Finite-dimensional von Neumann Algebras and the Basic Construction

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Abstract

We define the basic construction for finite-dimensional von Neumann algebras, and provide a non-standard proof that the basic construction acts on Bratteli diagrams by reflection. We will also discuss how one extends the trace under the basic construction, and the related Frobenius-Perron theory of matrices of connected bipartite graphs.

1 The basic construction

Our basic data will be M, a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a positive, faithful, normal, normalized tracial state tr. Why these properties? We're going to do GNS on M. Define a sesquilinear form on M by $\langle x, y\rangle = \mathrm{tr}(y^*x)$. Because tr is positive and faithful, this is an inner product. We call the completion $L^2(M, \mathrm{tr})$ or $L^2(M)$ when the trace is clear. If $x \in M$, we will let $\hat{x}$ denote the corresponding element of $L^2(M)$, and we let $\Omega = \hat{1}$. As usual, we get the left regular representation $L : M \to B(L^2(M))$ which is densely defined via $L_x(\hat{y}) = \widehat{xy}$. To check that this extends to M, we have

$$\|\widehat{xy}\|_2^2 = \mathrm{tr}(y^*x^*xy) \leq \|x\|_M^2 \,\mathrm{tr}(y^*y) = \|x\|_M^2 \|\hat{y}\|_2^2.$$

However, because we have a trace, it also holds that

$$\|\widehat{xy}\|_2^2 = \mathrm{tr}(y^*x^*xy) = \mathrm{tr}(xyy^*x) \leq \|y\|_M^2 \|x\|_2^2.$$

Thus L_x and R_y extended to bounded, commuting operators on $L^2(M)$. Because tr was assumed normal, the image of M under L is a von Neumann algebra (trust me) on $L^2(M)$. Esay to check that the representation is faithful, so we'll assume without loss of generality that M is given to us in "standard form" (i.e. acting on $L^2(M)$).

Under these circumstances, we have a symmetry of $L^2(M)$ called the *modular conjugation operator*, which is densely defined by $J\hat{x} = \widehat{x^*}$. This is a conjugate-linear "self-adjoint unitary." If $x \in M$, we have

$$JxJ\hat{y} = Jx\widehat{y^*} = J\widehat{xy^*} = \widehat{yx^*}.$$

Hence JxJ is right-multiplication by x^*, and in particular $JMJ \subseteq M'$. We have the following important result.

Theorem 1. *$JMJ = M'$.*

To see how to prove this, first observe that if $x' \in M'$, we need to show that $Jx'J \in M$. If it were to hold that $Jx'J \in M$, it would follow that $Jx'\Omega = (x')^*\Omega$. This is the first step.

Lemma 1. *$Jx'\Omega = (x')^*\Omega$*

Proof. If $y \in M$, then

$$\langle Jx'\Omega, y\Omega\rangle = \langle Jy\Omega, x'\Omega\rangle = \langle y^*\Omega, x'\Omega\rangle = \langle \Omega, yx'\Omega\rangle = \langle \Omega, x'y\Omega\rangle = \langle (x')^*\Omega, y\Omega\rangle$$

□

Proof of Theorem 1. We have that $JMJ \subseteq M'$, so it suffices to show that $M' \subseteq JMJ$, or equivalently $JM'J \subseteq M = M''$. Thus fix $x', y' \in M'$, and we will show that $Jx'J$ and y' commute. For $z \in M$, we have

$$Jx'Jy'(z\Omega) = Jx'Jzy'\Omega = Jx'(JzJ)(Jy'\Omega) = Jx'(JzJ)y'^*\Omega.$$

Since x', JzJ and $y'^* \in M'$, we have

$$Jx'(JzJ)y'^*\Omega = y'Jz^*Jx'^*\Omega = y'Jz^*x'\Omega = y'Jx'z^*\Omega = y'Jx'Jz\Omega = y'Jx'J(z\Omega).$$

□

Since $N \subseteq M$, we have $M' \subseteq N'$ and thus $M \subseteq JN'J$. The passage from $N \subseteq M$ to $M \subseteq JN'J$ is called the basic construction, and we write $M_1 = JN'J$.

Theorem 2. *$M_1 = (M \cup \{e_N\})''$, where $e_N \in B(L^2(M))$ is the projection onto $L^2(N)$.*

For this reason, we sometimes write $M_1 = \langle M, e_N\rangle$. Due to time restraints, we will not prove Theorem 2. Some natural questions to ask:

- What is the structure of M_1?
- When can we repeat the basic construction using $M \subseteq M_1$ as our initial data? That is, when can we extend tr to a trace on M_1?

The rest of the talk will be devoted to answering these questions in the case where M is finite-dimensional.

2 What is M_1?

If M is a finite-dimensional von Neumann algebra, then Wedderburn theory says that $M = \bigoplus M_i = \bigoplus_{i=1}^k M_{m_i}(\mathbb{C})$, where $m_i \in \{1, 2, \ldots\}$. We will specify M via a "dimension vector" $\overline{m} = (m_1, \ldots, m_k)$. Dimension vectors will be row vectors. For example, $\overline{m} = (2, 3)$ gives $M = M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$.

Assume that N is a finite-dimensional von Neumann algebra with dimension vector $\overline{n}$ and trace vector $\overline{s}$, and assume that we have a (unital) inclusion $N \hookrightarrow M$. What data is needed to specify this? The only inclusions of matrix algabras are of the form $X \mapsto X \oplus X \oplus \cdots \oplus X \oplus 0$ (not proven here). Thus the only inclusions of finite-dimensional von Neumann algebras are of the form [easier to say in words and handwave.] This may be specified via a matrix Λ_N^M, where

λ_{ij} is the number of times N_i is included in M_j. This is the matrix of a bipartite graph. For example, if $N = \mathbb{C} \oplus M_2(\mathbb{C})$ (i.e. $\overline{n} = (1, 2)$), then one possible inclusion is given by

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{[bratteli diagram]}$$

We must have $\overline{m} = \overline{n}\Lambda_N^M$ for the inclusion to be well-defined and unital.

Now let's look at the basic construction for $N \subseteq M$. First of all, we need to represent these algebras on $L^2(M)$. Lets do this explicitly when $M = M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ as before. Fix elements $X \oplus Y$ and $Z \oplus W$ in M, and consider the action of $X \oplus Y$ on $Z \oplus W$, where the second vector is regarded as being in $L^2(M)$. If z_1, z_2 are the columns of Z and w_1, w_2, w_3 are the columns of W, we have

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} (Xz_1 \mid Xz_2) & 0 \\ 0 & (Yw_1 \mid Yw_2 \mid Yw_3) \end{pmatrix}.$$

Since $z_i \in \mathbb{C}^2$ and $w_i \in \mathbb{C}^3$ are arbitrary, we see that M on $L^2(M)$ is isomorphic to M on $\mathbb{C}^{13}$ via

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \mapsto \begin{pmatrix} X & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 \\ 0 & 0 & Y & 0 & 0 \\ 0 & 0 & 0 & Y & 0 \\ 0 & 0 & 0 & 0 & Y. \end{pmatrix}$$

The more general statement:

Theorem 3. *If $M = \bigoplus M_{m_i}(\mathbb{C})$, then M on $L^2(M)$ is isomorphic to M on $\bigoplus \mathbb{C}^{m_i} \otimes \mathbb{C}^{m_i}$ via $\bigoplus X_i \mapsto \bigoplus X_i \otimes 1$.*

Using $N = \mathbb{C} \oplus M_2(\mathbb{C})$ as before, it would be easy to write down N in standard form. The next question: what is M_1? Well, $JN'J = (JNJ)'$, so lets find JNJ. First things first: what is J? We can compute

$$J \begin{pmatrix} x_1 & x_3 & 0 & 0 & 0 \\ x_2 & x_4 & 0 & 0 & 0 \\ 0 & 0 & y_1 & y_4 & y_7 \\ 0 & 0 & y_2 & y_5 & y_8 \\ 0 & 0 & y_3 & y_6 & y_9 \end{pmatrix} = \begin{pmatrix} \overline{x_1} & \overline{x_2} & 0 & 0 & 0 \\ \overline{x_3} & \overline{x_4} & 0 & 0 & 0 \\ 0 & 0 & \overline{y_1} & \overline{y_2} & \overline{y_3} \\ 0 & 0 & \overline{y_4} & \overline{y_5} & \overline{y_6} \\ 0 & 0 & \overline{y_7} & \overline{y_8} & \overline{y_9} \end{pmatrix}$$

Acting on $\mathbb{C}^13$, we can write $J = P_{(23)}C \oplus P_{(24)(37)(68)}C$, where C is elementwise complex conjugation and P_σ is the permutation matrix corresponding to the permutation σ. We can write a typical element of N as

$$\left(\begin{array}{cccc|cccccc} z & & & & & & & & & \\ & z & & & & & & & & \\ & & z & & & & & & & \\ & & & z & & & & & & \\ \hline & & & & z & & & & & \\ & & & & & X & & & & \\ & & & & & & z & & & \\ & & & & & & & X & & \\ & & & & & & & & z & \\ & & & & & & & & & X \end{array}\right).$$

If $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, then we get that a typical element of JNJ is

$$\begin{pmatrix} zI_7 & & \\ & x_1 I_3 & x_2 I_3 \\ & x_3 I_3 & x_4 I_3 \end{pmatrix}.$$

Hence $JN'J$ consists of matrices $T \oplus S \oplus S$ where T is 7×7 and S is 3×3. Thus $M_1 = JN'J \cong M_7(\mathbb{C}) \oplus M_3(\mathbb{C})$. Now let's look at how M is included in M_1. Recall that M is matrices of the form

$$\begin{pmatrix} X & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 \\ 0 & 0 & Y & 0 & 0 \\ 0 & 0 & 0 & Y & 0 \\ 0 & 0 & 0 & 0 & Y. \end{pmatrix}$$

So we can see that the inclusion matrix of M into M_1 is $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ (since the first summand of M_1 is two copies of X and one of Y, and the second is one copy of Y). We see $\Lambda_M^{M_1} = (\Lambda_N^M)^T$. This holds in general.

Theorem 4. *If $N \subseteq M$ is an inclusion of finite dimensional von Neumann algebras, and $M_1 = \langle M, e_N \rangle$, then $\Lambda_M^{M_1} = (\Lambda_N^M)^T$.*

First a lemma that helps us get a hold of λ_{ij}.

Lemma 2. *If $\{p_i\}$ and $\{q_i\}$ are the minimal central projections of N and M, respectively, then $\lambda_{ij} = (\dim_{\mathbb{C}} p_i q_j N' p_i q_j \cap p_i q_j M p_i q_j)^{1/2}$.*

Proof. First note that if $M \subseteq M_r(\mathbb{C})$, then $p_i q_j M_r(\mathbb{C}) p_i q_j = p_i q_j M p_i q_j$ is $\lambda_{ij} n_i \times \lambda_{ij} n_i$ matrices, which contains N_i along the diagonal as $X \oplus \cdots \oplus X$ for $X \in N_i$. The matrices that commute with these are isomorphic to $1 \otimes M_{\lambda_{ij}}(\mathbb{C})$, and thus they have dimension λ_{ij}^2. $\square$

As a corollary, we get that $(\Lambda_N^M)^T = \Lambda_{M'}^{N'}$.

Proof of Theorem 4. Let p_i and q_j be as before. Observe that since $p_i \in Z(M)$, right multiplication and left multiplication by p_i coincidide. That is, $Jp_iJ = p_i$. On the other hand, $N' \to JN'J$ is an (anti-)isomoprhism, and thus will take the minimal central projections of N' (and thus of N) to the minimal central projections of M_1. Thus the jith entry of $\Lambda_M^{M_1}$ is the square root of the dimension of

$$(Jq_jJ)(Jp_iJ)M'(Jp_iJ)(Jq_jJ)\cap(Jq_jJ)(Jp_iJ)(JN'J)(Jq_jJ)(Jp_iJ) = J(q_jp_iMp_iq_j\cap q_jp_iN'p_iq_j)J.$$

Since $x \mapsto JxJ$ is an automorphism of $B(\mathcal{H})$, it preserves dimension and the proof is complete. $\square$

So this lets us easily compute the basic construction of an inclusion of finite-dimensional von Neumann algebras..

3 When can we extend the trace from M to M_1?

Return to the setup $M = \bigoplus M_{m_i}(\mathbb{C})$ (so that M has dimension vector $\overline{m} = (m_1, \ldots, m_k)$. Since each summand of M admits a unique trace (up to multiplication by a scalar), the positive, faithful, normalized traces on M are in one-to-one correspondence with (column) vectors $\overline{t} \in \mathbb{R}^k_{>0}$ such that $\overline{m}\overline{t} = 1$. Here, t_i is the trace of a minimal projection in M_i (or the scaling factor appiled to the non-normalized trace). Returning to our example with, $\overline{m} = (2, 3)$ we can put $\overline{t} = (\frac{1}{3}, \frac{1}{9})^T$ and get

$$M = M_2(\mathbb{C}) \oplus M_3(\mathbb{C}), \qquad \mathrm{tr}(X \oplus Y) = \frac{1}{3}(x_{11} + x_{22}) + \frac{1}{9}(y_{11} + y_{22} + y_{33}).$$

First an easier question: what is the restriction of M's trace to N? Let's compute its trace vector $\overline{s}$. The trace of a minimal projection in the first slot is

$$\mathrm{tr}(1 \oplus 0) = \mathrm{tr}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = 2 \cdot \frac{1}{3} + 1 \cdot \frac{1}{9} = \frac{7}{9}.$$

Similarly, $\mathrm{tr}(0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{9} = \frac{1}{9}$. One can generalize from this example to get the following theorem.

Theorem 5. *A trace vector $\overline{s}$ for N is the restriction of the trace of M if and only if $\Lambda_N^M \overline{t} = \overline{s}$.*

Proof. A minimal projection in N_i is included in each M_j, λ_{ij} times, and thus is included in each M_j as the sum of λ_{ij} minimal projections in M_j. Hence the trace of such a projection is $\sum_j \lambda_{ij} t_j$, which is pricesely the ith entry of $\Lambda_N^M \overline{t}$. $\square$

Let $\Lambda = \Lambda_N^M$. Our question becomes, is there a vector $\overline{t_1} \in \mathbb{R}^l_{>0}$ such that $\Lambda^T \overline{t_1} = \overline{t}$? Equivalently, $\Lambda\Lambda^T \overline{t_1} = \Lambda \overline{t} = \overline{s}$. An elegent solution to this problem comes from the famous Perron-Frobenius Theorem in linear algebra. One consequence of this theorem is the following

Theorem 6. *If T is a square matrix with real nonnegative entries such that for some k, every entry of T^k is positive, then the following hold.*

i) There is an eigenvalue λ of Λ such that $\|\Lambda\| = \lambda$.

ii) The eigenspace of λ is one-dimensional, and it contains an eigenvector with all positive entries.

We wish to apply this theorem to $\Lambda^T\Lambda$, but first we need to verify that $(\Lambda^T\Lambda)^k$ has all non-zero entries for sufficiently large. It is intuitively obvious that this is equivalent to the Bratteli diagram of Λ being connected. We proceed under this assumption.

Now lets go back and choose $\bar{t}$ to be the unique P-F eigenvector for $\Lambda^T\Lambda$ such that $\overline{m}\bar{t} = 1$, and let $\bar{s} = \Lambda\bar{t}$. It is now easy to extend the trace on M to that of M_1 by putting $\overline{t_1} = \lambda^{-1}\bar{s} = \lambda^{-1}\Lambda\bar{t}$. We can check

$$\Lambda^T\overline{t_1} = \lambda^{-1}\Lambda^T\Lambda\bar{t} = \bar{t}.$$

Now that we have extended the trace to M_1, we have our original setup back with $M \subseteq M_1$. One then applies the basic construction again, and gets $M \subseteq M_1 \subseteq M_2$. We can, in fact, continue this process without end (for fun!). Simply observe that with each basic construction, we have an inclusion matrix of Λ or Λ^T. We have

- $\Lambda^T\bar{s} = \lambda\bar{t}$
- $\Lambda\bar{t} = \bar{s}$

So we get the tower:

$$N_{\bar{s}} \overset{\Lambda}{\subseteq} M_{0,\bar{t}} \overset{\Lambda^T}{\subseteq} M_{1,\lambda^{-1}\bar{s}} \overset{\Lambda}{\subseteq} M_{2,\lambda^{-1}\bar{t}} \overset{\Lambda^T}{\subseteq} M_{3,\lambda^{-2}\bar{s}} \subseteq \cdots$$

Interesting things to notice: each $M_{i+1} = \langle M_i, e_i\rangle$, and it turns out that $e_ie_{i\pm1}e_i = \lambda^{-1}e_{i\pm1}$ and that $e_ie_j = e_je_i$ when $|i-j| > 1$. Also, $\|\Lambda\|^2 = \lambda$, and it is known (Kroenecker) that the norms of such graphs are either ≥ 2, or of the form $2\cos(\pi/n)$ for $n = 3, 4, 5, \ldots$.