MATH 128A, SUMMER 2009: HOMEWORK 7 SOLUTIONS

(1) Let's use the notation w_{AB3} for the Adams-Bashforth(3) predictor's estimate of y_{i+1}, w_{AM2} for the Adams-Moulton(2) corrector's estimate of y_{i+1}, and τ_{AB3} and τ_{AM2} for their respective truncation errors.

From the textbook, we have the truncation error formulas

$$\begin{aligned} \tau_{\text{AB3}} &= \frac{y_{i+1} - w_{\text{AB3}}}{h} = && \frac{3}{8} y^{(4)}(\xi_{\text{AB3}}) h^3, && \xi_{\text{AB3}} \in [t_{i-2}, t_{i+1}] \\ \tau_{\text{AM2}} &= \frac{y_{i+1} - w_{\text{AM2}}}{h} \approx && \frac{-1}{24} y^{(4)}(\xi_{\text{AM2}}) h^3, && \xi_{\text{AM2}} \in [t_{i-1}, t_{i+1}] \end{aligned}$$

(Technically speaking, the equation in the second line is approximate: it gives the predictor-corrector method's truncation error as that of the corrector step. The difference ends up being $O(h^4)$, and may be ignored.)

Proceed by assuming that $y^{(4)}(\xi)$ is approximately constant on the relevant interval. When we subtract the equations, the y_{i+1} terms cancel and we're left with

$$\frac{w_{\text{AM2}} - w_{\text{AB3}}}{h} \approx \frac{10}{24} y^{(4)}(?) h^3$$

The left-hand side is -1, so $y^{(4)}(?)h^3 \approx -24/10$. Plugging back into the formula for τ_{AM2} gives $\tau_{\text{AM2}} \approx 0.1$.

(2) If $u_1 = y$, $u_2 = y'$, this system is equivalent: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} u_2 \\ -2u_1 - 2u_2 \end{bmatrix}$, $0 \le t \le 9.6$, $\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(3) Example solution with MATLAB:

```
>> for n=[16 12 8]
     [t,w]=euler( @(t,u)[u(2), -2*u(1)-2*u(2)], 0, 9.6, [1,-1], n);
     abs(w(end,1) - exp(-9.6)*cos(9.6))
   end

ans =

    0.0053


ans =

    0.0968


ans =

    1.1852
```

(4) (a) The method's characteristic polynomial (in the sense of §5.10) is $Q(\lambda) = \lambda^2 - 1$. This has roots $+1$ and -1, so by definition the method is weakly stable.

(b) The values of λ relevant to the hint are exactly these roots: $w_i = (1)^i = 1$ and $w_i = (-1)^i$ are both solutions to this recurrence. Hence the general solution is $w_i = c_1 + c_2(-1)^i$.

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(c) By part (b), in the small-h limit $\{w_i\}$ looks like an accurate (constant) approximation of y_i plus a deviation which changes sign on each step. The graph's jaggedness simply shows that such behavior is already visible when $h = 0.1$.

(d) Smoothing should preserve the trend in the $\{w_i\}$ because each w_i is averaged with its neighbors. [It also helps that the weights on w_{i-1} and w_{i+1} are equal.] On the other hand, it should roughly cancel the oscillations noticed in (b). For instance, $\frac{1}{4}((-1)^{i-1} + 2(-1)^i + (-1)^{i+1}) = 0$. The graph confirms this.

(5) (a) If we feed Euler's method $f(t, y) = \lambda y$, the steps will be $w_{i+1} = w_i + h\lambda w_i = (1 + h\lambda)w_i$. This decays to zero iff $|1 + h\lambda| < 1$.

(b) If we feed the backward Euler method $f(t, y) = \lambda y$, the steps will be $w_{i+1} = w_i + h\lambda w_{i+1}$. The solution to this equation is $w_{i+1} = \frac{1}{1-h\lambda} w_i$. This decays to zero iff $|1 - h\lambda|^{-1} < 1$, iff $|1 - h\lambda| > 1$. (In particular, this decays to zero when $h\lambda$ lies on the left side of the imaginary axis: the backward Euler method is *A-stable*.)

(c) The system from problem 2 has characteristic equation $\lambda^2 + 2\lambda + 2 = 0$. Thus its eigenvalues are $\lambda = -1 \pm i$. [This means that after a linear change of coordinates the system looks like $x_1' = (-1 - i)x_1$, $x_2' = (-1 + i)x_2$.] Therefore, we expect (roughly) that method can handle this problem without stability issues as long as the numbers $(-1 \pm i)h$ are in the stability region.

For the backward Euler method, this doesn't restrict the values of h at all.

For Euler's method,this does restrict h: We need $1 > |1 + (-1 + i)h| = |(1 - h) + (h)i|$. Squaring both sides, we get $1 > (1 - h)^2 + h^2 = 2h^2 - 2h + 1$. So we need $h < 1$.