Math 128A (Blanchard) Su09: **Quiz 2**, Jul 14, 2009 Solutions

PROBLEM ONE [4 pts.]

Let $M = f'(x_0)$. Consider the approximation $M \approx N(h) = \frac{1}{h}[-f(x_0 - h) + f(x_0)]$.
Show that $N(h) = M + K_1 h + O(h^2)$ (for some constant K_1) when f is a smooth function.

Solution: Expand f in a Taylor series centered at x_0, with at least this many terms:

$$f(x_0 - h) = f(x_0) + f'(x_0)(-h) + f''(x_0)(-h)^2/2! + O(h^3)$$

Plug this into the formula for $N(h)$:

$$N(h) = f'(x_0) - f''(x_0)/2 \cdot h + O(h^2)$$

This is an expansion of the form $M + K_1 h + O(h^2)$.

PROBLEM TWO [4 pts.]

Starting with the above formula, use extrapolation to derive an $O(h^2)$ formula for $f'(x_0)$.

Solution: Extrapolation means combining $N(h)$ and $N(2h)$ to form a new estimate for M, with the lowest-order error term cancelled. [Using $N(h)$ and $N(h/2)$ is okay too.] We need to combine

$$\begin{aligned} N(h) &= M \qquad +K_1 h + O(h^2) \\ N(2h) &= M \qquad +2K_1 h + O(h^2) \end{aligned}$$

in a way that keeps the "M" term and cancels the "$K_1 h$" term. Multiply the top equation by 2, and subtract the bottom: $2N(h) - N(2h) = M + 0 + O(h^2)$. So our $O(h^2)$ estimate is $\boxed{2N(h) - N(2h)}$. If we plug in $N(h) = \frac{1}{h}[-f(x_0 - h) + f(x_0)]$, we get an answer of $\boxed{\frac{1}{2h}[f(x_0 - 2h) - 4f(x_0 - h) + 3f(x_0)]}$.

PROBLEM THREE [4 pts.]

Find the degree of precision of the following quadrature rule (on the interval $[-3, 3]$):

$$\int_{-3}^{3} f(x)\,dx \approx \sqrt{3}f(-\sqrt{3}) + \sqrt{3}f(+\sqrt{3})$$

Solution: Plug in $f(x) = x^n$, $n = 0, 1, 2, \ldots$ to test whether $\int_{-3}^{3} f(x)\,dx = \sqrt{3}f(-\sqrt{3}) + \sqrt{3}f(+\sqrt{3})$ (exactly) until the equation breaks. The degree of precision is the largest n that worked.

This quadrature rule features a typo which makes this process very short: When $f(x) = x^0 = 1$, we have $\int_{-3}^{3} f(x) = 6 \neq \sqrt{3}f(-\sqrt{3}) + \sqrt{3}f(+\sqrt{3}) = 2\sqrt{3}$.

So the degree of precision is *not even zero*. (An answer of zero for the same reason is acceptable.)