

MATH 250A HOMEWORK SET #5

Problem 1. Define $\mathbb{Z}_{(p)} := \{\frac{a}{b} \in \mathbb{Q} : b \notin p\mathbb{Z}\}$ as an additive subgroup of \mathbb{Q} .

Verify that there is an exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \bigoplus_{p \text{ prime}} \mathbb{Q}/\mathbb{Z}_{(p)} \rightarrow 0$.

Problem 2. Let A be an abelian group. If Q is a subgroup of A isomorphic to \mathbb{Q} or $\mathbb{Q}/\mathbb{Z}_{(p)}$, show that Q is a direct summand of A (i.e., $A = Q \oplus R$ for suitable $R < A$).

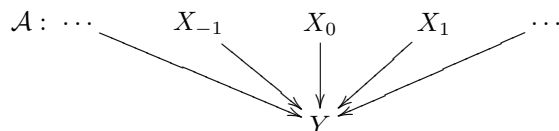
Hint: Use the divisibility of Q . An abelian group Q is called *divisible* if one can find $q' \in Q$ satisfying $q = n \cdot q' := \underbrace{q' + \dots + q'}_n$ for any $q \in Q$, $n \in \mathbb{Z}_{>0}$.

Note: An abelian group Q is called *injective* if it is a summand of any abelian group in which it is a subgroup. Observe that this property of abelian groups is dual to freeness. Thus \mathbb{Q} and $\mathbb{Q}/\mathbb{Z}_{(p)}$, in particular, are injective. The exact sequence from problem #1 is a so-called *injective resolution* of \mathbb{Z} .

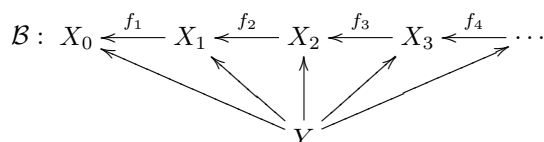
Problem 3. Show that abelianization (taking G to its quotient by its commutator subgroup) is a functor from **Grp** to **Ab**.

Problem 4. Which of these categories have initial objects? Final objects? Which are equivalent to their opposite categories?

(a)



(b)



(c) \mathcal{C} , the category of finitely generated abelian groups

(d) \mathcal{D} , the category of torsion abelian groups of finite exponent

Clarification: The morphisms of the categories \mathcal{A} and \mathcal{B} are those drawn in the corresponding *commutative* diagrams, along with the identity morphisms and compositions of the morphisms which have been drawn. The morphisms of the categories \mathcal{C} and \mathcal{D} are simply group homomorphisms.