MATH 54 S215/S205: DIFFERENTIAL EQUATIONS SO FAR

This is an outline of the major types of differential equation problems you should be able to solve now (and potentially on Wednesday's quiz.) Feel free to skip those with which you're most comfortable; a good target is probably finishing 6–8 of these.

Administrative Stuff

I apologize for missing my sections on Friday, for missing any appointments I had on that day, and for giving no warning. This was accidental. I will have **extra office hours** in 848 Evans between 1 and 5 PM today (Monday) and between 9 and 11:30 AM tomorrow (Tuesday).

I may have "lied" about Abel's formula. If I wrote down any formula for $W(y_1, y_2)$ (given two solutions to $y'' + p(t)y' + q(t)y = 0$) other than $W(t) = C \exp\left(-\int p(t)dt\right)$, you should forget the erroneous formula. It is stated correctly in Boyce & DiPrima.

Techniques and sample problems

(1) Given an "easy" linear, homogeneous ODE, find a fundamental set of solutions.
So far, the equations we should be able to solve are those with *constant coefficients*; for now we only fully know how to proceed when the *characteristic polynomial* has distinct (real) roots. (But you are expected to remember separable ODE's from Math 1B.) Warning: when writing down the characteristic polynomial, don't forget that $y = y^{(0)}$ translates to $r^0 = 1$.
Exercise: Do this for $y^{(3)} - y' = 0$.

(2) Given a fundamental set and suitable initial conditions (usually the values of $y(a), \dots, y^{(n-1)}(a)$), find the (hopefully unique) solution.
Exercise: Do this for the ODE from the previous exercise, with $y(0) = 12$, $y'(0) = 1$, $y''(0) = 3$.

(3) Given a set of functions, compute their Wronskian.
Exercise: Find $W(1-t, 1+t)$.

(4) Given a "wild type" linear, homogeneous ODE, be able to write it in the "standard" form $y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$ (for the purpose of doing the next two types of problems).
Exercise: do this for the associated Legendre equation (encountered during a quantum mechanical treatment of a hydrogen atom, for instance): $([1-x^2]y')' + \left(\ell[\ell-1] - \frac{m^2}{1-x^2}\right) y = 0$, $-1 < x < 1$.

(5) Given an equation in this "standard" form, determine the longest interval(s) on which you can expect a unique solution; also, determine the Wronskian $W(y_1, \dots, y_n)$ of any set of n solutions to the (nth order) equation. Which interval is longest might depend on your initial conditions (which specify a value of x where your solution should be defined).
Exercise: do this for the ODE from the previous exercise.

(6) Use Abel's formula, when applicable, to show a function (or set of functions) can't be a solution (or set of solutions) to a "nice" linear, homogeneous ODE.
Exercise: Prove that $f(t) = t\sin(t)$ cannot be a solution to any ODE $y'' + p(t)y' + q(t)y = 0$ in which $p(t)$, $q(t)$ are continuous. (Hint: this function satisfies $f(0) = f'(0) = 0$.) Also prove that $y_1(t) = 1$ and $y_2(t) = \cos(t)$ cannot both be solutions to such an ODE. (Hint: again consider $t = 0$. An easier, but less general, method is to think of it as the solution of an *initial value problem*.)

(7) (**important**) Be able to translate an nth-order linear ODE and its initial conditions to a system first-order linear ODE's. (Pretty much everything mentioned above has an analogue for such systems; these analogues often look more natural than their forms for higher-order ODE's!)
Exercise: Do this for $y'' - 2y' + y = 0$, with initial conditions $y(0) = 0$, $y'(0) = 1$.

(8) Finally, several techniques which are more closely related to linear algebra. First, be able to recognize whether an ODE, or a differential operator, is linear.

Exercise: Is $y'' - ty + t^{-1}y = 0$ a linear ODE? Is $L[y] = 2yy'' + y'y' + t$ a linear differential operator?

(9) Testing linear independence of functions, with and without Wronskians.

Exercise: If a family of functions is linearly dependent, is its Wronskian zero? If a family of functions has a Wronskian of zero, must it be linearly dependent? Explain why $y_1(t) = +t$ and $y_2(t) = |t|$ are linearly independent (when considered as functions defined on $\mathbb{R}$).

(10) Applying ideas of basis and coordinates to more abstract problems.

Past homework exercise: if $\{y_1, y_2\}$ is a linearly independent set of functions, under what conditions is $\{a_1y_1 + a_2y_2, b_1y_1 + b_2y_2\}$ also a linearly independent set of functions? (Your answer should be when the set of coefficient vectors, $\left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}$, is linearly independent.)

(11) Finding linear dependence relations, when they exist, among a set of functions.

Exercise: Do this for $\{1-t, 1+t, 2t\}$ using polynomial coefficients. Do this for $\{1, \cos(2t), \sin^2(t)\}$ using Wronskian-style matrices or any other method you can think of!