UCB MATH 128A-002, SUMMER 2009: MIDTERM 1 SOLUTIONS

1. (a) Use a third-order Taylor polynomial to estimate $\cos(0.1)$.
   Express the result as a decimal.
   (b) Find a bound for the error in this approximation. Express the result as a fraction.
   Solution:
   (a) The third-order Taylor polynomial (centered at $x = 0$) for $\cos(x)$ is $1 - \frac{x^2}{2!}$.
   (The coefficient to x^3 is zero.) Plugging in $x = 0.1$ yields $1 - .005 = \boxed{.995}$.
   (b) The remainder is $\frac{\cos^{(4)}(\xi)}{4!}(0.1)^4$, for $\xi \in [0, 0.1]$. The maximum value of $|\cos(\xi)|$ is 1, so a bound for the error is given by $\frac{1}{4!}(0.1)^4 = 1/(4! \cdot 10^4) = \boxed{1/240000}$.

2. Show that the interval $[0, 1]$ contains a root of $f(x) = 2x^3 - 4x + 1$.
   Solution: $f(x)$ is continuous, $f(0) = 1 > 0$ and $f(x) = -1 < 0$.
   Therefore, $f(x) = 0$ for some $x \in [0, 1]$ by the intermediate value theorem.

3. Find a quadratic polynomial $P(x)$ such that $P(0) = 0$, $P(1) = 3$, and $P(2) = 4$.
   Solution: The answer, $\boxed{P(x) = 4x - x^2}$, can be found in several different ways.

4. Let $f(x) = 2^x$. Compute the following divided differences:
   Solution:

(a) $f[0, 1] = \frac{2^1 - 2^0}{1 - 0} = \boxed{1}$ (b) $f[1, 2] = \frac{2^2 - 2^1}{2 - 1} = \boxed{2}$ (c) $f[0, 1, 2] = \frac{2 - 1}{2 - 0} = \boxed{1/2}$

5. (a) Show that the interval $[0, 1]$ contains a fixed point of $g(x) = (1/2)^x$.
   (b) Show that fixed-point iteration $p_{n+1} = (1/2)^{p_n}$ converges for any $p_0 \in [0, 1]$.
   Hint: $\ln(1/2) \approx -0.69$.
   (c) [Bonus] Is this convergence linear?
   Solution:
   (a) $g(x)$ is continuous on $[0, 1]$.
   Since $g(x)$ is decreasing on $[0, 1]$, so $g(x) \in [g(1), g(0)] = [1/2, 1] \subset [0, 1]$ for $x \in [0, 1]$.
   Therefore, $g(x)$ has a fixed point on $[0, 1]$.
   (b) As shown in (a), $g(x) \in [0, 1]$ for $x \in [0, 1]$.
   For $x \in [0, 1]$, $g'(x) = \ln(1/2) \cdot (1/2)^x$ exists, and $|g'(x)| \leq |\ln(1/2) \cdot (1/2)^0| < 1$.
   Therefore, fixed-point iteration converges for any $p_0 \in [0, 1]$.
   (c) Let p be the fixed point of $g(x)$.
   Then $g'(p) \neq 0$, since $\ln(1/2) \cdot (1/2)^p$ is non-zero for any value of p.
   Therefore, the sequence $\{p_n\}$ converges only linearly to p (for any $p_0 \in [0, 1]$).

Date: Tuesday 7/7.