MATH 128A, SUMMER 2009: HOMEWORK 5 SOLUTIONS

(1) Given a quadrature rule of the form $\int_a^b f(x)\,dx \approx \sum_{i=0}^n c_i f(x_i)$, let $f(x)$ be $[(x-x_0)\dots(x-x_n)]^2$, a polynomial of degree $2(n+1)$.

Since $f(x)$ is a continuous function, positive except at its $n+1$ roots, the exact value of $\int_a^b f(x)\,dx$ must be positive.

Since $f(x)$ is zero when $x = x_i$, the estimate for $\int_a^b f(x)\,dx$ given by the quadrature rule is zero.

Since the quadrature can't integrate $f(x)$ (a polynomial of degree $2(n+1)$) exactly, its degree of precision must be less than $2(n+1)$.

(2) Use the inner product $\langle f, g\rangle = \int_0^\infty e^{-x} f(x) g(x)\,dx$ for functions on $[0,\infty)$.

Let $f(x)$ be a polynomial of degree $2n-1$ (or less). use polynomial division to express $f(x)$ in the form $q(x)L_n(x)+r(x)$, where $q(x)$ (the quotient) is a polynomial of degree at most $(2n-1)-n = n-1$, and $r(x)$ (the remainder) is a polynomial of degree less than n.

The quadrature rule estimates $\int_0^\infty e^{-x} q(x) L_n(x)\,dx$ exactly due to the choice of nodes: Since $\deg q(x) \le n-1$, $q(x)$ is orthogonal to $L_n(x)$: the value of $\int_0^\infty e^{-x} q(x) L_n(x)\,dx$ is zero. Since $q(x)L_n(x) = 0$ when $x = x_i$, $\sum_{i=1}^n c_{n,i} f(x_i)$ is also zero.

The quadrature rule estimates $\int_0^\infty r(x)\,dx$ exactly due to the choice of coefficients: Let $P_{n,i}(x) = \prod_{\substack{j=1\\ j\ne i}}^n \frac{x-x_j}{x_i-x_j}$ be the Legendre interpolating polynomial (with $P_{n,i}(x_i) = 1$ and $P_{n,i}(x_j) = 0$ for $j \ne i$). Since $\deg r(x) \le n-1$, $r(x)$ equals the interpolating polynomial $\sum_{i=1}^n r(x_i) P_{n,i}(x)$. So $\int_0^\infty e^{-x} r(x)\,dx = \sum_{i=1}^n r(x_i) \int_0^\infty e^{-x} P_{n,i}(x)\,dx = \sum_{i=1}^n r(x_i) c_{n,i}$.

Therefore, the quadrature rule is exact if $f(x)$ is a polynomial of degree at most $2n-1$. So the degree of precision of this quadrature rule is at least $2n-1$. In fact it's exactly $2n-1$ by essentially the same argument from the previous problem.

The roots of $L_2(x) = x^2 - 4x + 2$ are $x_1 = 2 - \sqrt{2}$ and $x_2 = 2 + \sqrt{2}$. Thus, the polynomials $P_{n,i}$ above are

$$P_{2,1} = \frac{x-(2+\sqrt{2})}{-2\sqrt{2}} = \frac{-\sqrt{2}}{4}x + \frac{1+\sqrt{2}}{2}, \quad P_{2,2} = \frac{x-(2-\sqrt{2})}{+2\sqrt{2}} = \frac{+\sqrt{2}}{4}x + \frac{1-\sqrt{2}}{2}$$

We can compute $c_{n,i} = \int_0^\infty P_{n,i}(x) e^{-x}\,dx$ using $\int_0^\infty x^0 e^{-x}\,dx = 1$ and $\int_0^\infty x^1 e^{-x}\,dx = 1$.[1] $c_{2,1} = \frac{2+\sqrt{2}}{4}$, $c_{2,2} = \frac{2-\sqrt{2}}{4}$, and the quadrature rule is $\int_0^\infty e^{-x} f(x)\,dx \approx \frac{2+\sqrt{2}}{4} f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4} f(2+\sqrt{2}) \approx 0.854 f(0.586) + 0.146 f(3.414)$.

(Consider solutions correct if degree of precision $> 2n-1$ is not ruled out, or if $c_{2,i}$ are only estimated.)

(3) Using a suitable substitution, transform each of the following into an integral on the indicated interval.

(a) Example: using $x = (1+t)/2$, $dx = dt/2$: $\displaystyle\int_{-1}^1 \sin(\pi(1+t)^2/4)/2\,dt$.

(b) Example: using $x = 1/t$, $dx = -dt/t^2$: $\displaystyle\int_1^0 -\sin(1/t)\,dt = \int_1^0 \sin(1/t)\,dt$.

(c) Using $x = \frac{1+u}{1-u}$, $dx = 2(1-u)^{-2}du$: $\displaystyle\int_{-1}^1 2(1+u)^4(1-u)^{-6} e^{(u+1)/(u-1)}\,du$.

Date: Due Friday 7/24.

[1]The first integral is easy. For $n > 0$, integration by parts gives $\int_0^\infty x^n e^{-x}\,dx = [x^n \cdot -e^{-x}]_0^\infty - \int_0^\infty -e^{-x} n x^{n-1}\,dx = n\int_0^\infty x^{n-1} e^{-x}\,dx$. So $\int_0^\infty x^n e^{-x}\,dx = n!$.

(4) Laguerre quadrature gives 20 and Gaussian quadrature gives about 52. Laguerre quadrature came much closer to 24 (and impressively close for only two nodes). (This makes sense: Laguerre quadrature is estimating the kind of integral it was designed for, whereas Gaussian quadrature is seeing a fairly unfriendly integrand.)

(5) (a) The integrand x^{-1} is smooth on $[1, 4]$ (good for Romberg), and easily approximated by polynomials (good for Newton-Cotes and Gaussian).

(b) The adaptive method is free to spend less time on subintervals that affect the outcome less. For instance, the integral on $[2, 4]$ is only half as large as the integral on $[1, 2]$, but the fixed-step-size method must spend twice as much time on it.

(c) The composite midpoint and trapezoidal methods can't exploit the problem's smoothness to as high a degree as the others.

(6) (a) The function $|x|$ is difficult to approximate by polynomials on this domain, due to its corner. As the number of nodes increases, the Newton-Cotes estimates become integrals of very bad polynomial approximations.

(b) An adaptive method will notice that the integral is easy on any interval which doesn't contain the corner ($x = 0$). So it will always bisect the interval that contains the corner, effectively halving the error each time it adds nodes.

(7) (a) Gaussian quadrature performs well when the integrand is a high-degree polynomial, because it was specifically designed to integrate polynomials with a high degree of precision.

(b) An adaptive method can repeatedly refine its estimate on the leftmost and rightmost subintervals, where most of the error is being made. Newton-Cotes works well since it estimates a high-degree polynomial with a high-degree polynomial; absent round-off error, it would even be exact if it used more than 60 nodes.

(c) Although Gaussian quadrature with more than 30 nodes has a sufficient degree of precision to estimate the integral exactly, computer round-off error prevents this from happening in practice.

(8) Note that $\frac{\partial f}{\partial y} = -e^{t-y}$ exists for all (t, y). **(i)** No, $|-e^{t-y}|$ increases without bound as $y \to -\infty$. **(ii)** Yes, $|-e^{t-y}| \le e$ if $t \le 1$ and $y \ge 0$. **(iii)** **(i)** can't, but **(ii)** can: if $y(0) \approx 1$, then $y(0) > 0$ and $y'(t) = f(t, y(t)) > 0$ for all $t \in [0, 1]$ imply that $y(t) > 0$ for all $t \in [0, 1]$. Thus, the Lipschitz condition **(ii)** (and continuity of $f(t, y)$) is enough to show the problem is well-posed.

(9) The absolute value of $\frac{\partial}{\partial y}(1 + y/t) = 1/t$ is bounded by $L = 1$ on $[1, 2]$ (in fact, on each $[1, t_i]$). The absolute value of $y''(t) = \frac{d}{dt}(1 + y/t) = y'/t - y/t^2 = (1/t + y/t^2) - y/t^2 = 1/t$ (alternatively, $y''(t) = \frac{d^2}{dt^2}(t \ln t + 2t) = 1/t$) is bounded by $M = 1$ on $[1, 2]$ (in fact, on each $[1, t_i]$). So at $t = t_i$, a bound on the total error is given by $\frac{hM}{2L}(e^{(t_i - t_0)/L} - 1) = \frac{h}{2}(e^{ih} - 1)$.

i	t_i	w_i	y_i	$\lvert y_i - w_i \rvert$	error bound
0	1.00	2.0000000	2.0000000	0.0000000	0.0000000
1	1.25	2.7500000	2.7789294	0.0289294	0.0355032
2	1.50	3.5500000	3.6081977	0.0581977	0.0810902
3	1.75	4.3916667	4.4793276	0.0876610	0.1396250
4	2.00	5.2690476	5.3862944	0.1172467	0.2147852