MATH 128A-2 SUMMER 2009: MIDTERM 2 PRACTICE PROBLEMS

Questions

(1) Fill in the table as accurately as possible using two-point formulas (including the centered difference formula) for f':

$f(-1) = 6$	$f(0) = 1$	$f(1) = 2$
$f'(-1) =$	$f'(0) =$	$f'(1) =$

(2) Do you remember the definitions of the following quadrature methods? Newton-Cotes, composite trapezoidal, Romberg, adaptive Simpson, Gaussian.

(3) Find coefficients a, b, and c so that the following quadrature rule has the best possible degree of precision:

$$\int_0^1 f(x)\,dx \approx af(-1) + bf(0) + cf(+1).$$

What is the resulting rule's degree of precision?

(4) Use an easy substitution to transform the improper integral $\int_1^\infty \frac{1}{x^2+1}\,dx$ into an integral of the form $\int_0^1 f(u)\,du$. Transform the result into an integral which can be estimated with the following quadrature rule:

$$\int_{-1}^1 g(t)\,dt \approx f(-1) + f(+1).$$

(5) Use Euler's method with $h = 1$ to estimate the solution to $y' = 2t + y/t$, $1 \le t \le 3$, $y(1) = 1$.

(6) Find a Lipschitz constant L for the above problem.

(7) Find y'' in terms of t and (if necessary) y, and compute an upper bound M on $|y''(t)|$.

(8) Find a bound on the local truncation error τ when using Euler's method to solve this problem.

(9) Find a bound for the absolute error in estimating $y(3)$, using the error bound $\frac{hM}{2L}(e^{L(t_i - t_0)} - 1)$.

(10) Write down a step (w_{i+1} as a function of t_i, w_i, h) of the second-order Taylor method for solving the IVP from problem 5.

(11) Based on your answers to problem 5, estimate $y(1.5)$ based on $y(1)$, $y'(1)$, $y(2)$, and $y'(2)$.

(12) Express the following Runge-Kutta method in one line (i.e., as a single formula for w_{i+1}):

$$k_1 = hf(t_i, w_i)$$
$$k_2 = hf(t_i + h, w_i + k_1)$$
$$w_{i+1} = w_i + k2$$

(13) What is the order of the Runge-Kutta method in problem 12?

(14) (Skip if confused.) Consider Romberg integration with the following notation:

- $h_1 = h, h_2 = h, \ldots, h_i = h/2^{i-1}, \ldots;$
- $R_{i,1}$ is the composite trapezoidal estimate of $\int_a^b f(x)\,dx$ with step-size h_i;
- $R_{i+1,j+1}$ is the value of $\int_a^b f(x)\,dx$ extrapolated from $R_{i,j}$ and $R_{i+1,j}$, assuming $R_{i,j}$ has error $O(h_i^{2j})$.

Show that $R_{i+1,j+1}$ is equal to $R_{i+1,j} + \frac{h_i^2}{h_{i-j}^2 - h_i^2}(R_{i+1,j} - R_{i,j})$.

Date: Monday 7/26.

Answers

(1)

$f(-1) = 6$	$f(0) = 1$	$f(1) = 2$
$f'(-1) \approx -5$	$f'(0) \approx -2$	$f'(1) \approx 1$

(2) Yes.

(3) $(a, b, c) = \frac{1}{12}(-1, 8, 5)$; degree is 2.

(4) The substitution $x = 1/u$ produces $\int_0^1 \frac{1}{1+u^2}\, du$. Then $u = (t+1)/2$ produces $\int_{-1}^1 \frac{1}{2+(t+1)^2/2}\, dt \approx 0.75$.

(5) $y(2) \approx 4$, $y(3) \approx 10$.

(6) $\left|\frac{\partial}{\partial y}(2t + y/t)\right| = |1/t| \leq 1$.

(7) $y''(t) = 2 + y'/t - y/t^2 = 2 + (2t + y/t)/t - y/t^2 = 4$. We can take $M = 4$.

(8) Euler's method generally has $\tau_{i+1} = \frac{y_{i+1}-w_{i+1}}{h} = \frac{y(t_i+h)-[y(t_i)+hy'(t_i)]}{h} = \frac{y''(\xi)h^2/2}{h} = \frac{h}{2}y''(\xi)$. Thus we have $\tau = 2h = 2$, at every step.

(9) $2(e^2 - 1)$. (Numerically, this is something smaller than $2(3^2 - 1) = 16$.) The actual error is much better.

(10) $w_{i+1} = w_i + h[2t_i + w_i/t_i] + \frac{h^2}{2}[4]$.

(11) The divided difference table below (see section 3.3) says $y(t) \approx 1+3(t-1)+0(t-1)^2+3(t-1)^2(t-2)$:

$$
\begin{array}{r|llll}
t = 1 & 1 & & & \\
 & & 3 & & \\
t = 1 & 1 & & 0 & \\
 & & 3 & & 3 \\
t = 2 & 4 & & 3 & \\
 & & 6 & & \\
t = 2 & 4 & & &
\end{array}
$$

Plugging in $t = 1.5$ gives $y(1.5) \approx 2.125$.

(12) $w_{i+1} = w_i + hf(t_i + h, w_i + hf(t_i, w_i))$.

(13) The method is first-order (i.e., $\tau_{i+1} = O(h)$). In particular, the $O(h^2)$ Taylor coefficient to w_{i+1} is twice that of $y(t_i + h)$.

(14) This is algebra, but the important step is to realize $R_{i+1,j+1} = R_{i+1,j} - \frac{h_{i+1}^{2j}}{h_{i+1}^{2j}-h_i^{2j}}(R_{i+1,j} - R_{i,j})$.