Solutions to Homework 6

Lecture 16

3.7.34 Denote the columns of B by $B_1, \cdots B_n$. Then

$$0 = AB = A[B_1, \cdots, B_n] = [AB_1, \cdots, AB_n],$$

hence $AB_1 = \cdots = AB_n = 0$. Any vector $\mathbf{u}$ in $CS(B)$ is a linear combination of $B_1, \cdots B_n$, so it can be written as

$$\mathbf{u} = c_1 B_1 + \cdots c_n B_n.$$

Thus

$$A\mathbf{u} = A(c_1 B_1 + \cdots c_n B_n) = c_1(AB_1) + \cdots + c_n(AB_n) = 0 + \cdots + 0 = 0,$$

which means $\mathbf{u} \in NS(A)$. Since $\mathbf{u}$ is arbitrary, we have $CS(B) \subset NS(A)$.

Another method is as follows. Pick any vector $\mathbf{u} \in CS(B)$, then by Problem 3.7.33, there exists $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{u}$. Now

$$A\mathbf{u} = A(B\mathbf{x}) = (AB)\mathbf{x} = 0\mathbf{x} = 0,$$

so $\mathbf{u} \in NS(A)$. Therefore we have $CS(B) \subset NS(A)$.

3.7.43 ***(b)*** FALSE. A counterexample would be a square matrix, whose column space equals its null space. From Problem Review 3.16, we know the number of rows for such a matrix will be even number $2k$, and the dimension of the column space and null space are all k.

Just start with the simplest case $k = 1$ and try some 2×2 matrices where most of the entries are 0, then you will find

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is a suitable one, since $CS(A) = NS(A) = \{(x, 0)^T | x \in \mathbb{R}\}$.

(d) TRUE. The rank of a 3×4 matrix cannot exceed the number of rows 3, so the dimension of the column space is at most 3. However this matrix has 4 columns, they have to be linearly dependent.

3.7.49 ***(a)*** For any $x \in NS(A)$, we have $Ax = \mathbf{0}$. Hence

$$x = Ix = (BA)x = B(Ax) = B\mathbf{0} = \mathbf{0},$$

that is $NS(A) = \{\mathbf{0}\}$.

Clearly, $Ix = \mathbf{0}$ implies $x = \mathbf{0}$, so $NS(I) = \{\mathbf{0}\}$.

(b) Since $NS(A) = \{\mathbf{0}\}$, $\dim NS(A) = 0$. Then it follows from the formula

$$rk(A) + \dim NS(A) = n$$

that $rk(A) = n$. So A is invertible.

(c) From (b), there exists a matrix A^{-1} such that$AA^{-1} = A^{-1}A = I$. Multiplying A^{-1} on both sides of $BA = I$ from right, we get $B = B(AA^{-1}) = (BA)A^{-1} = IA^{-1} = A^{-1}$.

Therefore the left inverse of any square matrix is just the inverse of the original matrix.

Review 3.16 Let A be $m \times n$ matrix. Then $CS(A)$ is subspace of $\mathbb{R}^m$, while $NS(A)$ is a subspace of $\mathbb{R}^n$, so $m = n$.

We know that in general

$$rk(A) + \dim NS(A) = n,$$

and in this case $rk(A) = \dim(CS(A)) = \dim NS(A)$, so $n = 2rk(A)$, which shows that n is even.

Review 3.27 In the linear system $A\mathbf{x} = \mathbf{b}$, the number of equation is m while the number of unknowns is n.

If A has linearly independent rows, then in its row echelon form the number of pivots will be m. So $A\mathbf{x} = \mathbf{b}$ always has solution and $r = m$. If $n > m$, then there will be some free variables, so the solutions are infinite. While if $n = m$, the solution is unique.

If A has linearly dependent rows, then the rank r is less than m and some rows in its row echelon form will be zero rows. So for some $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ will contain inconsistent equations, which means sometimes it has no solution at all.

By the above analysis, we know the answer is

(b) $r = m = n$, ***(d)*** $r = m < n$.

Lecture 17

3.8.5 To find $[\mathbf{x}]_B$ is to solve the linear system

$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{x} = \begin{bmatrix} 3 \\ 14 \\ -4 \end{bmatrix},$$

where the coefficient matrix is obtained by writing down the coordinates of vectors in basis B as columns. The answer is

$$[\mathbf{x}]_B = [2, -2, 3]^T.$$

Since the vectors in basis C are just a reordering of vectors in B, reorder $[\mathbf{x}]_B$ in the same way gives

$$[\mathbf{x}]_C = [-2, 3, 2]^T.$$

3.8.11 To find $[\mathbf{x}]_B$, we need to solve the linear system

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ -2 \end{bmatrix}.$$

Here the columns of the coefficient matrix are the coordinates of the vectors in basis B under the standard basis for M_{22}. The answer is

$$[\mathbf{x}]_B = (c_1, c_2, c_3, c_4)^T = [2, -1, 3 - 6]^T.$$

Since the vectors in basis C are just a reordering of vectors in B, reorder $[\mathbf{x}]_B$ in the same way gives

$$[\mathbf{x}]_C = [3, -1, 2, -6]^T.$$

3.8.18 By the method to find the transition matrix, we first construct the following 2×4 matrix

$$\begin{bmatrix} 1 & 5 & 3 & -1 \\ -3 & 0 & 1 & -2 \end{bmatrix},$$

which can be abbreviated as $[C|B]$. Then applying elementary row operations on this big matrix, change the left part to I and the right part is the transition matrix

$$P = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

By the definition of transition matrix, we get

$$[\mathbf{x}]_C = P[\mathbf{x}]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

3.8.25 We first construct a 3×6 matrix which can be abbreviated as $[C|B]$. The two matrices B and C are obtained by writing down the coordinates of all the vectors in bases B and C respectively under the standard basis for P_2. Since C is just the standard basis, so the corresponding matrix C is I, while it is easy to see matrix

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Note that $[C|B]$ is already $[I|B]$, so the transition matrix

$$P = B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

By the definition of transition matrix, we get

$$[\mathbf{x}]_C = P[\mathbf{x}]_B = [5, 3, 0]^T.$$

3.8.31 By the definition of transition matrix, for any vector $\mathbf{x} \in V$, we have

$$[\mathbf{x}]_C = P[\mathbf{x}]_B, [\mathbf{x}]_D = Q[\mathbf{x}]_C.$$

Hence

$$[\mathbf{x}]_D = Q[\mathbf{x}]_C = Q(P[\mathbf{x}]_B) = (QP)[\mathbf{x}]_B,$$

QP should be the transition matrix from basis B to D.

Lecture 18

4.1.16 The unit vectors in three axis $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are transformed to $(0, 1, 0)$, $(-1, 0, 0)$ and $(0, 0, -1)$ respectively. The matrix where the columns are these three images is the required matrix. So the answer is

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

4.1.24 For any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, we have

$$T(\mathbf{u} + \mathbf{v}) = (1, 1, 1) \neq (2, 2, 2) = (1, 1, 1) + (1, 1, 1) = T(\mathbf{u}) + T(\mathbf{v}),$$

and

$$rT(\mathbf{u}) = r(1,1,1) = (r,r,r) \neq (1,1,1) = T(r\mathbf{u})$$

when $r \neq 1$. Hence T preserves neither the addition nor the scalar multiplication. T is NOT linear.

4.1.36 For any $C_1, C_2 \in M_{33}$ and $r \in \mathbb{R}$, we have

$$T(C_1 + C_2) = A(C_1 + C_2) = AC_1 + AC_2 = T(C_1) + T(C_2),$$

$$T(rC) = A(rC) = r(AC) = rT(C).$$

Hence T IS linear.

4.1.34 T can be expressed as $T(f(x)) = f(x) - 1$ for any polynomial $f(x)$ in P_2.

So for $f(x)$, $g(x) \in P_2$ and $r \in \mathbb{R}$, we have

$$T(f(x) + g(x)) = f(x) + g(x) - 1 \neq f(x) + g(x) - 2,$$

$$T(f(x)) + T(g(x)) = (f(x) - 1) + (g(x) - 1) = f(x) + g(x) - 2,$$
$$rT(f(x)) = r(f(x) - 1) = rf(x) - r \neq rf(x) - 1 = T(rf(x)),$$

when $r \neq 1$. Hence T preserves neither the addition nor the scalar multiplication. T is NOT linear.

4.7.11 ***(a)*** Since $f(x) = 2 \cdot 1 + 3 \cdot x + (-1) \cdot e^x + (-2) \cdot xe^x$,

$$[f]_B = \begin{bmatrix} 2 & 3 & -1 & -2 \end{bmatrix}^T.$$

(b) Routine calculation gives

$$D(1) = 0, D(x) = 1, D(e^x) = e^x, D(xe^x) = e^x + xe^x.$$

Thus

$$[D(1)]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T, [D(x)]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$[D(e^x)]_B = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T, [D(xe^x)]_B = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T.$$

These 4 coordinates are exactly the columns of $[D]_B$, so the answer is

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) Since $D(f) = 3 - 3e^x - 2xe^x$, so

$$[D(f)]_B = \begin{bmatrix} 3 & 0 & -3 & -2 \end{bmatrix}^T.$$

OR

$$[D(f)]_B = [D]_B[f]_B = \begin{bmatrix} 3 & 0 & -3 & -2 \end{bmatrix}^T.$$

NOTE

The 5 problems that will be graded are 3.7.34, 3.7.49, 3.8.18, 4.1.36 and 4.7.11. Each graded problem is worth 5 points and each ungraded problem is worth 1 point. The full mark is 35 points.