

MATH 128A, SUMMER 2009: HOMEWORK 4 SOLUTIONS

3.4.11. $S(x)$, $S'(x)$, and $S''(x)$ must be continuous at $x = 1$; so $2 = S_1(1) = S_0(1) = 2$, $b = S'_1(1) = S'_0(1) = -1$, and $2c = S''_1(1) = S''_0(1) = -6$. The natural boundary condition requires $S''(x)$ to be zero at $x = 0$ and $x = 2$, so $0 = S''_0(0) = 0$ and $0 = S''_1(2) = 2c + 6d$. This gives us coefficients $(b, c, d) = (-1, -3, 1)$.

4.1.6a+8a. At the endpoints, we use the one-sided three-point formulas (equation (4.4) in the section and its mirror-image). At the other points, we use the centered three-point formula (4.5), which is typically more accurate.

All error bound formulas involve $f^{(3)}(\xi) = 8(e^{2\xi} - \sin(2\xi))$, where ξ is an x -value between the three which are sampled. $f^{(3)}$ is a positive decreasing function on $[-0.3, 0]$, so $8(e^{2\xi_{\min}} - \sin(2\xi_{\min}))$ is the best possible upper-bound for $|f^{(3)}(\xi)|$. A less careful bound is $8(e^{2(0)} + 1) = 16$.

x	$f(x)$	Approx. $f'(x)$	Exact $f'(x)$	Err.	Bound (good)	Bound (easy)
-0.3	-0.27652	-0.06030	-0.03167	0.02864	0.02969	0.05333
-0.2	-0.25074	0.57590	0.56180	0.01410	0.01485	0.02667
-0.1	-0.16134	1.25370	1.24012	0.01358	0.01413	0.02667
0	0	1.97310	2.00000	0.02690	0.02826	0.05333

4.1.28. The fourth-order Taylor polynomials are

$$\begin{aligned}
 f(x_0 - 2h) &= \underline{f(x_0)} - 2h\underline{f'(x_0)} + \frac{4h^2}{2!}\underline{f''(x_0)} - \frac{8h^3}{3!}f'''(x_0) + \frac{16h^4}{4!}f^{(4)}(x_0) + O(h^5) \\
 f(x_0 - h) &= \underline{f(x_0)} - h\underline{f'(x_0)} + \frac{h^2}{2!}\underline{f''(x_0)} - \frac{h^3}{3!}f'''(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) + O(h^5) \\
 f(x_0 + h) &= \underline{f(x_0)} + h\underline{f'(x_0)} + \frac{h^2}{2!}\underline{f''(x_0)} + \frac{h^3}{3!}f'''(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) + O(h^5) \\
 f(x_0 + 2h) &= \underline{f(x_0)} + 2h\underline{f'(x_0)} + \frac{4h^2}{2!}\underline{f''(x_0)} + \frac{8h^3}{3!}f'''(x_0) + \frac{16h^4}{4!}f^{(4)}(x_0) + O(h^5)
 \end{aligned}$$

If we ignore all $O(h^4)$ (and higher) terms, this is a system of linear equations which can be “solved” for the underlined unknowns. One way to isolate $f'''(x_0)$ is to first cancel the even-order terms:

$$\begin{aligned}
 f(x_0 + h) - f(x_0 - h) &= 2h\underline{f'(x_0)} + \frac{2h^3}{3!}f'''(x_0) + O(h^5) \\
 f(x_0 + 2h) - f(x_0 - 2h) &= 4h\underline{f'(x_0)} + \frac{16h^3}{3!}f'''(x_0) + O(h^5)
 \end{aligned}$$

We can then eliminate $f'(x_0)$:

$$f(x_0 + 2h) - 2f(x_0 + h) + 2f(x_0 - h) - f(x_0 - 2h) = \frac{12h^3}{3!}f'''(x_0) + O(h^5) = 2h^3 f'''(x_0) + O(h^5).$$

Dividing the left-hand side by $2h^3$ yields a formula for $f'''(x_0)$ with error $O(h^5/2h^3) = O(h^2)$.

4.1.29. $e(h)$ has a critical point on $(0, \infty)$ when $0 = e'(h) = -eh^{-2} + Mh/3$. The unique solution to this equation is $h = \sqrt[3]{3\epsilon/M} \in (0, \infty)$. (Since $e(h) \rightarrow +\infty$ as $h \rightarrow 0^+$ and $h \rightarrow \infty$, $e(h)$ must take its minimum value at this critical point.)

4.2.10. (As a matter of taste, I'm changing the signs of the constants K_i .)

We could write out the three power series and try to find a linear combination in which the h^2 and h^4 terms cancel. An easier method is to extrapolate M from $N(h)$ and $N(h/3)$:

$$\begin{array}{rcccccc} N(h) = M & +K_1h^2 & +K_2h^4 & +K_3h^6 & + & \dots \\ N(h/3) = M & +\frac{K_1}{9}h^2 & +\frac{K_2}{81}h^4 & +\frac{K_3}{3^6}h^6 & + & \dots \end{array}$$

$$9N(h/3) - N(h) = 8M \quad -0 \quad -\frac{8K_2}{9}h^4 \quad +\frac{80K_3}{81}h^6 \quad + \quad \dots$$

So set $N_2(h) = \frac{9N(h/3) - N(h)}{8}$ Its expansion is $M - \frac{K_2}{9}h^4 + \frac{10K_3}{81}h^6 + \dots$. Notice that $N_2(h/3)$ involves $N(h/9)$. So we can get the desired approximation through one more step of extrapolation.

$$\begin{array}{rcccccc} N_2(h) = M & +\frac{K_2}{9}h^4 & +\frac{10K_3}{81}h^6 & + & \dots \\ N_2(h/3) = M & +\frac{K_2}{9 \cdot 81}h^4 & +\frac{10K_3}{3^{10}}h^6 & + & \dots \end{array}$$

$$81N_2(h/3) - N_2(h) = 80M \quad +0 \quad -\frac{80K_3}{729}h^6 \quad + \quad \dots$$

Then take $N_3(h) = \frac{81N_2(h/3) - N_2(h)}{80}$ and we're done.

4.3.15.

n	$\int_{-1}^1 x^n dx$	$(-\sqrt{3}/3)^n + (+\sqrt{3}/3)^n$
0	2	2
1	0	0
2	2/3	2/3
3	0	0
4	2/5	2/9

We see from the table that the indicated quadrature rule exactly integrates x^0, \dots, x^3 but not x^4 . Thus, its degree of precision is 3.

4.3.24. **Note:** The problem is more reasonable if you are asked to plug in $f(x) = x^n$ for $n = 0, 1$, and 2.

Possible solution to the original problem:

Plug $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$ into the equation $\int_{x_0}^{x_2} f(x) dx = a_0f(x_0) + a_1f(x_1) + a_2f(x_2) + kf^{(4)}(\xi)$. This produces

$$\begin{aligned} \frac{1}{2}(x_2^2 - x_0^2) &= a_0x_0 + a_1x_1 + a_2x_2 \\ \frac{1}{3}(x_2^3 - x_0^3) &= a_0x_0^2 + a_1x_1^2 + a_2x_2^2 \\ \frac{1}{4}(x_2^4 - x_0^4) &= a_0x_0^3 + a_1x_1^3 + a_2x_2^3 \end{aligned}$$

Since the nodes are separated by a distance of h , we may plug in $x_0 = x_1 - h$ and $x_2 = x_1 + h$. After expanding, we get

$$\begin{aligned} a_0(x_1 - h) + a_1x_1 + a_2(x_1 + h) &= \frac{1}{2}(4x_1h) = 2x_1h \\ a_0(x_1 - h)^2 + a_1x_1^2 + a_2(x_1 + h)^2 &= \frac{1}{3}(6x_1^2h + 2h^3) = 2x_1^2h + 2h^3/3 \\ a_0(x_1 - h)^3 + a_1x_1^3 + a_2(x_1 + h)^3 &= \frac{1}{4}(8x_1^3h + 8x_1h^3) = 2x_1^3h + 2x_1h^3 \end{aligned}$$

Eliminate a_1 from the second equation (by subtracting x_1 times the first) and the third (by subtracting x_1 times the second).

$$\begin{aligned} (1) \quad & a_0(x_1 - h) + a_1x_1 + a_2(x_1 + h) = 2x_1h \\ (2) \quad & a_0(x_1 - h)(-h) + 0 + a_2(x_1 + h)(+h) = 2h^3/3 \\ (3) \quad & a_0(x_1 - h)^2(-h) + 0 + a_2(x_1 + h)^2(+h) = 4x_1h^3/3 \end{aligned}$$

Eliminate a_2 from (3): $a_0(x_1 - h)(-2h)(-h) = -2(-x_1 + h)h^3/3$. So $a_0 = h/3$. Substitute into (2) to get $a_0 = h/3$. Substitute into (1) to get $a_1 = 4h/3$.

Possible solution to the modified problem:

Plug $f(x) = 1$, $f(x) = x$, and $f(x) = x^2$ into the equation $\int_{x_0}^{x_2} f(x) dx = a_0f(x_0) + a_1f(x_1) + a_2f(x_2) + kf^{(4)}(\xi)$. This produces

$$\begin{aligned} \frac{1}{1}(x_2 - x_0) &= a_0 + a_1 + a_2 \\ \frac{1}{2}(x_2^2 - x_0^2) &= a_0x_0 + a_1x_1 + a_2x_2 \\ \frac{1}{3}(x_2^3 - x_0^3) &= a_0x_0^2 + a_1x_1^2 + a_2x_2^2 \end{aligned}$$

Since the nodes are separated by a distance of h , we may plug in $x_0 = x_1 - h$ and $x_2 = x_1 + h$. After expanding, we get

$$\begin{aligned} \frac{1}{1}(2h) &= a_0 + a_1 + a_2 \\ \frac{1}{2}(4x_1h) &= a_0(x_1 - h) + a_1x_1 + a_2(x_1 + h) \\ \frac{1}{3}(6x_1^2h + 2h^3) &= a_0(x_1 - h)^2 + a_1x_1^2 + a_2(x_1 + h)^2 \end{aligned}$$

Eliminate a_1 from the second equation (by subtracting x_1 times the first) and the third (by subtracting x_1 times the second).

$$\begin{aligned} a_0 + a_1 + a_2 &= 2ha_0(-h) + a_2(+h) = 0 \\ a_0(x_1 - h)(-h) + 0 + a_2(x_1 + h)(+h) &= 2h^3/3 \end{aligned}$$

From the middle equation, $a_0 = a_2$. Plug this into the last equation to get $2h^2a_2 = 2h^3/3$, so $a_0 = a_2 = h/3$. Substitute into the top equation to get $a_1 = 4h/3$.

Finding k :

Plug $f(x) = x^4$ into the equation $\int_{x_0}^{x_2} f(x) dx = (h/3)f(x_0) + (4h/3)f(x_1) + (h/3)f(x_2) + kf^{(4)}(\xi)$. This is simpler if we put the center node, x_1 , at 0: $2h^5/5 = 2h^5/3 + k \cdot 4!$, so $k = h^5(2/5 - 2/3)/4! = h^5(-4/15)/24 = -h^5/90$.

4.4.11a. To attain this accuracy, we must ensure that $|\frac{2-0}{12}h^2f''(\mu)| \leq 10^{-4}$ for every $\mu \in [0, 2]$. Since $f''(\mu) = \frac{d}{dx}(e^{2x}(2 \sin(3x) + 3 \cos(3x))) = e^{2x}(-5 \sin(3x) + 12 \cos(3x))$, we can bound this error term by maximizing this function on $[0, 2]$ (it's clear from the graph that this occurs at $x = 2$) or by overestimating it as $e^{2 \cdot 2}(5 + 12)$.

Using the former estimate, we need $117.56h^2 \leq 10^{-4}$, so $h \geq 9.2230 \cdot 10^{-4}$ and $n = \frac{2-0}{h} \geq 2168.5$. (Since n must be an integer, this actually means $n \geq 2169$ and $h \leq 9.2208 \cdot 10^{-4}$.)

Using the latter estimate, we need $154.69h \leq 10^{-4}$, so $h \geq 8.0401 \cdot 10^{-4}$ and $n = \frac{2-0}{h} \geq 2487.5$. (Since n must be an integer, this actually means $n \geq 2488$ and $h \leq 8.0386 \cdot 10^{-4}$.)

Note: Finding the best bound on n is hardly a big deal; a laptop computer can run the calculation with $n = 1,000,000$ in under a quarter-second.

4.5.8. Since $R_{2,1}$ is just the composite trapezoidal approximation to $\int_a^b f(x) dx$ with $n = 2$, we can just compute $R_{2,1}$ is $\frac{b-a}{4} [f(a) + f(b) + 2f(\frac{a+b}{2})]$. Or we can notice that $R_{2,2}$ is calculated from the formula $R_{2,2} = R_{2,1} + \frac{R_{2,1} - R_{1,1}}{3}$; since $R_{2,2}$ and $R_{1,1}$ are given, we can solve for $R_{2,1}$. Both methods give us $R_{2,1} \approx 0.2361$.