1} exists, then $(B^{-1})^2$ is clearly an inverse of B^2 —that is, $(B^{-1})B^2 = B^2(B^{-1})^2 = I$. On the other hand, if B^2 is invertible, there are a few ways to proceed. You can try showing $A = B(B^2)^{-1}$ or $A = (B^2)^{-1}B$ is an inverse for B —but be careful, because you need both AB and BA to equal I . So you have to argue either that a “left inverse” or “right inverse” is always an inverse, or else argue that $B(B^2)^{-1}B = I$ by other means. A method which bypasses the question altogether is

noticing the only solution to $(B^2)x = 0$ is $x = 0$. So in particular, if $Bx = 0$, then $B(Bx) = 0$, so $x = 0$ is also the only solution to $Bx = 0$. Thus, B is nonsingular. (Similarly, you can argue $\mathbb{R}^n = CS(B^2) \supset CS(B)$, and conclude $\text{rk}(B) = n$.)

- (6) What is the inverse of ABC^{-1} , and when is it defined?

Solution: $CB^{-1}A^{-1}$.

- (7) Suppose $A = [x|y|z]$ is a 3×3 matrix built from column vectors.

Explain why $A \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a linear combination of $\{x, y, z\}$ but $[a, b, c]A$ may not be.

If $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in NS(A)$, find a linear dependence relation for $\{x, y, z\}$.

If $\left\{ \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} \right\}$ is a basis for $NS(A)$, find a system of equations describing the span of $\{x, y, z\}$.

Solution: The first product is $[x(a) + y(b) + z(c)]$, more commonly written as $ax + by + cz$. The second matrix product is not; instead, it forms a combination of the *rows* of $[x|y|z]$, which is definitely not what you'd want.

Based on this observation, it's clear that a dependence relation is $ax + by + cz = 0$ in the second question. (This doesn't work if $a = b = c = 0$, of course.)

Third question: trick question: $NS(A)$ doesn't determine the span of $\{x, y, z\}$, or vice-versa. However, if you instead had a basis for $NS(A^T)$, then the equations $\{a_i x_1 + b_i x_2 + c_i x_3 = 0\}$ would do!

- (8) If $\{Ax, Ay, Az\}$ is a linearly independent set, show $\{x, y, z\}$ is. Show that the rank of A is at least 3.

Solution: If $ax + by + cz = 0$, then $0 = A(ax + by + cz) = a(Ax) + b(Ay) + c(Az)$; in this case, $a = b = c = 0$ by linear independence of $\{Ax, Ay, Az\}$. Since $ax + by + cz = 0 \implies a = b = c = 0$, the vectors x, y, z are linearly independent. For the second question, notice that $Ax, Ay, Az \in CS(A)$. Since the column space of A contains three linearly independent vectors, $\text{rk}(A) = \dim CS(A) \geq 3$.

- (9) State the rank-nullity theorem.

(See Hill.)

- (10) Let X be a 2-dimensional vector space, and let B be a basis for X . Is B a vector space? How large, as a set, is B ? If $x, y \in B$, and $ax + by = 0$, what can you say about a and b ?

Solution: No, B is not a vector space. Far from being closed under addition and scalar multiplication, it is linearly independent: the sum of two elements of B , or a scalar multiple (besides $1x$) of an element, will *never* be in B !

B contains two elements. If $ax + by = 0$, then (by linear independence) $a = b = 0$. (Taken literally the question might allow $x = y$, in which case $a = -b$ could work. But that's not what I meant.)

- (11) When is the solution set for $Ax = b$ a vector space?

Do the 9×9 matrices A satisfying $A^2 = A$ form a vector space?

What about the invertible 6×6 matrices?

Solution: When $b = 0$, no because $I^2 = I$ but $(2I)^2 \neq (2I)$, no because 0 is not invertible.