UCB MATH 128A-2, SUMMER 2009: FINAL EXAM SOLUTIONS

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1. (a) Prove that the bisection method on $[1, 2]$ will converge to a root of $x^3 - x - 1$.
   (b) Find a bound on the absolute error in approximating this root after n iterations.
   (c) How many iterations are necessary to ensure the relative error is at most 2^{-5}?
   (Your answer does not have to be exact.)

   Solution:

   (a) This function is continuous. Its values at the endpoints are $-1 < 0$ and $5 > 0$. Therefore, the interval brackets a root and the bisection method will converge.
   (b) On the nth iteration, the root is known to lie in an interval of width $\frac{2-1}{2^{n-1}}$. Its distance from this interval's midpoint can be at most $\boxed{1/2^n}$. (The answer $1/2^{n+1}$ is also acceptable, depending on how the phrase "after n iterations" is interpreted.)
   (c) Since the root lies in $[1, 2]$, the relative error is at most equal to the absolute error. By (b), we must use $\boxed{n = 5}$ iterations.

2. (a) Find an algebraic expression for the unique root (in $\mathbb{R}$) of $f(x) = x^2 - 2/x$.
   (b) Newton's method searches for this root using the iteration $p_{n+1} = g(p_n)$.
   Find $g(x)$. Simplify your answer.
   (c) Is the convergence linear? Is it at least quadratic?
   (d) Find the exact order of convergence.

   Solution:

   (a) $x^2 - 2/x = 0 \iff x^3 - 2 = 0 \iff \boxed{x = 2^{1/3}}$.
   (b) $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2-2/x}{2x+2/x^2} = x - x\frac{x^3-2}{2x^3+2} = \boxed{\dfrac{x(x^3 + 4)}{2(x^3 + 1)}}$
   (c) The convergence is at least quadratic, for either of the following reasons: $2^{1/3}$ is a simple root of $f(x)$, as $f'(2^{1/3}) = 2 \cdot 2^{1/3} + 2/2^{2/3} \neq 0$. Alternatively, $g'(2^{1/3}) = 0$ after the calculation $g'(x) = \frac{(4x^3+4)2(x^3+1)-x(x^3+4)(6x^2)}{2^2(x^3+1)^2} = \frac{8(x^3+1)^2-6x^3(x^3+4)}{2^2(x^3+1)^2}$.
   (d) (This was intended to be a harder problem.)
   By definition, the order of convergence is α iff $\lim_{n\to\infty} \left|\frac{p_{n+1}-2^{1/3}}{p_n-2^{1/3}}\right| = \lambda \neq 0$. If $p_n = x$, this fraction's numerator is $g(x) - 2^{1/3} = \frac{x^4-2^{4/3}x^3+4x-2}{2(x^3+1)}$. Factor out $(x - 2^{1/3})$ as many times as possible: $\frac{(x-2^{1/3})^3(x+2^{1/3})}{2(x^3+1)}$. (Alternatively, Taylor-expand $x^4 - 2^{4/3}x^3 + 4x - 2$ with center $x = 2^{1/3}$.) It follows that the order of convergence is $\boxed{3}$

3. (a) Estimate $y(1/2)$ with polynomial interpolation given the data $y(0) = 1$, $y(1) = 2$.
   (b) Repeat part (a) with Hermite interpolation, if $y(t)$ is a solution to $y' = y$.

   Solution:

   (a) Polynomial interpolation in this case is linear interpolation, so $y(0.5) \approx \boxed{1.5}$
   (b) Hermite interpolation, Newton form:

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$$
\begin{array}{c|ccccccc}
x_0 = 0 & \boxed{f[x_0] = 1} & & & & & & \\
 & & \boxed{f[x_0, x_1] = 1} & & & & & \\
x_1 = 0 & f[x_1] = 1 & & \boxed{f[x_0, x_1, x_2] = 0} & & & & \\
 & & f[x_1, x_2] = 1 & & \boxed{f[x_0, x_1, x_2, x_3] = 1} & & & \\
x_2 = 1 & f[x_2] = 2 & & f[x_0, x_1, x_2] = 1 & & & & \\
 & & f[x_2, x_3] = 2 & & & & & \\
x_3 = 1 & f[x_3] = 2 & & & & & &
\end{array}
$$

Using the coefficients above, we can read off the polynomial $P(x) = 1+1(x-0)+0(x-0)^2+1(x-0)^2(x-1) = 1 + x - x^2 + x^3$. Plugging in $x = 0.5$ yields $\boxed{1.375}$

4. (a) Transform $\int_{-3}^{3} \frac{1}{t^2+1}\, dt$ to an integral of the form $\int_{-1}^{1} f(x)\, dx$.
   (b) Find the coefficients below for the three-point Gaussian quadrature rule:

$$\int_{-1}^{1} f(x)\, dx \approx af\left(-\sqrt{\frac{3}{5}}\right) + bf(0) + cf\left(+\sqrt{\frac{3}{5}}\right).$$

   (c) What is this rule's degree of precision?
   (d) Estimate the integral from (a) using this rule. Express your answer as a fraction.

Solution:

(a) Let $t = 3x$ to get $\boxed{\int_{-1}^{1} \frac{3}{9x^2+1}\, dx}$

(b) Method one:

$$
\begin{array}{llllll}
f(x) = x^0: & \int_{-1}^{1} x^0\, dx = \frac{2}{1} & = +a\left(\frac{3}{5}\right)^{0/2} & + b + c\left(\frac{3}{5}\right)^{0/2} & \implies a + b + c = 2 & \\
f(x) = x^1: & \int_{-1}^{1} x^1\, dx = 0 & = -a\left(\frac{3}{5}\right)^{1/2} & + c\left(\frac{3}{5}\right)^{1/2} & \implies a = c & \\
f(x) = x^2: & \int_{-1}^{1} x^2\, dx = \frac{2}{3} & = +a\left(\frac{3}{5}\right)^{2/2} & + c\left(\frac{3}{5}\right)^{2/2} & \implies \boxed{a = c = \frac{5}{9}} & \implies \boxed{b = \frac{8}{9}}
\end{array}
$$

Method two: Integrate Lagrange polynomials: for instance, $a = \int_{-1}^{1} \frac{(x-0)(x-\sqrt{3/5})}{(-\sqrt{3/5}-0)(-\sqrt{3/5}-\sqrt{3/5})}\, dx$, etc. (This strategy leads to a mess, and will not be elaborated here.)

(c) Method one: It's $\boxed{5}$ because Gaussian quadrature achieves degree of precision 2(# nodes) − 1. Method two: Continue the tests above. The first one that fails is $f(x) = x^6$, so the degree is 5.

(d) $\frac{5}{9}\frac{3}{9(3/5)+1} + \frac{8}{9}(3) + \frac{5}{9}\frac{3}{9(3/5)+1} = \frac{2\cdot 5\cdot 3}{9\cdot 32/5} + \frac{8\cdot 3}{9} = \frac{25}{3\cdot 16} + \frac{8}{3} = \frac{153}{48} = \boxed{\frac{51}{16}}$.

5. (a) Find the local truncation error for the method $w_{i+1} = 4w_i - 3w_{i-1} - 2hf(t_{i-1}, w_{i-1})$.
   (b) Classify this multistep method as strongly stable, weakly stable, or unstable.

Solution:

(a) Note: This may be clearer if written out, long-hand.
Let $y(t) = \sum c_n (t - t_i)^n$. Then $y'(t) = \sum nc_n (t - t_i)^{n-1}$, and

$$\tau_{i+1} = \frac{y(t_i + h) - w_{i+1}}{h} = -4c_0 h^{-1} + \sum c_n[1 + 3(-1)^n + 2n(-1)^{n-1}]h^{n-1} = 0 + 0 + 0 + 4c_3 h^2 + \ldots$$

Thus, the local truncation error is $\boxed{O(h^2)}$

(b) The characteristic polynomial for this method is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$. Since $|3| > 1$, the method is unstable.

6. Describe (algebraically) the region of absolute stability for *Heun's Method*,

$$w_{i+1} = w_i + \frac{h}{4}[f(t_i, w_i) + 3f(t_i + \frac{2}{3}h, w_i + \frac{2}{3}hf(t_i, w_i))].$$

Solution: Plug in the test equation $f(t, y) = \lambda y$. This produces

$$w_{i+1} = w_i + \frac{h}{4}[\lambda w_i + 3\lambda(w_i + \frac{2}{3}h\lambda w_i)] = w_i[1 + \frac{1}{4}[h\lambda + 3h\lambda(1 + \frac{2}{3}h\lambda)]] = w_i[1 + h\lambda + \frac{1}{2}(h\lambda)^2].$$

Therefore, this method's stability region is $\boxed{\left\{h\lambda \in \mathbb{C} : \left|1 + h\lambda + \frac{1}{2}(h\lambda)^2\right| < 1\right\}}$

7. (a) Explain how the second-order IVP $y'' + 4y' - 5y = 0$, $y(0) = 2$, $y'(0) = -3$ is equivalent to this system of first-order IVPs:

$$\mathbf{u}' = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \mathbf{u}, \quad \mathbf{u}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

(b) If A is the above matrix, find $\|A\|_\infty$.

Solution:

(a) If we let $\boxed{u_1 = y}$ and $\boxed{u_2 = y'}$, the second-order IVP is equivalent to $u_1' = u_2$, $u_2' = 5u_1 - 4u_2$, $u_1(0) = 2$, $u_2(0) = -3$. This is exactly the system above.

(b) $\|A\|_\infty$ is the largest possible value of $\|Ax\|_\infty$ when $\|x\|_\infty$ is 1. Let $x = [x_1\ x_2]^*$. Then $\|x\|_\infty = 1$ iff $\max\{|x_1|, |x_2|\} = 1$, and $\|Ax\|_\infty = \|[x_2\ 5x_1 - 4x_2]^*\|_\infty = \max\{|x_2|, |5x_1 - 4x_2|\}$. When $|x_1|, |x_2| \leq 1$, this is at most $5 + 4 = \boxed{9}$, achieved when $x_1 = 1$ and $x_2 = -1$.

8. Solve this system via elimination *with partial pivoting* followed by back substitution.

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ -2 & 5 & 1 & 2 \\ -1 & 4 & 5 & -2 \end{array}\right]$$

Solution:

$$\underbrace{\left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ -2 & 5 & 1 & 2 \\ -1 & 4 & 5 & -2 \end{array}\right]}_{|-2|>|1|,|-1|} \longrightarrow \left[\begin{array}{rrr|r} -2 & 5 & 1 & 2 \\ 1 & -1 & 0 & 0 \\ -1 & 4 & 5 & -2 \end{array}\right] \longrightarrow \underbrace{\left[\begin{array}{rrr|r} -2 & 5 & 1 & 2 \\ 0 & 1.5 & 0.5 & 1 \\ 0 & 1.5 & 4.5 & -3 \end{array}\right]}_{\text{No pivoting needed}} \longrightarrow \boxed{\left[\begin{array}{rrr|r} -2 & 5 & 1 & 2 \\ 0 & 1.5 & 0.5 & 1 \\ 0 & 0 & 4 & -4 \end{array}\right]} \Longrightarrow \boxed{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$$

9. Let $A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$

(a) Is A strictly diagonally dominant?
(b) Perform elimination with *no* pivoting. Compute $\det(A)$ using the result.
(c) Is A a symmetric, positive definite matrix?

Solution:

(a) Yes: $|4| > |2| + |0| + |0|$, $|5| > |2| + |2| + |0|$, $|5| > |0| + |2| + |2|$, $|0| + |0| + |5| > |2|$.

(b)

$$\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The above operations didn't change the determinant, so $\det(A) = 4 \cdot 4 \cdot 4 \cdot 4 = \boxed{256}$

(c) Yes, because part (b) ended with all diagonal entries positive.

10.

$$\text{Find a Cholesky } (LL^*) \text{ factorization for this matrix: } A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 6 \end{bmatrix}.$$

Solution: If you remember (or re-derive, as described in class) a Cholesky factorization algorithm, you can practically write down the answer.

If not, there is still hope: $A = LL^*$ must take the form

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{bmatrix} \begin{bmatrix} a & b & d & g \\ 0 & c & e & h \\ 0 & 0 & f & i \\ 0 & 0 & 0 & j \end{bmatrix} = \begin{bmatrix} a^2 & ab & ad & ag \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Without further computation, you can find $a = 1$, $b = 0$, $d = 1$, $g = 1$, and write

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 1 & e & f & 0 \\ 1 & h & i & j \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & c & e & h \\ 0 & 0 & f & i \\ 0 & 0 & 0 & j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & c^2 & ce & ch \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Write down $c = 1$, $e = 0$, $h = 2$, and

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & f & 0 \\ 1 & 2 & i & j \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & f & i \\ 0 & 0 & 0 & j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1+f^2 & 1+fi \\ ? & ? & ? & ? \end{bmatrix}$$

Write down $f = 1$, $i = 0$, and

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & j \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 5+j^2 \end{bmatrix}$$

Finally, $j = 1$, and the desired factorization has

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}}$$