MATH 128A, SUMMER 2009: HOMEWORK 5

(1) (This is a slightly modified version of 4.7.8.)

Show that the quadrature rule $\int_a^b f(x)\,dx \approx \sum_{i=0}^n c_i f(x_i)$ cannot have degree of precision greater than $2n+1$, regardless of the choice of $c_0, \ldots, c_n$ and $x_0, \ldots, x_n$.

Hint: Suppose the nodes (x_i) and coefficients (c_i) are fixed. Construct a polynomial $f(x)$ of degree $2(n+1)$ such that $f(x_i) = 0$ for each i, but $f(x) > 0$ for other values of x.

(2) **4.9**: 6–7a [reworded] The Laguerre polynomials $\{L_0(x), L_1(x) \ldots\}$ form an orthogonal set on $[0, \infty)$ with respect to the inner product

$$\langle f, g\rangle = \int_0^\infty e^{-x} f(x) g(x)\,dx$$

(i.e., $\langle L_i, L_j\rangle = 0$ for $i \neq j$). The polynomial $L_n(x)$ has n distinct zeros $x_1, \ldots, x_n$ in $[0, \infty)$. Let

$$c_{n,i} = \int_0^\infty e^{-x} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}\,dx.$$

Show that the quadrature formula

$$\int_0^\infty e^{-x} f(x)\,dx \approx \sum_{i=1}^n c_{n,i} f(x_i)$$

has degree of precision $2n-1$. (*Hint*: Follow the steps in the proof of Theorem 4.7.)

Hint: The degree of precision is the largest n such that the quadrature rule is exact when $f(x)$ is a polynomial of degree n.

Derive the quadrature formula using $n = 2$ and the zeros of $L_2(x) = x^2 - 4x + 2$.

(3) Using a suitable substitution, transform each of the following into an integral on the indicated interval.

(a) $\displaystyle\int_0^1 \sin(\pi x^2)\,dx$, $[-1, 1]$.

(b) $\displaystyle\int_1^\infty \frac{\sin(x)}{x^2}\,dx$, $[0, 1]$.

(c) $\displaystyle\int_0^\infty x^4 e^{-x}\,dx$, $[-1, 1]$. (Let $x = \frac{1+u}{1-u}$.)

(4) Use the quadrature formula from problem 2 (with two nodes) to estimate $\int_0^\infty x^4 e^{-x}\,dx$.

Use problem 3(c) to estimate the same integral with a two-point Gaussian quadrature rule.

Which is more accurate? (The exact answer is 24.)

(5) In figure 1 is a graph of the absolute error in computing $\int_1^4 \frac{1}{x}\,dx$ using the methods we have seen, versus the number of nodes used.

(a) Why did the Gaussian, Romberg, and (closed) Newton-Cotes methods outperform the others?

(b) Why is the adaptive Simpson method outperforming the composite Simpson rule?

(c) Why are the composite midpoint and trapezoid rules less accurate than the others?

(6) In figure 2 is a graph of the absolute error in computing $\int_{-1}^e |x|\,dx$ using the methods we have seen, versus the number of nodes used.

(a) Why are the (closed) Newton-Cotes formulas giving such bad results?

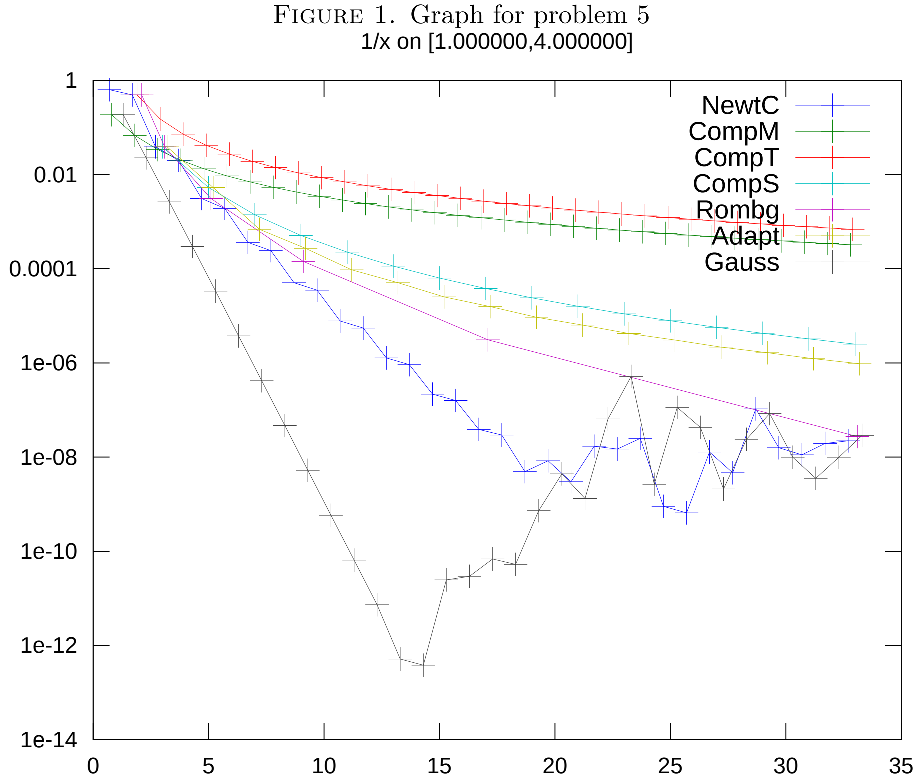
(b) Why is the adaptive Simpson method so much more accurate than the others?

(7) In figure 3 is a graph of the absolute error in computing $\int_{-1}^1 x^{60}\,dx$ using the methods we have seen, versus the number of nodes used.

(a) Why is Gaussian quadrature so effective?

Date: Due Thursday 7/23 or Friday 7/24, 4PM, 1008 Evans.

FIGURE 1. Graph for problem 5



(b) Why did the adaptive Simpson and Newton-Cotes methods take second and third place?

(c) Why isn't Gaussian quadrature giving the exact value when more than 30 nodes are used?

(8) (This is a slightly modified version of 5.1.4a.) For $f(t, y) = e^{t-y}$, **(i)** Does f satisfy a Lipschitz condition on $D = \{(t, y) \mid 0 \leq t \leq 1, -\infty < y < \infty\}$? **(ii)** Does f satisfy a Lipschitz condition on $D = \{(t, y) \mid 0 \leq t \leq 1, 0 < y < \infty\}$? **(iii)** Can **(i)** or **(ii)** be used to show that the initial-value problem $y' = f(t, y)$, $0 \leq t \leq 1$, with $y(0) = 1$, is well-posed?

(9) (5.2.1c+3c)

Use Euler's method to approximate the solutions for the initial value problem $y' = 1 + y/t$, $1 \leq t \leq 2$, $y(1) = 2$, with $h = 0.25$. The actual solution is $y(t) = t \ln t + 2t$. Compare the actual error at each step to the error bound.

FIGURE 2. Graph for problem 6

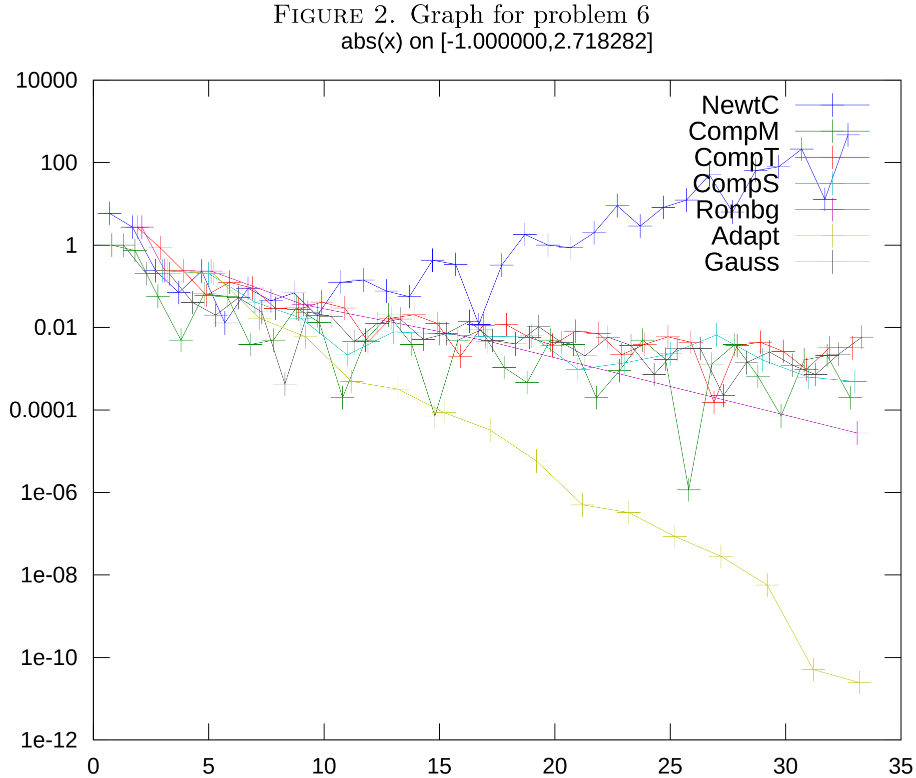


FIGURE 3. Graph for problem 7