Math 128A (Blanchard) Su09: **Quiz 3**, Jul 21, 2009 Solutions

PROBLEM ONE [4 PTS.]

Find the composite trapezoidal estimates $R_{1,1}$ and $R_{2,1}$ to $\int_0^2 x^4\,dx$. Compute the Romberg estimate $R_{1,2} = \frac{4R_{2,1}-R_{1,1}}{3}$. Which estimate came closest to the exact value, 6.4?

Solution: $R_{1,1} = \frac{2}{2}(0^4 + 2^4) = 16$. $R_{2,1} = \frac{1}{2}(0^4 + 2\cdot 1^4 + 2^4) = 9$. So $R_{1,2} = 20/3 \approx 6.7$. $R_{1,2}$ is easily the closest.

PROBLEM TWO [4 PTS.]

Find a C^2 function $r(x)$ and a number s, such that $\int_0^1 \frac{\cos(x)}{x^{1/2}}\,dx = \int_0^1 r(x)\,dx + s$.
(Pretend that you want to use a composite trapezoidal estimate.)

Solution: Write $\cos(x) = 1 - x^2/2 + R(x)$. Then $R(x)/x^{1/2}$ is C^2 (since $R(x) = O(x^4)$ for $x \approx 0$), so we can take

$$\int_0^1 \frac{R(x)}{x^{1/2}}\,dx + \int_0^1 \frac{1-x^2/2}{x^{1/2}}\,dx = \int_0^1 \frac{R(x)}{x^{1/2}}\,dx + \int_0^1 (x^{-1/2} - x^{3/2}/2)\,dx$$
$$= \int_0^1 \frac{R(x)}{x^{1/2}}\,dx + [2x^{1/2} - \frac{1}{5}x^{5/2}]_0^1\,dx = \int_0^1 \frac{\cos(x)-(1-x^2/2)}{x^{1/2}}\,dx + \frac{9}{5}$$

(One student noticed an alternate solution: after the substitution $x = t^2$, the integral becomes $\int_0^1 2\cos(t^2)\,dt$.)

PROBLEM THREE [4 PTS.]

Find a quadratic polynomial q satisfying the following orthogonality relations on $[0,1]$ (not $[-1,1]$): $\int_0^1 x^0 q(x)\,dx = 0, \quad \int_0^1 x^1 q(x)\,dx = 0$.

Solution 1: Define $\langle f,g\rangle$ as $\int_0^1 f(x)g(x)\,dx$. Apply the Gram-Schmidt process to $\{1, x, x^2\}$.

Start with 1.

Compute $x - \frac{\langle x,1\rangle}{\langle 1,1\rangle}1 = x - \frac{1/2}{1} = x - 1/2$.

Compute $x^2 - \frac{\langle x^2,1\rangle}{\langle 1,1\rangle}1 - \frac{\langle x^2, x-\frac{1}{2}\rangle}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle}(x-\frac{1}{2}) = x^2 - \frac{1/3}{1} - \frac{1/4-1/6}{1/3-1/2+1/4}(x-\frac{1}{2}) = x^2 - \frac{1}{3} - (x-\frac{1}{2}) = x^2 - x + \frac{1}{6}$.

Solution 2: Let $q(x) = ax^2 + bx + c$. Then we need $0 = \int_0^1 (ax^2+bx+c)\,dx = a/3 + b/2 + c$ and $0 = \int_0^1 (ax^3 + bx^2 + cx)\,dx = a/4 + b/3 + c/2$. Eliminate c to get $-a/6 - b/6 = 0$. So $b = -a$, and then $c = a/6$. This shows that any multiple of $x^2 - x + \frac{1}{6}$ works.