

MATH 128A, SUMMER 2009: HOMEWORK 1 SOLUTIONS

1.1

- 1(d).  $f(x) = x - (\ln x)^x$  is continuous when  $x > 1$  (since  $\ln(x) > 0$ ), hence on  $[4, 5]$ .  $f(4) \approx 0.3066 > 0$  and  $f(5) \approx -5.799 < 0$ , so by the intermediate value theorem there is an  $x \in [4, 5]$  with  $f(x) = 0$ .
- 4(a). First note that  $f'(x) = (-e^x + 2)/3$ , so the only critical point of  $f$  occurs at  $x = \ln 2$ , which lies in the interval  $[0, 1]$ . The maximum for  $|f(x)|$  must consequently be

$$\max\{|f(0)|, |f(\ln 2)|, |f(1)|\} = \max\{1/3, (2 \ln 2)/3, (4 - e)/3\} = (2 \ln 2)/3.$$

10. The first two derivatives of  $f(x) = e^x \cos x$  are  $f'(x) = e^x(\cos x - \sin x)$  and  $f''(x) = e^x(-2 \sin x)$ . Thus the second Taylor polynomial centered at  $\pi/6$  is

$$P_2(x) = \frac{\sqrt{3}}{2}e^{\pi/6} + \frac{\sqrt{3}-1}{2}e^{\pi/6}(x - \pi/6) - \frac{1}{2}e^{\pi/6}(x - \pi/6)^2.$$

Since  $f'''(x) = e^x(-2 \cos x - 2 \sin x)$ , the error  $R_2(x)$  is given by  $\frac{-2e^{\xi(x)}(\cos \xi(x) + \sin \xi(x))}{3!}(x - \pi/6)^3 = -\frac{1}{3}e^{\xi(x)}(\cos \xi(x) + \sin \xi(x))(x - \pi/6)^3$ , where  $\xi(x)$  is a number between  $\pi/6$  and  $x$ .

- (a)  $P_2(0.5) \approx 1.44688$ . The error is  $\frac{1}{3}e^{\xi}(\cos \xi + \sin \xi)(\frac{\pi}{6} - \frac{1}{2})^3$ , for some  $\xi \in [0.5, \pi/6]$ . This quantity's maximum value is  $1.01019 \cdot 10^{-3}$ , obtained when  $\xi = \pi/6$ ; one may also overestimate it to find an error bound: for example,  $|R_2(0.5)| < \frac{1}{3}e^{\pi/6}(\cos 0 + \sin \frac{\pi}{6})(\frac{\pi}{6} - \frac{1}{2})^3 < \frac{1}{3}(3)^1(1 + \frac{1}{2})(\frac{4}{6} - \frac{1}{2})^3 \approx 6.94444 \cdot 10^{-3}$ . The actual error is  $|R_2(0.5)| \approx 1.00265 \cdot 10^{-5}$ .

- (b) There are again multiple correct ways to find a bound on  $|R_2(x)|$  for  $x \in [0, 1]$ . Without any knowledge of  $\xi(x)$ , the most precise approach is to notice  $e^{\xi}(\cos \xi + \sin \xi)$  is a positive, increasing function of  $\xi$  on  $[0, 1]$ . Thus,  $|R_2(x)| \leq \frac{1}{3}e^{\pi/6}(\cos \frac{\pi}{6} + \sin \frac{\pi}{6})(\frac{\pi}{6} - x)^3 \leq \frac{1}{3}e^{\pi/6} \frac{\sqrt{3}+1}{2}(\frac{\pi}{6})^3 \approx 1.10339 \cdot 10^{-1}$  when  $x \in [0, \pi/6]$ , and  $|R_2(x)| \leq \frac{1}{3}e^x(\cos x + \sin x)(x - \frac{\pi}{6})^3 \leq \frac{1}{3}e^1(\cos 1 + \sin 1)(1 - \frac{\pi}{6})^3 \approx 1.35372 \cdot 10^{-1}$  when  $x \in [\pi/6, 1]$ . Thus, the latter is a bound on  $|R_2(x)|$  for  $x \in [0, 1]$ . This strategy is overkill; it's also correct to make an estimate like  $|R_2(x)| \leq \frac{1}{3}e^1(\cos 0 + \sin 1)|0 - \frac{\pi}{6}|^3 \leq \frac{1}{3}3^1(1 + 1)|0 - \frac{4}{6}|^3 \approx 5.92593 \cdot 10^{-1}$ .

- (c)  $\int_0^1 P_2(x) dx = 1.37654$ .

- (d) Integrating the "overkill" estimates for  $R_2(x)$  from (b) yields  $2.93300 \cdot 10^{-2}$ . It's also correct to integrate the answer to (b) (i.e., repeat it). The actual error,  $1.48276 \cdot 10^{-3}$ , is much smaller. (This is because  $P_2(x)$  both overestimates and underestimates  $f(x)$  on part of  $[0, 1]$ .)

23. Since  $2.5 = 1 + 1 + \frac{1}{2}$ , the Maclaurin polynomial in use is  $P_2(x)$ . This makes an error of  $\frac{e^{\xi(1)}}{3!}(1)^3 \leq e/6$ . Whoever gave an error bound of  $1/6$  should be fired, as it's too low by up to  $(e - 1)/6$  [and it's smaller than the true error].

1.2

- 6(c).  $(121 \ominus 0.327) \ominus 119 = 120.7 \ominus 119 = 1.7$ . Absolute error is .027; relative error is .016139.
- 6(d).  $(121 \ominus 119) \ominus 0.327 = 2 \ominus 0.327 = 1.673$ . This is exact.
- 12(a) Using L'Hospital's Rule, we have

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{1} = \frac{1 + 1}{1} = 2.$$

- 12(b) With three-digit rounding arithmetic we have  $e^{0.100} = 1.11$  and  $e^{-0.100} = 0.905$ , so  $f(0.100) = \frac{1.11 - 0.905}{0.100} = \frac{0.205}{0.100} = 2.05$ .

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Date: Due Thursday 6/25.

12(c) The third Maclaurin polynomials give

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \text{ and } e^{-x} \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3,$$

so

$$f(x) = \frac{(e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3) - (e^{-x} \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3)}{x} = \frac{2x + \frac{1}{3}x}{x} = 2 + \frac{1}{3}x^2.$$

Thus, with three-digit rounding, we have

$$f(0.100) \approx 2 + \frac{1}{3}(0.100)^2 = 2 + (0.333)(0.001) = 2.00 + 0.000333 = 2.00.$$

12(d) The relative error for the approximation  $y$  is  $\frac{|y - 2.003335000|}{2.003335000}$ . For (b) and (c) this gives relative error of  $2.329 \times 10^{-2}$  and  $1.665 \times 10^{-3}$ .

Note: Replacing the exponentials with Maclaurin polynomials *improved* accuracy by preventing the cancellation of significant digits. Simplifying the expression in part (c) is essential; omitting this step results in the same answer found in (b).

16(c). This number has a sign of + and exponent of  $1023 - 1023 = 0$ . By inspection, the same is true for the next smallest or largest machine number. Thus, these numbers are  $2^{-52}$  smaller and  $2^{-52}$  larger than the original, respectively. In exact decimals:

$$c^- = 1.3242187499999997779553950749686919152736663818359375$$

$$c = 1.32421875$$

$$c^+ = 1.3242187500000002220446049250313080847263336181640625$$

25(a). If overflow occurs at all, it will overflow when computing  $m!$ . ( $k!(m-k)!$  is smaller.) Trial and error shows that up to  $m = 17$  is safe, but  $m = 18$  overflows.

25(b).  $m!/k!(m-k)! = (1) \dots (m-k)(m-k+1) \dots (m)/(1) \dots (m-k) = (m-k+1) \dots (m)$ . So  $m!/k!(m-k)! = (m-k+1) \dots (m)/(1) \dots (k) = \frac{m}{k} \cdot \frac{m-1}{k-1} \dots \frac{m-k+1}{1}$ .

25(c). The calculation would be  $\frac{m}{3} \cdot \frac{m-1}{2} \cdot \frac{m-2}{1}$ . A search near  $\sqrt[3]{6 \cdot 0.9999 \cdot 10^{15}}$  finds that up to  $m = 181707$  works. (What's important is that this is much larger.)

### 1.3

6(a)–(c).  $O(n^{-1})$ ,  $O(n^{-2})$ , and  $O(n^{-2})$ , respectively, since  $\sin h = O(h)$  as  $h \rightarrow 0$ .

6(d).  $\ln(n+1) - \ln(n) = \ln((n+1)/n) = \ln(1 + 1/n) = (1/n) - (1/n)^2/2 + (1/n)^3/3 - \dots = O(n^{-1})$ .

8(a). There are  $\sum_{i=1}^n \sum_{j=1}^i 1 = n(n+1)/2$  multiplications and  $(n-1) + \sum_{i=1}^n (i-1) = (n-1)(n+1)/2$  additions. (Any answer that implies  $O(n^2)$  for both is acceptable.)

8(b).  $\sum_{i=1}^n \sum_{j=1}^i a_i b_j = \sum_{i=1}^n a_i \sum_{j=1}^i b_j$ . The latter summation involves only  $n$  multiplications.

14(a). If  $F(h) = L + O(h^p)$ , there are constants  $K$  and  $\delta$  such that  $|F(h) - L| \leq K|h^p|$  whenever  $|h| < \delta$ . If  $q < p$ , we also have  $|F(h) - L| \leq K\delta^{p-q}|h|^q$  whenever  $|h| < \delta$ , so  $F(h) = L + O(h^q)$  as well.

### EXTRA

(a) The semicolons suppressed output; without them, MATLAB would have printed the values of  $f$  and  $g$ . The  $\wedge$  means to exponentiate each entry of  $x$  if  $x$  is a vector or matrix— $g$  can be called on a list of data.

(b) Multiply  $f(x)/1$  by  $\sqrt{x^2+1} + x$  on top and bottom. The numerator expands as  $(\sqrt{x^2+1})^2 - x^2 = x^2 + 1 - x^2 = 1$ .

(c) When a computer calculates  $f(x)$ , for large  $x$ , it must subtract  $x$  from  $\sqrt{x^2+1}$ . When  $x$  is large, the first several significant (binary) digits of these numbers are the same; they are cancelled, and only the remaining bits are significant in the answer. The computation of  $g(10^7)$  has no such problem with catastrophic cancellation, and is therefore more accurate.