Math 128A (Blanchard) Su09: **Quiz 1**, July 1, 2009 Solutions

PROBLEM ONE [4 PTS.]

The function $f(x) = x^3 - 7$ has one real root. Find an interval which brackets it. Starting with this interval, how many iterations of the bisection method are needed to ensure a relative error of less than 0.25 in approximating the solution?

Solution: The interval $[1, 2]$ brackets a root of $f(x)$, because $f(1) = -6 < 0$ and $f(2) = 1 > 0$. After n iterations of the bisection method, the *absolute* error is $|p_n - p| \leq (b-a)/2^n = 1/2^n$. Since the solution p (to $f(x) = 0$) is greater than 1, the relative error $|p_n - p|/|p|$ is even smaller. So we are guaranteed a relative error of less than 0.25 if $1/2^n \leq 0.25$, which happens after 2 steps.

(Note: details may vary depending on the interval chosen, the way of estimating relative error, or even which step is considered the "first" iteration of the bisection method.)

PROBLEM TWO [4 PTS.]

Find the rate of convergence of $f(h) = 1 - \cos(h)$ as $h \to 0$. Express your answer in the form $f(h) = O(h^r)$, where r is a number.

Solution: As $h \to 0$, we have $\cos(h) = 1 - \frac{h^2}{2!} + \ldots$, so $f(h) = \frac{h^2}{2!} - \ldots$. Looking at the lowest-degree term of this series, we see $f(h) = O(h^2)$.

PROBLEM THREE [4 PTS.]

Let $g(x) = \frac{2}{3}(x + x^{-2})$. Show that fixed-point iteration, $p_{n+1} = g(p_n)$, converges for any $p_0 \in [1, 2]$.

Solution: We must show, for all $x \in [1, 2]$, that $g(x) \in [1, 2]$ and that $|g'(x)| \leq k$ for some $k < 1$.

(Note: this is the same as saying that $|g'(x)| < 1$ for all $x \in [1, 2]$, **if** $g'(x)$ is continuous. In this case, it is, but a correct proof should either say so or find a k.)

Since $g'(x) = \frac{2}{3}(1 - 2x^{-3})$ is an increasing function, $g'(x) \in [g'(1), g'(2)] = [\frac{2}{3}(1-2/1), \frac{2}{3}(1-2/8)] = [-\frac{2}{3}, \frac{1}{2}]$. So $|g'(x)| \leq k = 2/3$.

The min and max values of $g(x)$ must occur at endpoints or critical points. $g(1) = \frac{2}{3}(1 + 1/1) = \frac{4}{3}$, and $g(2) = \frac{2}{3}(2 + 1/4) = \frac{3}{2}$. Finally, the only critical point of g is $x = \sqrt[3]{2}$, at which $g(x) = \frac{2}{3}(\sqrt[3]{2} + \sqrt[3]{2}/2) = \sqrt[3]{2}$. All of these numbers are in $[1, 2]$, so $g(x) \geq 1$ and $g(x) \leq 2$ for all $x \in [1, 2]$.