MATH 128A, SUMMER 2009: FINAL REVIEW PROBLEMS

(1) Find the largest n for which $\cos(x) = 1 - \frac{x^2}{2} + O(x^n)$. Write the corresponding Taylor remainder.

(2) Match each of (1, 2, 3) below with the correct value (a, b, c).

1. Absolute error for the estimate $\sqrt{5} \approx 2$	2. Relative error for the estimate $\sqrt{5} \approx 2$	3. Neither
a. $[\sqrt{5} - 2]/2$	b. $[\sqrt{5} - 2]/\sqrt{5}$	c. $\sqrt{5} - 2$

(3) (a) Show that the bisection method will converge to a root of $f(x) = x^2 - 2$ in $[1, 2]$.
(b) Approximately how many iterations are needed to ensure two significant digits of accuracy?

(4) (a) Find the unique fixed point of the following function: $g(x) = x/2 + 5/x^2$.
(b) Show that the corresponding fixed point iteration $p_{n+1} = g(p_n)$ converges for any $p_0 \in [2, 3]$.
(c) Determine whether the convergence is linear or (at least) quadratic.
(d) Does your answer to 4c change if Steffensen iteration is used?

(5) (a) Find the unique root of the following function: $f(x) = x/2 - 5/x^2$.
(b) Show that Newton's method converges to this root if p_0 is sufficiently close.
(c) Determine whether the convergence is linear or (at least) quadratic.

(6) (a) How many roots (in $\mathbb{C}$) does $p(x) = x^4 + x^2 + 2$ have?
(b) Why will Newton's, with any initial guess $p_0 \in \mathbb{R}$, fail to find any of them?
(c) Why doesn't Müller's method have this problem?

(7) (a) State the definition of a sequence's *order of convergence*.
(b) Is a sequence's convergence always "linear or (at least) quadratic"?

(8) Assume $f(x)$ passes through the points $(-1, 0)$, $(0, 0)$, and $(1, 2)$.
(a) Find a polynomial $p(x)$ of minimal degree which also passes through these points.
(b) If $|f'''(x)| \leq 6$ for all $x \in [-1, 1]$, find an upper bound for $|f(0.5) - p(0.5)|$.
(c) Compute the divided difference $f[-1, 0, 1]$.

(9) Background: You're solving the IVP $y' = 2ty + 1$, $y(0) = 0$. `rk4` with $h = 1$ says $y(1) \approx 2$.
Problem: Use Hermite interpolation to estimate $y(0.5)$.

(10) (a) What's the order of the (absolute) error in using $[f(x_0 + h) - f(x_0 - h)]/2h$ to estimate $f'(x_0)$?
(b) If this formula gives you the following estimates of $f'(x_0)$, what's the highest order estimate given by Richardson's extrapolation? 1.00 when $h = 0.4$, 1.09 when $h = 0.2$, 1.90 when $h = 0.1$.

(11) State the midpoint rule estimate for $\int_{-1}^{1} f(x)\,dx$. What is this rule's degree of precision?

(12) Find the composite trapezoidal rule estimate (with $n = 2$) for $\int_{-1}^{1} x^2\,dx$.

(13) Would Romberg integration or adaptive quadrature perform better evaluating $\int_0^3 |x^2 - 2|\,dx$?

(14) (a) Find the roots x_1 and x_2 of the Legendre polynomial of degree 2.
(b) Find α_1 and α_2 maximizing this rule's degree of precision: $\int_{-1}^{1} f(x)\,dx \approx \alpha_1 f(x_1) + \alpha_2 f(x_2)$.
(c) Is it possible for a 2-point quadrature rule to have a larger degree of precision?

(15) Explain a way to estimate $\int_0^1 \frac{e^x}{x^{1/2}} dx$ numerically.

(16) (a) Show that the IVP $y' = 1 + \sin y$, $y(0) = 0$ is well-posed.
(b) Find a formula for y'' and a numerical upper bound for $|y''|$.
(c) Find a bound on the local truncation error made by Euler's method with step size h.
(d) Find a bound on the error in estimating $y(1)$ using Euler's method with step size h.
(e) What is this Euler's method estimate for $y(1)$ if $h = 1$?
(f) What is the second-degree Taylor method estimate for $y(1)$ if $h = 1$?

(17) (a) Write the Runge-Kutta midpoint method, $w_{i+1} = w_i + hf(t_i + \frac{1}{2}h, w_i + \frac{1}{2}hf(t_i, w_i))$, in the form $k_1 = hf(t_i, w_i)$, $k_2 = hf(t_i + ch, w_i + ak_1)$, $w_{i+1} = w_i + b_1 k_1 + b_2 k_2$.
(b) How would you show this method has $O(h^2)$ (or better) local truncation error?

Date: Tuesday 8/11.

(18) A RKF method with `tol` $= 0.1$ computes the 4th- and 5th-order steps $w_{i+1} = 1.00$, $\tilde{w}_{i+1} = 1.04$ using $h = 0.1$. Does it have to redo the step with a smaller h?

(19) (a) Find coefficients b_1 and b_0 optimizing this multistep method's local truncation error: $w_{i+1} = w_{i-1} + h[b_1 f(t_i, w_i) + b_0 f(t_{i-1}, w_{i-1})]$.

(b) Will the resulting method be strongly stable, weakly stable, or unstable?

(20) What is a predictor-corrector method?

(21) Why do an Adams-Bashforth m-step predictor and Adams-Moulton $m-1$-step corrector work well together in an adaptive multistep method?

(22) Transform $y'' - 3y' + 2y = 0$ into an equivalent system of first-order ODEs.

(23) If R is the region of absolute stability for the Runge-Kutta midpoint method of problem 17,

(a) Describe this set algebraically.

(b) Find the largest value of h for which this method solves $y' = -2y$ stably.

(24) Find the region of absolute stability for the backward Euler method, $w_{i+1} = w_i + hf(t_{i+1}, w_{i+1})$. (This was a homework problem.)

(25) Today's quiz, problem 1.

(26) Today's quiz, problem 2.

(27) Today's quiz, problem 3.

(28) Find the determinant of the matrix $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(29) Is $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 7 & 3 \\ 1 & 0 & 3 \end{bmatrix}$ strictly diagonally dominant? Why might we care?

(30) Is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ symmetric? Positive definite? A band matrix?

(31) If $x = [0\ 1\ 0]^*$, find $\|x\|_1$, $\|x\|_2$, and $\|x\|_\infty$.

(32) If $A = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right]$, show that $\|A\|_2 = \sqrt{5}$ and $\|A\|_\infty = 3$. (This problem might be harder than the rest.)