MATH 113 Midterm 1 – Fall 2003

Give complete solutions/proofs to the following problems about group theory. You have 80 minutes. The exam total is 45 marks.

1. Define or state the following [10 marks total]:
   (i) Normal subgroup N of G. [2 marks]
   (ii) Lagrange's theorem. [2 marks]
   (iii) The fundamental homomorphism theorem. [3 marks]
   (iv) Group action G on a set X. [3 marks]

2. Prove that if A and B are normal subgroups of a group G, then [6 marks total]:

(i) $A \cap B$ is a subgroup. [4 marks]

(ii) $A \cap B$ is a normal subgroup. [2 marks]

3. Prove that if G is a group of even order $2n$, then [7 marks total]
   (i) it contains an element $a \neq e$ such that $a^2 = e$. [4 marks]
   (ii) if furthermore n is odd, then the element a in (i) is unique. (Hint, if a is not unique, construct a group of order 4) [3 marks]

4. Show that a group with at least two elements but with no proper nontrivial subgroups must be finite and of prime order. [5 marks]

5. Let H be a subgroup of G and let it act on $X = G/H$, the set of left cosets of H in G, by left multiplication, i.e., $h \star (gH) = (hg)H$ for all $h \in H$ and $g \in G$. Prove that the stabilizer of a specific coset $gH \in G/H$, i.e., all $h \in H$ such that $h \star gH = gH$, is the subgroup gHg^{-1} of G. [7 marks]

6. Let G be a group. Let $\mathrm{Aut}(G)$ be the set of one-to-one, onto group homomorphisms from $G \to G$. [10 marks total]

(i) Show $\mathrm{Aut}(G)$ is a subgroup of S_G, the group of permutations of the elements of G. [3 marks]

(ii) Define $\Phi : G \to S_G$ by $g \mapsto (h \mapsto ghg^{-1})$. Show that each $\Phi(g)$, $g \in G$ is actually in $\mathrm{Aut}(G)$. [2 marks]

(iii) Φ is a group homomorphism. What is its kernel? [2 marks]

(iv) Show that the image of Φ is a normal subgroup of $\mathrm{Aut}(G)$. [3 marks]