MATH 54 S215/S205: REVIEW FOR BOYCE&DIPRIMA

Note: It may also be a good idea to consult the handout on sections 3.1–7.1.

(1) Solve: $y'' - 4y = 0$.

Solve: $y'' - 4y = 4e^{2x}$, given $y = xe^{2x}$ is a solution.

Solve the above equations subject to the initial conditions $y(0) = 0$, $y'(0) = 4$.

Solution: The first equation is a linear, homogeneous ODE with constant coefficients. It is also an eigenfunction problem, looking for an eigenfunction of y'' corresponding to the eigenvalue 4. Either one of these ideas leads to a solution.

Following the first idea, you can set up the characteristic equation $r^2 - 4 = 0$. Since its solutions are $r = \pm 2$, you find a general solution of $y = a_1 e^{2x} + a_2 e^{-2x}$ (a_1, a_2 constant).

Following the second idea, you can simply write down the general solution $y = b_1 \cosh(2x) + b_2 \sinh(2x)$ (b_1, b_2 constant).

The second equation is inhomogeneous; its general solution takes the form $y_p + y_c$, where y_p is some solution to the original equation (given) and y_c is the general solution to the homogeneous equation $y'' - 4y = 0$ (just solved). So the general solution to this ODE can be expressed as either $y = a_1 e^{2x} + a_2 e^{-2x} + xe^{2x}$ or $y = b_1 \cosh(2x) + b_2 \sinh(2x) + xe^{2x}$.

Finally, imposing initial conditions will give a system of equations for the coefficients. For the first ODE, you should be getting $y = e^{2x} - e^{-2x} = 2\sinh(2x)$. For the second ODE, you should be getting $y = \frac{3}{4}(e^{2x} - e^{-2x}) + xe^{2x} = \frac{3}{2}\sinh(2x) + xe^{2x}$.

(2) When do the solutions of $y'' - 2y' = f(t)$ form a vector space? When they do, what is the general solution? What is a fundamental set?

Solution: When $f(t) = 0$ (in other words, when the equation is homogeneous), the solution set is just the nullspace of a linear operator. (It is this observation that led to the idea of a fundamental set of solutions.) You can prove it is a vector space by plugging in $y = 0$, $y = y_1 + y_2$, and $y = ry_1$ (with y_1, y_2 solutions and r any scalar). This would show that the set of solutions contains zero and is closed under addition and scalar multiplication.

If f is not zero, then certainly $(0)'' - 2(0)'$ is not equal to $f(t)$ (for all values of t); this implies that the solution set is *only* a vector space when $f = 0$.

(3) Show that $W(af_1 + cf_2, bf_1 + df_2) = W(f_1, f_2) \cdot \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Assume that $\{f_1, f_2\}$ is a linearly independent set. Why does this equation not prove that $\{af_1 + cf_2, bf_1 + df_2\}$ is linearly independent if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$?

Explain why this statement is true anyway.

Solution: This is because of a matrix relation involving the "Wronskian matrices" (whose determinants are $W(af_1 + cf_2, bf_1 + df_2)$ and $W(f_1, f_2)$). $\begin{bmatrix} af_1(x) + cf_2(x) & bf_1(x) + df_2(x) \\ af_1'(x) + cf_2'(x) & bf_1'(x) + df_2'(x) \end{bmatrix} = \begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Remember that $\det(AB) = \det(A)\det(B)$. So, taking determinants of both sides, you get the equation I was looking for.

This equation does tell us that $W(af_1 + cf_2, bf_1 + df_2)(t) = 0$ if and only if either $W(f_1, f_2)(t) = 0$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$. However, if f_1 and f_2 are just some two functions, the Wronskian is not a perfect test of linear independence. In fact, the example of $f_1(t) = t$ and $f_2(t) = |t|$ showed that two linearly independent functions can have a Wronskian equal to zero at every point where it is defined! [On the other hand, if the Wronskian is never non-zero *or* undefined, then it can prove linear dependence.]

The statement is true regardless. Effectively, it says that two vectors in a vector space are linearly independent if and only if their coordinate vectors (with respect to a given basis) are linearly independent. This should make sense. A possible proof along these lines:

Assume $c_1(af_1 + cf_2) + c_2(bf_1 + df_2) = 0$. Then $(c_1a + c_2b)f_1 + (c_1c + c_2d)f_2 = 0$; by linear independence of $\{f_1, f_2\}$, $c_1a + c_2b = c_1c + c_2d = 0$. In other words:

$$c_1 \begin{bmatrix} a \\ c \end{bmatrix} + c_2 \begin{bmatrix} b \\ d \end{bmatrix} = 0.$$

So the set $\{af_1 + cf_2, bf_1 + df_2\}$ is linearly independent iff the only solution to this equation is $c_1 = c_2 = 0$. This is the same as requiring the two vectors $\langle a, c\rangle$ and $\rangle b, d\rangle$ to be linearly independent. That's equivalent to having $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

(4) Consider the ODE $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous functions.

If y is a solution, $y(t_0) = 0$, $y'(t_0) = 0$; show $y(t) = 0$.

Explain why $y(t) = \cos^2(t)$ cannot be a solution.

If y_1, y_2 are solutions, $W(y_1, y_2)(t_0) = 0$; show $\{y_1, y_2\}$ is a linearly dependent set.

Explain why $y_1(t) = t^2 + t + 4$, $y_2(t) = t$ cannot both be solutions.

Solution: Under these hypotheses, some helpful things are true. First, any corresponding initial value problem (to find a solution y to the equation with prescribed valued of $y(t_0)$ and $y'(t_0)$) has a unique solution. Second, Abel's theorem applies: for any solutions f_1 and f_2, $W(f_1, f_2) = C \exp\left(-\int p(t)dt\right)$, for some constant C.

Notice that $y = 0$ is a solution to the IVP $y'' + p(t)y' + q(t)y = 0$, $y(t_0) = 0$, $y'(t_0) = 0$. Therefore it is the only one, justifying the first statement. This reasoning rules out $y = \cos^2(t)$ (no matter *what* $p(t)$ and $q(t)$ are) because it would be a nonzero function satisfying $y(\pi/2) = y'(\pi/2) = 0$.

The only way to have $0 = W(y_1, y_2)(t_0) = C \exp\left(-\int p(t)dt\right)|_{t=t_0}$ is to have $C = 0$. This makes the Wronskian defined and equal to zero everywhere; this makes y_1 and y_2 linearly dependent. In particular, $W(t^2 + t + 4, t) = \begin{vmatrix} t^2 + t + 4 & t \\ 2t + 1 & 1 \end{vmatrix} = -t^2 + 4$. This is equal to zero at the point $t_0 = 2$, but the functions $t^2 + t + 4$ and t are linearly independent. Thus they cannot both be solutions to any ODE of the form we're talking about.

(5) The system $x' = Ax$ has a spiral-shaped phase portrait. Explain why A cannot be a symmetric matrix.

The system $x' = Bx$ has a solution of the form $x(t) = \eta e^{\lambda t} + \xi t e^{\lambda t}$. Find $(B - \lambda I)\eta$, $(\lambda I - B)\eta$, and (most importantly) a second, linearly independent solution.

Solution: Spirals only occur in 2×2 systems which have complex (non-real, non-imaginary) eigenvalues. Symmetric matrices, on the other hand, always have real eigenvalues.

Using algebra or your knowledge of repeated-eigenvalue situations, you should find that $(B - \lambda I)\eta = \xi$. (That's how you would usually find η in the first place.) On the other hand, $(\lambda I - B)\eta = -\xi$, so be careful not to make a sign error. The other linearly independent solution is $x(t) = \xi e^{\lambda t}$; don't forget it.

(6) If $\Phi(t)$ is a fundamental matrix for an $n \times n$ system and t_0 is some value of t, what are $\text{rk}(\Phi(t_0))$ and $\dim NS(\Phi(t_0))$?

Solution: n and zero, respectively!

(7) Fill in the blank: if $y'' = \lambda y$, $y'' - 9y =$ ______y.

Solve: $y'' - 4y = \cosh(x)$, $y'(0) = 0$, $y'(1) = m$. Does the value of m affect the number of solutions?

Solution: The blank should hold the number $1 - 9$ (i.e, -8). The fact that you can even answer the question is a reminder that y'' and $y'' - 9y$ have *the same* eigenfunctions.

$\cosh(x)$ is an eigenfunction of $y'' - 4y$: $\cosh'' - 4\cosh = -3\cosh$. So a particular solution is $y_p = \cosh(x)/(-3)$. There are other solutions, differing from this one by $y_c = c_1 \cosh(2x) + c_2 \sinh(2x)$ (the general solution to $y'' - 4y = 0$).

So we must solve for the coefficients c_1 and c_2 which make $y = c_1\cosh(2x)+c_2\sinh(2x)-\frac{1}{3}\cosh(x)$ satisfy the boundary conditions. Plugging in y, you should get $c_2 = 0$, $2c_1\sinh(2) + 2c_2\cosh(2) - (1/3)\cosh(1) = 0$. This has a unique solution, no matter what m is.

(8) $f(x) = x^2$, $0 \le x < \pi$. Set up one integral each for the coefficients of the sine series and the cosine series (each of period 2π).

Solution:

Cosine series: $c_n = \frac{2}{\pi}\int_0^\pi x^2\cos(nx)dx$, $n \ge 0$.

Sine series: $c_n = \frac{2}{\pi}\int_0^\pi x^2\sin(nx)dx$, $n \ge 1$.

(9) If $a_n = \frac{1-\cos(n\pi)}{n}$ and $b_n = \sin(n\pi/2)$, plug in $n = 2m - 1$ and simplify.

Solution: $a_n = \frac{2}{n}$, $b_n = (-1)^n$.

(10) Separate: $u_{xx} + u_{yy} + xu = 0$.

Solution: Let $u(x, y) = X(x)Y(y)$.

Plug in to get $X''(x)Y(y) + X(x)Y''(y) + xX(x)Y(y) = 0$.

Divide by $X(x)Y(y)$ to get $\frac{X''}{X} + x = -\frac{Y''}{Y}$. You should then set both sides equal to a constant, and simplify to your heart's content, to get a pair of ordinary differential equations.