

Galois Representations with Prescribed Restriction to Inertia

Jared Weinstein, UC Berkeley

Red Lodge, 6 April 2006

Basic question:

Let F be a totally real number field. For each prime \mathfrak{p} of F , let $I_{\mathfrak{p}} \subset \text{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ be the inertia group. Given a family of continuous complex representations $\rho_{\mathfrak{p}}: I_{\mathfrak{p}} \rightarrow \text{GL}_2(\mathbb{C})$, when does there exist a geometric object (e.g., a GL_2 -abelian variety over F) whose associated Galois representation restricts to each $I_{\mathfrak{p}}$ as $\rho_{\mathfrak{p}}$?

Contents:

1. Review of Weil-Deligne representations
2. An “inertial” Langlands Correspondence
3. The Equivariant Riemann-Roch formula

Notation, Weil-Deligne Representations

Let K be a p -adic field, with residue field \mathbf{F}_q , and let $G_K = \text{Gal}(\overline{K}/K)$. The Weil group W_K fits into the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & I_K & \longrightarrow & W_K & \longrightarrow & \mathbf{Z} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & I_K & \longrightarrow & G_K & \longrightarrow & \widehat{\mathbf{Z}} \longrightarrow 0 \end{array}$$

Local class field theory furnishes an isomorphism $\text{Art} : K^* \xrightarrow{\sim} W_K^{\text{ab}}$, which sends a uniformizer in \mathcal{O}_K to a geometric Frobenius element Φ . We define an absolute value $|\cdot|$ on W_K^* by $|\sigma| = |\text{Art}^{-1}(\sigma)|_K$.

A *Weil-Deligne representation* of K is a pair (ρ, N) , where

$$\rho : W_K \rightarrow \text{GL}_n(\mathbf{C})$$

is semisimple and continuous and $N \in \text{End } \mathbf{C}^n$ satisfies

$$\rho(\sigma) N \rho(\sigma)^{-1} = |\sigma| N, \quad \sigma \in W_K$$

(N is the “monodromy operator”; it is necessarily nilpotent.)

In the wild, we normally encounter l -adic Galois representations. The formalism of the Weil-Deligne representation allows us to pass between the l -adic and complex worlds, at least when $l \neq p$:

Theorem. (Grothendieck) *For $l \neq p$, there exists a bijection*

$$\left\{ \begin{array}{l} \text{WD reps } (\rho, N) \\ \text{of dimension } n \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{continuous reps} \\ \rho_l: W_K \rightarrow \text{GL}_n(\overline{\mathbf{Q}}_l) \end{array} \right\}$$

Here one makes use of an isomorphism $\iota: \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_l$, whence (ρ, N) and ρ_l are related by

$$\rho_l(\sigma) = \iota(\rho(\sigma)) \exp(t_l(\sigma)\iota(N)),$$

for σ in an open subgroup of I_K . Here $t_l: I_K \rightarrow \mathbf{Q}_l$ is an l -adic tame character.

(Thus N measures how far away ρ_l is from being semisimple. Sometimes we refer to the pair (ρ, N) as simply ρ .)

Now let A/\mathbb{Q} be an abelian variety of GL_2 -type. This means there is a number field E with $[E : \mathbb{Q}] = \dim A$ and an action $E \hookrightarrow \text{End}_{\mathbb{Q}} A \otimes_{\mathbb{Z}} \mathbb{Q}$. Then for each prime $\lambda|l$ in E , the action of Galois on the λ -adic Tate module

$$T_{\lambda}A := \varprojlim_{\infty \leftarrow n} A[\lambda^n] \otimes \mathbb{Q}$$

gives a representation

$$\begin{aligned} \rho_{A,\lambda} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) &\rightarrow \text{Aut}_{E_{\lambda}} T_{\lambda}A \approx GL_2(E_{\lambda}) \\ &\hookrightarrow GL_2(\overline{\mathbb{Q}}_l) \end{aligned}$$

Previous theorem yields a family of Weil-Deligne representations

$$\rho_{A,p} : W_{\mathbb{Q}_p} \rightarrow GL_2(\mathbb{C})$$

for each prime p . The restrictions to inertia

$$\rho_{A,p}|_{I_{\mathbb{Q}_p}} : I_{\mathbb{Q}_p} \rightarrow GL_2(\mathbb{C})$$

are trivial at all but finitely many primes. The $\{\rho_{A,p}\}$ might be called the “bad reduction type” of A .

Example: Elliptic Curves

If $A = E$ is an elliptic curve and $p > 2$, there are three possibilities for the restriction of the WD representation $(\rho_{E,p}, N)$ to $I_{\mathbb{Q}_p}$.

- E has good reduction at p . Then $\rho_{E,p} = 1 \oplus 1$ and $N = 0$.
- E has multiplicative reduction at p . Then $\rho_{E,p} = 1 \oplus 1$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
- E has additive reduction at p . Then either:
 - E is a twist of a curve of multiplicative reduction by a ramified quadratic character χ , so that $\rho_{E,p} = \chi \oplus \chi$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, or
 - E is not such a twist: then there is a quadratic extension L/\mathbb{Q}_p , and a character $\theta: W_L \rightarrow \mathbb{C}^*$, for which $\rho_{E,p}$ is the restriction to $I_{\mathbb{Q}_p}$ of $\text{Ind}_{W_L}^{W_K} \theta$. ($N = 0$.)

We ask: Suppose we have a family of inertial representations

$$\rho_p: I_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(\mathbb{C})$$

such that

1. Each ρ_p admits an extension to a Weil-Deligne representation of \mathbb{Q}_p ,
2. All but finitely many ρ_p are trivial,
3. $\prod_p \det \rho_p(\mathrm{Art}(-1)) = 1$.

Then does there exist an abelian variety A/\mathbb{Q} of GL_2 type for which $\rho_{A,p}|_{I_p} \approx \rho_p$?

We ask: Suppose we have a family of inertial representations

$$\rho_p: I_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(\mathbb{C})$$

such that

1. Each ρ_p admits an extension to a Weil-Deligne representation of \mathbb{Q}_p ,
2. All but finitely many ρ_p are trivial,
3. $\prod_p \det \rho_p(\mathrm{Art}(-1)) = 1$.

Then does there exist an abelian variety A/\mathbb{Q} of GL_2 type for which $\rho_{A,p}|_{I_p} \approx \rho_p$?

Theorem. Such an A exists for all but finitely many families $\{\rho_p\}$ and their twists.

In fact, one can give a formula for the number of *modular* abelian varieties A which match $\{\rho_p\}$ this way. (It is conjectured that all abelian varieties of GL_2 type are modular.)

The Langlands Correspondence for $GL_n(K)$

Let $\text{Irr}(GL_n(K))$ denote the set of isomorphism classes of smooth admissible irreducible complex representations of $GL_n(K)$.

Ex. For $n = 1$, an irreducible representation of $GL_1(K) = K^*$ is just a homomorphism $K^* \rightarrow \mathbb{C}^*$. But we have the isomorphism $\text{Art} : K^* \rightarrow W_K^{\text{ab}}$, so $\text{Irr}(GL_1(K))$ is identified with one-dimensional WD representations of K .

The generalization of this to $n \geq 1$ is embodied in the

Theorem. There is a bijection

$$\text{rec} : \{n\text{-dim'l WD reps of } K\} \xrightarrow{\sim} \text{Irr}(GL_n(K))$$

which extends class field theory in dimension 1.

Theorem. There is a bijection

$\text{rec} : \{n\text{-dim'l WD reps of } K\} \xrightarrow{\sim} \text{Irr}(\text{GL}_n(K))$
which extends class field theory in dimension
1.

For $n = 1$, this is local class field theory, established by Hasse (1930).

The existence of rec as above was conjectured by Langlands (1970).

For $n = 2$, the theorem was established by Kutzko (1980).

Proofs for all n were given by Harris/Taylor (1998) (local methods) and Henniart (2000) (global methods).

The Category of Inertial WD representations Let K once again be a local field. An *inertial WD representation* of K is a pair (ρ, N) consisting of a continuous rep. $\rho: I_K \rightarrow \mathrm{GL}_n(\mathbb{C})$ and an endomorphism $N \in \mathrm{End} \mathbb{C}^n$ such that ρ admits an extension to $\tilde{\rho}: W_K \rightarrow \mathrm{GL}_n(\mathbb{C})$ which makes $(\tilde{\rho}, N)$ a WD representation.

Ex. For $n = 1$, a WD rep. of W_K is a map $K^* \rightarrow \mathbb{C}^*$ (class field theory). The image of I_K in $W_K^{\mathrm{ab}} \rightarrow K^*$ is \mathcal{O}_K^* , so an inertial WD representation of K is simply a homomorphism $\mathrm{GL}_1(\mathcal{O}_K) \rightarrow \mathrm{GL}_1(\mathbb{C})$.

The generalization of this to higher dimension should put into correspondence inertial WD representations and certain irreducible representations of $\mathrm{GL}_n(\mathcal{O}_K)$.

An inertial Langlands Correspondence?

Let $\text{Irr}(\text{GL}_n(\mathcal{O}_K))$ denote the set of isomorphism classes of irreducible representations of $\text{GL}_n(\mathcal{O}_K)$ which are *smooth*, i.e. which factor through $\text{GL}_n(\mathcal{O}_K/\mathfrak{p}^N)$ for some N , and *typical*, in the sense of Bushnell-Kutzko.

Conjecture: There is a diagram

$$\begin{array}{ccc} \{\text{WD reps of } K\} \xrightarrow{\sim}^{\text{rec}} \text{Irr}(\text{GL}_n(K)) & & \\ \text{res} \downarrow & & \downarrow \\ \{\text{IWD reps of } K\} \xrightarrow{\sim} \text{Irr}(\text{GL}_n(\mathcal{O}_K)) & & \end{array}$$

Proven for $n = 2$ by Henniart (2002), and for all n in “supercuspidal case” by Paskunas (2005).

Need to describe map

$$\text{rec} : \text{Irr}(\text{GL}_2(K)) \rightarrow \text{Irr}(\text{GL}_2(\mathcal{O}_K)).$$

Let $\Pi : \text{GL}_2(K) \rightarrow \text{Aut } V$ be an element of $\text{Irr}(\text{GL}_2(\mathcal{O}_K))$. Then there is a unique (up to scalar) *new vector* $v \in V$, fixed by the largest subgroup of the form

$$\Gamma_1(\mathfrak{p}_K^n) := \left\{ \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}_K^n} \right\} \subset \text{GL}_2(\mathcal{O}_K).$$

Let π be the $\text{GL}_2(\mathcal{O}_K)$ -span of v .

Then π is the image of Π in our map. Note that π is necessarily finite dimensional.

Back to the original problem. Let $\{\rho_p\}$ be a collection of inertial WD representations of \mathbf{Q}_p for each prime p , almost all of which are trivial. Then by “inertial Langlands correspondence” get family of finite-dimensional representations

$$\pi_p := \text{rec}(\rho_p): \text{GL}_2(\mathbf{Z}_p) \rightarrow \text{Aut } V_p.$$

Tensor these together to form

$$\pi := \bigotimes_p \pi_p: \text{GL}_2(\hat{\mathbf{Z}}) \rightarrow \text{Aut } V,$$

which is once again finite dimensional.

The condition $\prod_p \det \rho_p(\text{Art}(-1)) = 1$ means that the action of $\text{GL}_2(\hat{\mathbf{Z}})$ factors through $G := \text{GL}_2(\hat{\mathbf{Z}}) / \{\pm 1\}$.

We are on the hunt for a abelian variety A of GL_2 type whose “bad reduction type” is $\{\rho_p\}$. A supply of GL_2 abelian varieties is given by cuspidal eigenforms

$$f \in S_2(X(\infty), \mathbf{C}) \circlearrowleft \text{GL}_2(\hat{\mathbf{Z}}) / \{\pm 1\}$$

Each such f yields an abelian variety quotient A_f of $J(N)$ (Shimura). How to read off the bad reduction type of A_f from f ?

Claim. Let f be a weight 2 cuspidal eigenform. The bad reduction type of A_f is $\{\rho_p\}$ iff the $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ -span of f is \cong to $\pi = \otimes_p \mathrm{rec}(\rho_p)$.

Explanation. The automorphic form attached to f is a square-integrable function

$$\phi_f: \mathrm{GL}_2(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbf{Q}}) \rightarrow \mathbf{C}.$$

The *automorphic representation* Π_f associated to f is the $\mathrm{GL}_2(\mathbb{A}_{\mathbf{Q}})$ -span of ϕ_f . Then

$$\Pi_f = \otimes_p \Pi_{f,p}$$

and we have

$$\mathrm{rec}(\rho_{A,p}) \cong \Pi_{f,p}.$$

If f starts out life as a newform, then $\phi_f|_{\mathrm{GL}_2(\mathbf{Q}_p)}$ is a new vector for $\Pi_{f,p}$. Let π_f be the $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ -span of ϕ , with $\phi_f = \otimes_p \pi_{f,p}$. Then by commutativity of the “reciprocity diagram” get

$$\mathrm{rec}(\rho_{A,p}|_{I_{\mathbf{Q}_p}}) \cong \pi_{f,p}.$$

as required.

So, we are trying to count how many times $\pi = \otimes_p \mathrm{rec}(\rho_p)$ appears as a $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ -module inside of $S_2(X(\infty), \mathbf{C})$.

The Equivariant Riemann-Roch Formula

Context: Let X be a smooth projective (pro-)curve over an algebraically closed field k admitting a faithful action by a (pro-)finite group G , with quotient Y . Let \mathcal{L} be a G -equivariant invertible sheaf on X , meaning that for each $g \in G$, there is an isomorphism $\phi_g: g_*\mathcal{L} \rightarrow \mathcal{L}$ satisfying an appropriate cocycle condition. (Such bundles can be identified with classes of G -invariant divisors on X .)

Then G acts on the cohomology groups $H^i(X, \mathcal{L})$ ($i = 0, 1$). Can define the Euler characteristic

$$\chi(\mathcal{L}) = [H^0(X, \mathcal{L})] - [H^1(X, \mathcal{L})]$$

in the ring $R_k(G)$ of virtual representations of G over k .

Can also define the “equivariant degree” $\deg_{\text{eq}} \mathcal{L} \in R_k(G)$, a virtual representation whose dimension is $\deg \mathcal{L}$.

Theorem (Borne, 2003) The equality

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \deg_{\text{eq}} \mathcal{L}$$

holds in $R_k(G)$.

Further, suppose the quotient map $X \rightarrow Y$ is ramified over the set Y_{ram} . Then

$\deg_{\text{eq}} \Omega_X = (2g_Y - 2 + \#Y_{\text{ram}}) [k[G]] - [X_{\text{ram}}]$,
 where X_{ram} is the permutation representation on the ramified points of X in $X \rightarrow Y$.

We apply the theorem to the data

$$\begin{array}{c} X(\infty) \\ \downarrow G := \text{GL}_2(\hat{\mathbf{Z}})/\pm 1 \\ X(1) \end{array} .$$

This map is ramified over the points 1728, 0, and ∞ , with inertia groups $\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle$, and $\begin{pmatrix} 1 & \hat{\mathbf{Z}} \\ 0 & 1 \end{pmatrix}$, respectively. We are interested in the canonical bundle $\Omega_{X(\infty)}$ because

$$H^0(X(\infty), \Omega_{X(\infty)}) = S_2(X(\infty), \mathbf{C}).$$

The Riemann-Roch theorem gives

$$2 \left([S_2(X(\infty))] - [H^0(\mathcal{O}_{X(\infty)})] \right) = \deg \Omega_{X(\infty)}$$

Meanwhile, a little calculation yields

$$\begin{aligned} \deg \Omega_{X(\infty)} = & \mathbf{C}[G] - \mathbf{C}[G/G_0] \\ & - \mathbf{C}[G/G_{1728}] - \mathbf{C}[G/G_\infty], \end{aligned}$$

where G_0 is the stabilizer of a point in $X(\infty)$ above 0, etc.

Conclusion: If $\rho = \{\rho_p\}$ is a collection of inertial WD representations which are almost all trivial and which satisfy $\prod_p \det \rho_p(\text{rec}(-1)) = 1$, then

$$\# \left\{ \begin{array}{l} \text{Modular AVs with} \\ \text{bad reduction type } \{\rho_p\} \end{array} \right\} = \frac{1}{12} \dim V_\rho + o(1),$$

where $V_\rho = \otimes_p \text{rec}(\rho_p)$. (The $o(1)$ is as $\dim V_\rho \rightarrow \infty$; up to twisting there are only finitely many ρ with bounded $\dim V_\rho$.)