MATH 250A: LANG EXERCISE I.45 SOLUTION SKETCH

Lang is assuming a group action $G \to \mathrm{Aut}(A)$. This implies $f, g \in \mathrm{End}(A)$ and these crucial properties: $f \circ g = g \circ f = 0$ (implying $A^f \subseteq A_g$, $A^g \subseteq A_f$— and (b)); f and g commute with any $x \in G$ (i.e., $f \circ x = x \circ f$, $g \circ x = x \circ g$—used in (c)).

In (a), each $x \in G$ defines a map $\pi_x : A \xrightarrow{x} A \xrightarrow{q} A/B$. Since $GB \subset B$, $B \subseteq \ker(\pi_x)$ for each x; thus these maps factor as $A \xrightarrow{q} A/B \to A/B$. In other words, the G-action $x[a] = [a]$ on A/B is well-defined.

Consider two cases for (c). If A has a G-invariant subgroup $B \notin \{0, A\}$, then under a suitable induction hypothesis we can assume $q(B) = q(A/B) = 1$. Then $q(A) = 1$ by (b). If not, then in particular all four of A^f, A_f, A^g, and A_g must be either 0 or A. (These subgroups are G-invariant because f and g commute with G.) Knowing that $A/A_f \cong A^f$, $A/A_g \cong A^g$, $A^g \subseteq A_f$, and $A^f \subseteq A_g$; we see there are only three possibilities for (A_f, A^f, A_g, A^g). In each case we verify that $q(A) = 1$.

Proving (b) is more difficult. We should somehow prove the equality of *cardinals* $[B_f : A^g][A_g : A^f][(A/B)_f : (A/B)^g] = [B_g : A^f][A_f : A^g][(A/B)_g : (A/B)^f]$. Otherwise, our result might depend on some indices being finite.

These groups will show up sooner or later, so let's name them: $Q := A/B$; for $X = A, B, Q$, $H_f^X := X_f/X^g$ and $H_g^X := X_g/X^f$. We want $|H_f^B||H_g^A||H_f^Q| = |H_g^B||H_f^A||H_g^Q|$. I'll present one possible proof. It may look poorly motivated now, but it shows a preview of an important construction in algebraic topology.

The ingredients of this proof are the exact sequence $0 \to B \to A \to Q \to 0$, and the maps f, g which are defined compatibly on each group and satisfy $f \circ g = g \circ f = 0$. The situation is summarized in commutative diagram 1. I will construct an exact sequence, illustrated in diagram 2. The map from each H will then have image I and kernel K, with $H/K \cong I$, and exactness makes these right-hand sides equal:

$$|H_f^B||H_g^A||H_f^Q| = |K_f^B||I_f^B\,||K_g^A||I_g^A\,||K_f^Q||I_f^Q\,|$$
$$|H_g^B||H_f^A||H_g^Q| = |I_g^Q\,||K_f^A||I_g^B\,||K_g^Q||I_f^A\,||K_g^B|$$

It remains to construct these maps and verify the sequence is exact. Since the roles of f and g are interchangeable, I construct three maps and compute their images and kernels.

FIGURE 1. Commutative diagram describing the setup for 45(b)

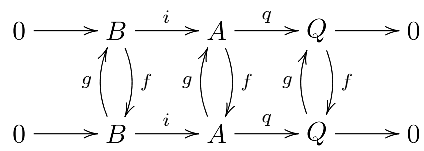
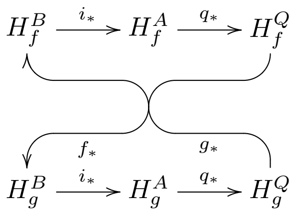


FIGURE 2. Resulting exact sequence



Warning: $q^{-1}(Q^f)$ is just $A^f + B$ (the subgroup of elements which are equivalent in A/B to a member of A^f), but $q^{-1}(Q_f) = f^{-1}(B)$ need not equal $A_f + B$.

i_*:
- Mapping $B_f/B^g \to A_f/A^g$.
- Rule: $[x] \mapsto [x]\ \forall x \in B_f$.
- Well-defined: Yes, $B^g \subseteq A^g$.
- Kernel: $(A^g \cap B_f)/B^g$.
- Image: $B_f/(A^g \cap B_f)$.

q_*:
- Mapping $A_f/A^g \to Q_f/Q^g$.
- Rule: $[x] \mapsto [x]\ \forall x \in A_f$.
- Well-defined: Yes, $q(B^g) \subseteq Q^g$.
- Kernel: $q^{-1}(Q^g)/A^g = (A^g + B_f)/A^g$.
- Image: $A_f/q^{-1}(Q^g) = A_f/(A^g + B_f)$.

f_*:
- Mapping $Q_f/Q^g \to B_g/B^f$.
- Rule: $[x] \mapsto [f(x)]\ \forall x \in q^{-1}(Q_f)$.
- Well-defined: Yes:
  1. $x \in q^{-1}(Q_f) = f^{-1}(B) \implies f(x) \in B \implies f(x) \in B_g$;
  2. $x \in q^{-1}(Q^g) = A^g + B \implies f(x) \in 0 + B^f$.
- Kernel: $f^{-1}(B^f)/q^{-1}(Q^g) = (A^g + A_f)/(A^g + B_f) = A_f/(A^g + B_f)$.
- Image: $f(f^{-1}(B))/B^f = (A^f \cap B)/B^f = (A^f \cap B_g)/B^f$.

Exactness may be verified from here (keeping in mind isomorphisms of the form $(X + Y)/X \cong Y/(X \cap Y)$).