

UCB MATH 128A-002, SUMMER 2009: MIDTERM 2 SOLUTIONS

1. Fill in the blanks below as accurately as possible.

$x$	0	0.5	+1
$f(x)$	1		3
$f'(x)$			
$\int_0^1 f(x) dx \approx$	□		

**Solution:** All answers end up being 2.

The easy way: the best we can do is assume  $f(x)$  is a line, so let  $f(x) = 1 + 2x$  and fill the boxes.

Harder method:

Estimate  $f(0.5)$  by linear interpolation (i.e., averaging):  $f(0.5) \approx \frac{1+3}{2} = \boxed{2}$ .

Estimate  $f'(0)$ ,  $f'(0.5)$ , and  $f'(1)$  with the forward-, centered-, and backward-difference formulas (with  $h = 0.5$  in the case of  $f'(0.5)$ ). They all end up calculating the same thing: the slope  $\frac{3-1}{1} = \boxed{2}$ .

Estimate  $\int_0^1 f(x) dx$  with the trapezoidal rule:  $\frac{1}{2}(1+3) = \boxed{2}$ .

2. Let  $T(h)$  denote the composite trapezoidal estimate of  $\int_0^1 f(x) dx$ , with step size  $h$ .

You have measured  $T(0.2) = 3.0$  and  $T(0.1) = 3.3$ . Assume that the composite trapezoidal rule makes error  $\int_0^1 f(x) dx - T(h) \approx Kh^2$  (for a non-zero constant  $K$ ).

(a) Estimate the exact value of  $\int_0^1 f(x) dx$  as accurately as possible.

(b) Estimate the absolute error  $\left| \int_0^1 f(x) dx - T(0.1) \right|$  as accurately as possible.

(c) Estimate how small  $h$  must be for  $T(h)$  to approximate  $\int_0^1 f(x) dx$  to within  $10^{-3}$

**Solution:**  $3.0 - 3.3 \approx K(0.2)^2 - K(0.1)^2 = 0.03K$ , so  $K \approx \frac{-0.3}{0.03} = -10$ .

(a)  $\int_0^1 f(x) dx \approx T(0.1) - K(0.1)^2 = \boxed{3.4}$ .

(b)  $\left| \int_0^1 f(x) dx - T(0.1) \right| \approx |K(0.1)^2| = \boxed{0.1}$ .

(c)  $|Kh^2| < 10^{-3} \iff h^2 < 10^{-4} \iff \boxed{h < 10^{-2}}$ .

3. Find the largest possible degree of precision for the following quadrature rule, and the corresponding values of  $a$  and  $b$ :

$$\int_{-1}^1 f(x) dx \approx af\left(-\frac{5}{6}\right) + bf(0) + af\left(+\frac{5}{6}\right).$$

**Solution:**

$$\text{Degree of precision } \geq 0, \text{ if } \quad 2a + b = \int_{-1}^1 x^0 dx = 2 \quad (*)$$

$$\text{Degree of precision } \geq 1, \text{ if } \quad 0 = \int_{-1}^1 x^1 dx = 0 \quad (\text{trivial})$$

$$\text{Degree of precision } \geq 2, \text{ if } \quad 2\left(\frac{5}{6}\right)^2 a = \int_{-1}^1 x^2 dx = \frac{2}{3} \quad (**)$$

By (\*\*),  $\boxed{a = \frac{12}{25}}$ . By (\*),  $\boxed{b = \frac{26}{25}}$ . We may now check:

Degree of precision  $\geq 3$ , if  $0 = \int_{-1}^1 x^3 dx = 0$  (true)

Degree of precision  $\geq 4$ , if  $2 \left(\frac{5}{6}\right)^4 \frac{12}{25} = \int_{-1}^1 x^4 dx = \frac{2}{5}$  (false)

Therefore, the degree of precision is  $\boxed{3}$ .

4. Consider the initial value problem  $y'(t) = t^2 + 2y$  ( $t \in [0, 4]$ ) with  $y(0) = 0$ .

(a) Fill in the blanks using Euler's method: 

$t$	0.0	1.0	2.0	3.0	4.0
$y(t)$	0.0				

(b) Find a Lipschitz constant  $L$  for the problem.

(c) Find a formula for  $y''$  in terms of  $t$  and  $y$ .

(d) Redo (a) with a Taylor method, order 2.

**Solution:**

(a) The iteration is  $w_{i+1} = w_i + (1)(t_i^2 + 2w_i) = t_i^2 + 3w_i$ . Result:

$t$	0.0	1.0	2.0	3.0	4.0
$y(t)$	0	0	1	7	30

(b)  $\left| \frac{\partial}{\partial y}(t^2 + 2y) \right| = 2$  is bounded above by  $\boxed{L = 2}$ .

(c)  $y'' = \frac{d}{dt}(t^2 + 2y) = 2t + 2y' = 2t + 2(t^2 + 2y) = \boxed{2(t^2 + t + 2y)}$ .

(d) The iteration is  $w_{i+1} = w_i + (1)(t_i^2 + 2w_i) + \frac{(1)^2}{2} 2(t_i^2 + t + 2y) = 2t_i^2 + t_i + 5w_i$ . Result:

$t$	0.0	1.0	2.0	3.0	4.0
$y(t)$	0	0	3	25	146

5. Find the order of the local truncation error  $\tau_{i+1}$  in this rule for solving  $y' = f(t, y)$ :

$$w_{i+1} = w_i + hf(t_i + h, w_i).$$

**Solution:** Compute Taylor terms...

... for  $y(t_i + h)$  (using Taylor series for  $y(t)$  centered at  $t_i$ ):

$$y(t) = y(t) \\ \rightarrow w_i$$

$$y'(t) = f(t, y(t)) \\ \rightarrow f(t_i, w_i)h$$

$$y''(t) = f_t(t, y(t)) + f_y(t, y(t))f(t, y(t)) \\ \rightarrow [f_t(t_i, w_i) + f_y(t_i, w_i)f(t_i, w_i)] \frac{h^2}{2}$$

... for  $w_{i+1}$  (as a function of  $h$ ):

$$w_{i+1} = w_i + hf(t_i + h, w_i) \\ \rightarrow w_i$$

$$w'_{i+1} = f(t_i + h, w_i) + hf_t(t_i + h, w_i) \\ \rightarrow f(t_i, w_i)h$$

$$w''_{i+1} = 2f_t(t_i + h, w_i) + hf_{tt}(t_i + h, w_i) \\ \rightarrow [2f_t(t_i, w_i)] \frac{h^2}{2}$$

Since the degree-2 Taylor terms don't match,  $y_{i+1} - w_{i+1} = O(h^2)$ . Thus  $\tau_{i+1} = \boxed{O(h^2)}$ .