

MATH 250A MIDTERM SOLUTIONS, FALL 2007

1. Let R be a c. ring containing a field K . Suppose that R is 2-dimensional as a K -vector space, but not a field. Prove that $R \cong K \times K$ or $R \cong K[t]/(t^2)$ as rings.

Solution. For $t \notin K$, the 2-element set $\{1, t\} \subset R$ is linearly independent, hence a basis. So the ring homomorphism $K[t] \rightarrow R$ (given by evaluation) is surjective, and in particular one has $t^2 = a_0 1 + a_1 t$ for some $a_0, a_1 \in K$. Thus the kernel I of this map contains $f = t^2 - a_1 t - a_0$ but no linear polynomials. This forces $I = (t^2 - a_1 t - a_0)$, giving $R \cong K[t]/(t^2 - a_1 t - a_0)$.

Next, factor $f = (t - r_1)(t - r_2)$. This is possible unless f is irreducible. Argue either that this would make f a maximal ideal (hence R a field) or that this would make f a prime ideal (hence R an entire ring, and field by homework exercise III.3). Note: The latter strategy also lets you take t to be a zero divisor above, giving $f = t(t - r)$ and simplifying the proof a bit.

Finally, consider two cases: if $r_1 = r_2$, then after a change of variables one has $R \cong K[t]/(t - 0)^2$. Otherwise, by the Chinese remainder theorem, $R \cong K[t]/(t - r_1) \times K[t]/(t - r_2) \cong K \times K$. Alternatively, set up the map $(a, b) \mapsto a\alpha(t - r_1) + b\beta(t - r_2)$, where α, β are wisely chosen scalars. \square

2. Let $G = S_7$, and let X be G -set G/H , consisting of the cosets of H , where $H = \langle (1, 2, 3, 4, 5, 6, 7) \rangle$. Show that there is a subset $Y \subset X$ (specifically, $\emptyset \subsetneq Y \subsetneq X$) such that for every $\sigma \in G$ either $\sigma(Y) = Y$ or $\sigma(Y) \cap Y = \emptyset$.

Solution. Any singleton set $Y = \{gH\}$ works. The answer was trivial but (mistakenly) allowed, so it got full credit. An “interesting” answer is to take all cosets consisting of even (or odd) permutations for Y . This makes $\sigma Y = Y$ if σ is even, and $\sigma Y = X \setminus Y$ if σ is odd. This makes sense precisely because the 7-cycle generating H is an even permutation, so a coset’s elements all have the same sign. \square

3. Define the coproduct of two objects in a category. Use the definition to prove that in the category of (not necessarily abelian) groups, the direct product $\mathbb{Z}/2 \times \mathbb{Z}/3$ is not equal to the coproduct of $\mathbb{Z}/2$ and $\mathbb{Z}/3$.

Solution. The coproduct of A, A' is an object C (and morphisms $i : A \rightarrow C$ and $i' : A' \rightarrow C$) with the following universal property: for any object D with maps $j : A \rightarrow D, j' : A' \rightarrow D, \exists! f : C \rightarrow D$ making this diagram commute:

$$\begin{array}{ccc}
 & D & \\
 j \nearrow & \uparrow f & \nwarrow j' \\
 A & \xrightarrow{i} C \xleftarrow{i'} & B
 \end{array}$$

Take $A = \mathbb{Z}/2, A' = \mathbb{Z}/3$, and $C = A \times A'$ (with the usual inclusions i, i'). The above diagram commutes iff $j = f \circ i, j' = f \circ i'$. Explicitly, this means $f(a, 0) = j(a)$ and $f(0, a') = j'(a')$, giving $j(a)j'(a') = f(a, a') = j'(a')j(a)$. If one chooses (D, j, j') not satisfying the identity $j(a)j'(a') = j'(a')j(a)$, this is impossible. For instance, take $A \rightarrow \langle (12) \rangle$ and $A' \rightarrow \langle (123) \rangle$ inside $D = S_3$. [The true coproduct is the free product $\langle x, y \mid x^3 = y^2 = 1 \rangle$, which is an infinite group.] \square

4. Give a definition of a projective module. Let K be a field and let $I = (x, y)$ be the ideal of $R = K[x, y]$ generated by the elements x and y . Show that I is not a projective R -module.

Solution 1. State any of the criteria on page 137 of Lang, other than **P3**. (As a shortcut you can even state **P2** without the “ $0 \rightarrow M' \rightarrow$ ” bit, although it’s an abuse of terminology.)

Conclude from your definition that there is a surjection f from $F = \oplus R$ to I , and a map $g : I \rightarrow F$, with $f \circ g = \text{id}_I$. (If necessary, take $F = R \oplus R$, $f(p, q) = xp + yq$.) With respect to a basis $\{e_i\}$ for F , let $g(x) = \sum p_i e_i$, $g(y) = \sum q_i e_i$. Then $xg(y) = g(xy) = yg(x)$, giving $xp_i = yq_i$ for each i . By unique factorization, then, $p_i = xp'_i$ for some $p'_i \in R$. Finally, $x = f \circ g(x) = f(x \sum p'_i e_i) = xf(\sum p'_i e_i)$. On the other hand, there is no element $r \in I$ such that $xr = x$, contradiction. \square

Solution 2. State **P1** or **P4**. Consider the sequence $R \rightarrow R/I \rightarrow 0$. Conclude from your definition that any map $I \rightarrow R/I$ has a lift $I \rightarrow R$. (Notice R/I is just K with the trivial scalar law $xk = yk = 0$.) Define a map from I to R/I , taking $p(x, y)$ to its $x^1 y^0$ -coefficient. Call it f and show no h makes this diagram commute:

$$\begin{array}{ccc} & P & \\ & \swarrow h & \downarrow f \\ R & \xrightarrow{q} & R/I \longrightarrow 0 \end{array}$$

One has $xh(y) = h(xy) = yh(x)$, so $x \mid h(x)$. Then $q \circ h(x) = 0 \neq 1 = f(x)$. \square

Solution 3. Based on material covered in class, it is also enough to show that the ideal I is not invertible. Suppose a fractional ideal J (living inside the field of fractions of R) satisfies $IJ = R$; let $f \in J$. Then $xf = p$ and $yf = q$ for some polynomials $p, q \in R$. So $yp = xq$, and by unique factorization get $x \mid p$, $y \mid q$. This implies $f \in R$, whence $J \subset R$. But then $IJ \subseteq IR = I \subsetneq R$, contradiction. \square

5. Classify the groups of order 4 and 8 up to isomorphism.

Solution. You’ve shown that the only groups of order 4 (up to isomorphism) are $\mathbb{Z}/4$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$ twice on homework, so it’s OK just to list them.

For order 8, the *answer* should be the abelian groups $\mathbb{Z}/8$, $\mathbb{Z}/4 \times \mathbb{Z}/2$, and $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$; the dihedral group D_8 , and the group Q_8 of quaternion units. The abelian case is covered by the structure theorem; it remains to show that any nonabelian G , $|G| = 8$, is isomorphic to D_8 or Q_8 (and make it clear that these two are non-isomorphic). One strategy is given below.

If any $x \in G$ has order 8, G is cyclic. On the other extreme, one might have $x^2 = e, \forall x \in G$. Then for $x, y \in G$ one has $(xy)(xy) = e = xx = x(yy)x = (xy)(yx)$, whence $xy = yx$. Either way, G is abelian.

Otherwise, fix $a \in G$ of order 4 and $b \notin \langle a \rangle$. Since $[G : \langle a \rangle] = 2$, $\langle a \rangle \triangleleft G$ and $G/\langle a \rangle \cong \mathbb{Z}/2$. Thus $b^2 \in \langle a \rangle$. Since $|b| < 8$, either $b^2 = e$ or $b^2 = a^2$.

In the former case, G is a semidirect product of $\mathbb{Z}/4$ with $\mathbb{Z}/2$. Identifying $\text{Aut}(\mathbb{Z}/4)$ with $\{\pm 1\}$, obtain two possibilities: one trivial ($\mathbb{Z}/4 \times \mathbb{Z}/2$), the other nonabelian (D_8). Note D_8 has two order-4 elements; those below have ≥ 4 .

In the latter case, either $ab \neq ba$ or G is generated by commuting elements (hence abelian). If $ab \neq ba$, then $G = \{e, a^{\pm 1}, a^2, b^{\pm 1}, ab, ba\}$ (i.e., these 8 elements are distinct). Map cautiously to $\{1, \pm i, -1, \pm j, k, -k\}$. \square