Section 101 of Math 54, Summer 2007
Solution to Sample Quiz 8

15 minutes

1. *(5 points)* Consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}.$$

(a) *(3 points)* Find all the eigenvalues of A and a basis for each eigenspace.
Solution. The characteristic polynomial is

$$\det(\lambda I - A) = \begin{bmatrix} \lambda - 1 & 2 \\ -4 & \lambda + 3 \end{bmatrix} = (\lambda - 1)(\lambda + 3) - (-4) \cdot 2 = \lambda^2 + 2\lambda + 5.$$

The two eigenvalues are the roots of $0 = \lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4$, which are $\lambda_1 = \sqrt{-4} - 1 = -1 + 2\mathrm{i}$ and $\lambda_1 = -\sqrt{-4} - 1 = -1 - 2\mathrm{i}$.

For $\lambda_1 = -1 + 2\mathrm{i}$, the eigenspace is

$$NS((-1 + 2\mathrm{i})I - A) = NS \begin{bmatrix} -2 + 2\mathrm{i} & 2 \\ -4 & 2 + 2\mathrm{i} \end{bmatrix}.$$

So a basis for this eigenspace is $\{[2, 2 - 2\mathrm{i}]^T\}$.

For $\lambda_2 = -1 - 2\mathrm{i}$, the eigenspace is

$$NS((-1 - 2\mathrm{i})I - A) = NS \begin{bmatrix} -2 - 2\mathrm{i} & 2 \\ -4 & 2 - 2\mathrm{i} \end{bmatrix}.$$

So a basis for this eigenspace is $\{[2, 2 + 2\mathrm{i}]^T\}$.

(b) *(2 points)* Find an invertible matrix S and a diagonal matrix Λ, such that $A = S\Lambda S^{-1}$.

Solution. The columns of S are two linearly independent eigenvectors of A and the corresponding eigenvalues are put on the diagonals of Λ. Hence

$$S = \begin{bmatrix} 2 & 2 \\ 2 - 2\mathrm{i} & 2 + 2\mathrm{i} \end{bmatrix}, \Lambda = \begin{bmatrix} -1 + 2\mathrm{i} & 0 \\ 0 & -1 - 2\mathrm{i} \end{bmatrix}.$$

2. *(5 points)* Find all the pairs (a, b), such that matrix $A = \begin{bmatrix} 2 & a \\ 0 & b \end{bmatrix}$ is diagonalizable.

Solution. First note that A is an upper triangular matrix. The characteristic polynomial of A is $(\lambda - 2)(\lambda - b)$.

If $b \neq 2$, then A has two distinct eigenvalues 2 and b, so it is diagonalizable.

If $b = 2$, then the only eigenvalue of A is 2. A is diagonalizable if and only if $\dim NS(A - 2I) = 2$. Clearly $A - 2I = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, the above condition is equivalent to $a = 0$.

Hence the answer is $(a, b) = (0, 2)$, or $(s, t), t \neq 2$.