Quiz 4 Solutions

1) $f(-1) = 1$, $f(0) = 0$, $f(1) = 1$.

Let $S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$

$S_i'(x) = 3a_i x^2 + 2b_i x + c_i$

$S_i''(x) = 6a_i x + 2b_i$

we require:

$$\begin{array}{ll} S_1''(-1) = 0 & \\ S_1(-1) = f(-1) & \\ S_1(0) = f(0) & \\ S_2''(1) = 0 & \Longleftrightarrow \\ S_2(0) = f(0) & \\ S_2(1) = f(1) & \\ S_2'(0) = S_1'(0) & \\ S_2''(0) = S_2''(0) & \end{array} \qquad \begin{array}{rll} -6a_1 + 2b_1 & = 0 & (1) \\ -a_1 + b_1 - c_1 + d_1 & = 1 & (2) \\ d_1 & = 0 & \\ 6a_2 + 2b_2 & = 0 & (3) \\ d_2 & = 0 & \\ a_2 + b_2 + c_2 + d_2 & = 1 & (4) \\ c_2 & = c_1 & \\ 2b_2 & = 2b_1 & \end{array}$$

(1), (3) $\Rightarrow$ $b_1 = 3a_1$, $b_2 = -3a_2$

$b_1 = b_2 \Rightarrow 3a_1 = -3a_2 \Rightarrow a_1 = -a_2$

$(2) + (4) = 2a_2 + 2b_2 = 2 \Rightarrow a_2 + b_2 = 1$

$\Rightarrow a_2 - 3a_2 = 1 \Rightarrow a_2 = -\frac{1}{2}$

$\Rightarrow a_1 = \frac{1}{2}$, $b_1 = b_2 = \frac{3}{2}$

$\Rightarrow c_1 = c_2 = 1 - a_2 - b_2 - d_2 = 1 + \frac{1}{2} - \frac{3}{2} = 0$

$$\Rightarrow S(x) = \begin{cases} \frac{1}{2}x^3 + \frac{3}{2}x^2 & -1 \le x \le 0 \\ -\frac{1}{2}x^3 + \frac{3}{2}x^2 & 0 \le x \le 1 \end{cases}$$

2)
$$\begin{array}{rl} -4f(x-h) = & -4f(x) + 4f'(x)h - 2f''(x)h^2 + \frac{2}{3}f'''(\xi_1)h^3 \\ f(x+2h) = & f(x) + 2f'(x)h + 2f''(x)h^2 + \frac{4}{3}f'''(\xi_2)h^3 \\ + \quad 3f(x) = & 3f(x) \\ \hline & 0 + 6f'(x)h + 0 + Kh^3 \end{array}$$

$$\Rightarrow \frac{-4f(x-h) + 3f(x) + f(x+2h)}{6h} = f'(x) + \hat{K}h^2$$

$$= f'(x) + O(h^2)$$

Hence, this approximation is of order 2.