The Action of SL_2 on cusp forms

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Abstract

Let $N > 1$. The space $S_k(\Gamma(N))$ of cusp forms on the full modular group of level N admits an action of the quotient $\mathrm{PSL}_2(\mathbf{Z})/\Gamma(N) = \mathrm{PSL}_2(\mathbf{Z}/NZ)$ when k is even. Using the equivariant Riemann-Roch formula, we will compute the action of $\mathrm{PSL}_2(\mathbf{Z}/N\mathbf{Z})$ on $S_k(\Gamma(N))$.

1 Generalities

Much of the following is adapted from [Bo]. See also...

Let $p\colon X \mapsto Y$ be a morphism of projective curves with action by a finite group G over an algebraically closed field K. A *G-equivariant line bundle* is a pair $(\mathcal{L}, \{\phi_g\})$ consisting of a line bundle $\mathcal{L}$ on X together with isomorphisms

$$\phi_g\colon g^*\mathcal{L} \to \mathcal{L}$$

for each $g \in G$, satisfying the cocycle condition

$$\phi_{gh} = h^*(\phi_g) \circ \phi_h.$$

We can recast the notion in terms of divisor classes as follows. The group $\operatorname{Div} X$ admits an action of G. If $\mathcal{L}$ is a G-equivariant line bundle, let s be a nonzero meromorphic section of $\mathcal{L}$. Then for each $g \in G$, the quotient $\phi_g(s(gx))/s(x)$ is a meromorphic function on X. By the cocycle condition, $g \mapsto \phi_g(s(gx))/s(x)$ is a cocycle ξ in $H^1(G, K(X)^*)$, which vanishes by Hilbert's Theorem 90. Therefore there is a meromorphic function $h \in K(X)^*$ which splits the ξ, meaning that

$$\frac{\phi_g(s(gx))}{s(x)} = \frac{h(g(x))}{h(x)}.$$

Thus $x \mapsto s(x)/h(x)$ is another meromorphic section whose divisor D is visibly G-invariant. The choice of another section s or another splitting of ξ replaces D by the divisor of an element in $K(X)^G = k(Y)$. The line bundle $\mathcal{L}$ can be recovered from D as the line bundle whose fibers are $\mathcal{L}(U) = \{f \in k(X)^* | \operatorname{div} f \geq D \text{ on } U\}$, with transition maps ϕ_g given by $\phi_g(f) = f \circ g$. Therefore the group of G-equivariant line bundles can be identified with the quotient $(\operatorname{Div} X)^G / \operatorname{Div} K(Y)^*$.

For each $k \geq 0$, the line bundle of differentials Ω_X^k is a G-equivariant line bundle. In the case of $k = 0$, $\Omega_X^0 = \mathcal{O}_X$ has associated divisor class 0.

In this context, the sheaf cohomology groups $H^i(G, \mathcal{L})$ admit an action of G, $i = 0, 1$. Let $R_k(G)$ be the representation ring of G. Then we can define the *equivariant Euler characteristic*

$$\chi_{\mathrm{eq}}(\mathcal{L}) = \left[H^0(X, \mathcal{L})\right] - \left[H^1(X, \mathcal{L})\right].$$

The *equivariant degree* is defined as follows: Let D be the G-invariant divisor attached to $\mathcal{L}$. First assume D is a multiple of an orbit $\sum_{\sigma \in G/G_P} \sigma P$ with multiplicity μ_P, where G_P is the decomposition group at $P \in X$. The group G_P acts on the tangent space $T_P X$ through a character ψ_X with values in k^*. Then define

$$\deg_{\mathrm{eq}} D = \begin{cases} \sum_{c=1}^{\mu_P} \operatorname{Ind}_{G_P}^G \psi_P^{-c}, & \mu_P > 0, \\ 0, & \mu_P = 0, \\ -\sum_{c=0}^{-(\mu_P+1)} \operatorname{Ind}_{G_P}^G \psi_P^{c}, & \mu_P < 0. \end{cases}$$

Extend $\deg_{\mathrm{eq}}$ to general D in such a way that $\deg_{\mathrm{eq}}(D + D') = \deg_{\mathrm{eq}} D + \deg_{\mathrm{eq}} D'$ when D and D' have disjoint support.

The association $\deg_{\mathrm{eq}} : (\operatorname{Div} X)^G \to R_k(G)$ is *not* a linear map. However, the composition of $\deg_{\mathrm{eq}}$ with the dimension map $R_k(G) \to \mathbf{Z}$ is the usual degree map on divisors. Further, the equivariant degree vanishes on the group $\operatorname{Div} f^*(K(Y)^*)$ of principal divisors, so it really is well-defined on G-equivariant line bundles.

We are now ready to state the

Theorem 1. *In the ring $R_k(G)$:*

$$\chi_{eq}(\mathcal{L}) = \chi_{eq}(\mathcal{O}_X) + \deg_{eq}(\mathcal{L}).$$

2 Application To Spaces of Cusp Forms

Here, we follow [Sh]. The map of modular curves $f\colon X(N) \to X(1)$ (the latter being the projective j-line) plays the role of $X \to Y$ in the previous section, with $G = \mathrm{PSL}_2(\mathbf{Z}/N\mathbf{Z})$. We assume $N > 1$, so that $\Gamma(N)$ has no elliptic points. The map f is ramified over the points $j = 1728, 0, \infty$ with inertia groups (conjugate to) $G_{1728}, G_0, G_\infty \subset G$ having generators $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of order $2, 3, N$, respectively. The $\omega_1 \in H^0(X(1), \Omega_{X(1)})$ be a nonzero meromorphic differential with divisor $-2S$, for a point $S \in X(1) \setminus \{0, 1728, \infty\}$. Then the pullback $\omega = f^*(\omega_1)$ has divisor

$$\operatorname{div} \omega = -2 \sum_{g \in G} gR + \sum_{g \in G/G_0} gP_0 + 2 \sum_{g \in G/G_{1728}} gP_{1728} + (N-1) \sum_{g \in G/G_\infty} gP_\infty,$$

where R is in the preimage of S and P_i is fixed by G_i for $i \in \{0, 1728, \infty\}$.

Let $A_k(N)$ be the space of automorphic forms of weight k: these are meromorphic functions F on the upper half-plane with $F|_k\gamma = F$ for each $\gamma \in \Gamma(N)$. Then $A_k(N)$ is a one-dimensional vector space over $\mathbf{C}(X(N))$. Recall that k even. We choose a basis vector F by setting $\omega^{k/2} = F(z)dz^{k/2}$. Then the divisor of F is $\operatorname{div} F = k \operatorname{div} \omega + (k/2) \sum_{g \in G/G_\infty} gP_\infty$. (Indeed, the coordinate in the neighborhood of P_∞ is $q = \exp(2\pi i z/N)$, whereby $dz = N dq/2\pi i q$ has a simple pole at P_∞ and therefore at all of its G-translates.) Then

$$\begin{aligned} S_k(\Gamma(N)) &= \left\{ F' \in A_k(N) \middle| \operatorname{div} F' \geq \sum_{g \in G/G_\infty} gP_\infty \right\} \\ &\xrightarrow{\sim} \left\{ f \in \mathbf{C}(X(N)) \middle| \operatorname{div} f \geq -\operatorname{div} F + \sum_{g \in G/G_\infty} gP_\infty \right\} \end{aligned}$$

via the map $F' \mapsto f = F'/F$. Therefore $S_k(\Gamma(N)) \cong H^0(X(N), \mathcal{L}_k)$, where $\mathcal{L}_k$ is the line bundle with G-invariant divisor

$$-k \sum_{g \in G} gR + \frac{k}{2} \sum_{g \in G/G_{1728}} gP_{1728} + k \sum_{g \in G/G_0} gP_0 + \left(\frac{k}{2}(N-1) - 1 \right) \sum_{g \in G/G_\infty} gP_\infty.$$

For each $i \in \{0, 1728, \infty\}$, the character ψ_{G_i} is the one taking the generator of G_i to $\exp(2\pi i/\#G_i)$. Let $\Psi_{i,j} = \operatorname{Ind}_{G_i}^{G} \psi_{G_i}^{-j}$ for j taken modulo $\#G$.

Note that $\sum_{j=0}^{\#G_i} \Psi_{i,j} = \mathbf{C}G$, the regular representation of G. The equivariant degree of $\mathcal{L}_k$ is roughly $-k + \frac{k}{4} + \frac{k}{6} + \frac{k}{2N}(N-1) = k\left(\frac{1}{12} - \frac{1}{2N}\right)$ copies of $\mathbf{C}G$ plus an error term whose dimension can be bounded independently of k. The precise statement is

Theorem 2.

$$\deg_{eq} \mathcal{L}_k = \left(\left\lfloor \frac{k}{12} \right\rfloor - \left\lfloor \frac{k}{2N} \right\rfloor \right) \mathbf{C}G + \varepsilon_{elliptic} + \varepsilon_\infty,$$

where $\varepsilon_{elliptic}$ *is the sum of terms*

$$\begin{cases} 0, & k \equiv 0 \pmod{12} \\ \Psi_{1728,1} - \Psi_{0,0}, & k \equiv 2 \pmod{12} \\ \Psi_{0,1}, & k \equiv 4 \pmod{12} \\ \Psi_{1728,1}, & k \equiv 6 \pmod{12} \\ \Psi_{0,1} + \Psi_{0,2}, & k \equiv 8 \pmod{12} \\ \Psi_{1728,1} + \Psi_{0,1}, & k \equiv 10 \pmod{12} \end{cases}$$

and $\varepsilon_\infty = -\sum_{j=0}^{k/2 \bmod N} \Psi_{\infty,-j}$.

Note that the error term $\varepsilon = \varepsilon_{\text{elliptic}} + \varepsilon_\infty$ only depends on the image of k in $(\mathbf{Z}/12\mathbf{Z}) \times (\mathbf{Z}/N\mathbf{Z})$. For $k > 2$, the line bundle $\mathcal{L}_k$ has vanishing H^1, so that the Riemann-Roch formula gives

$$[S_k(\Gamma(N))] = \deg_{\text{eq}} \mathcal{L}_k + \chi_{\text{eq}}(\mathcal{O}_{X(N)}),\ k > 2.$$

It follows immediately that

Corollary 1. *Let $N > 1$ be fixed and let π be an irreducible complex representation of* $\mathrm{PSL}_2(\mathbf{Z}/NZ)$. *Let $\mu_\pi(k)$ be the multiplicity of π in $S_k(\Gamma(N))$. Then as $k \to \infty$, $\mu_\pi(k) = \frac{N-6}{12N} k \dim \pi + O(1)$.*

3 Galois structure of differentials

When $k = 2$, $\mathcal{L}_k = \Omega^1_{X(N)}$ and the Riemann-Roch formula gives $\chi(\Omega^1_{X(N)}) = \chi(\mathcal{O}_{X(N)} + \deg_{\text{eq}}(\Omega^1_{X(N)})$. As Ω^1 and $\mathcal{O}_{X(N)}$ are dual line bundles, their Euler characteristics are inverse to each other, so that

$$2\chi(\Omega^1_{X(N)}) = 2\left([S_2(\Gamma(N))] - [1]\right) = \deg_{\text{eq}}\left(\Omega^1_{X(N)}\right).$$

On the other hand, a calculation using Theorem 1 in [Kö] shows that

$$\deg_{\mathrm{eq}}\left(\Omega^1_{X(N)}\right) = \mathbf{C}G - \Phi_{0,0} - \Phi_{1728,0} - \Phi_{\infty,0}.$$

This, together with Theorem 2, gives the structure of $S_k(\Gamma(N))$ for each even $k \geq 2$.

References

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