

Midterm Exam #2 - Solutions

1. (a) (5 points) True or False? $x^3 + 15x^2 + 17x + 29$ is irreducible in $\mathbb{Q}[x]$.
 TRUE: using the change of coefficients morphism $\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, we get the new polynomial $f = x^3 + 2x + 2$. Note $\deg(f) = 3$, so we can check irreducibility in $\mathbb{Z}/3\mathbb{Z}[x]$ by testing for roots: $f(0) = f(1) = f(2) = 2$, so there are no roots, and f is irreducible.
- (b) (5 points) Let G be the symmetry group of the tetrahedron. How many elements of order 3 are there in G ?
 There are 8 such elements. Through the center of any face, there is a rotation axis - there are two rotations of order 3 (and the identity) about this axis. The remaining 4 symmetries of the tetrahedron are the identity and three elements of order 2.
- (c) (5 points) Give a complete description of the group D_7 .
 D_7 is the symmetry group of the regular heptagon (7-gon). It's generators are σ and τ , with relations $\sigma^7 = \tau^2 = e$ and $\sigma\tau = \tau\sigma^{-1}$.
- (d) (5 points)
- i. True or False? Every integral domain is a field.
 False: \mathbb{Z} is an integral domain and not a field.
 - ii. True or False? Every field is an integral domain.
 True: Suppose a and b are elements of a field F . If $ab = 0$, and $a \neq 0$, then a has an inverse $a^{-1} \in F$, and $b = a^{-1}ab = a^{-1}0 = 0$. So there are no zero-divisors in F .
- (e) (20 points) For parts (i) to (v), NO explanation is required. Give an example of:
 (There are many possible solutions)
- i. a non-commutative ring - $M_{2 \times 2}(\mathbb{R})$
 - ii. a ring R that contains exactly 4 invertible elements - $\mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}$
 - iii. two non-isomorphic rings that are isomorphic as groups - \mathbb{Z} and $2\mathbb{Z}$
 - iv. three non-isomorphic Principal Ideal Domains - $\mathbb{Z}, n\mathbb{Z}, \mathbb{Q}[x], \mathbb{Z}/p\mathbb{Z}[x], \mathbb{R}[x]$
 - v. five non-isomorphic groups of order 24
 $\mathbb{Z}/24\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}/3\mathbb{Z}, D_{12}, S_4$

2. Let $\phi : R \rightarrow S$ be a homomorphism of rings.

- (a) (10 points) Show that the kernel of ϕ is an ideal of R .
 (A)+closure: Let $a, b \in \ker(\phi)$. Then $\phi(a+b) = \phi(a) + \phi(b) = 0 + 0 = 0$, so $a+b \in \ker(\phi)$.
 (B)associativity is free. (C)identity: $\phi(0) = 0$, so $0 \in \ker(\phi)$. (D)inverses: if $a \in \ker(\phi)$, then $\phi(-a) = -\phi(a) = 0$, so $-a \in \ker(\phi)$. So the kernel of ϕ is an additive subgroup of R . Now let $r \in R$ and $k \in \ker(\phi)$. Then $\phi(rk) = \phi(r)\phi(k) = 0\phi(k) = 0$ so $rk \in \ker(\phi)$. Similarly $kr \in \ker(\phi)$. So $\ker(\phi)$ is extra-super closed under \cdot and thus an ideal
- (b) (10 points) Suppose ϕ is invertible and $\psi : S \rightarrow R$ is its inverse. Show that ψ is a ring homomorphism.
 Suppose $x, y \in S$. Since ϕ is invertible, it is onto, so we can write $x = \phi(a)$ and $y = \phi(b)$ for some $a, b \in R$. Since ϕ and ψ are inverses, we also have $\psi(x) = a$ and $\psi(y) = b$. Then, using the homomorphism properties of ϕ , we get

$$\psi(xy) = \psi(\phi(a)\phi(b)) = \psi(\phi(ab)) = ab = \psi(x)\psi(y)$$

$$\psi(x+y) = \psi(\phi(a) + \phi(b)) = \psi(\phi(a+b)) = a+b = \psi(x) + \psi(y)$$

So ψ is a ring homomorphism.

3. Consider the action of a group G on a set S .

(a) (5 points) Define the terms 'Orbit' and 'Stabilizer'.

Denote the action of g on s by gs . Then for any $s \in S$, we have $\text{Orb}(s) = \{gs | g \in G\}$ and $\text{Stab}(s) = \{g \in G | gs = s\}$.

(b) (15 points) Let $|G| = p^n m$, where p is prime and $(p, m) = 1$. Let H be a Sylow- p -subgroup of G , and let H act on G/H . Show that there is at least one orbit of size 1.

If the action is by multiplication, then the solution is trivial since for any $h \in H$ we have $h(eH) = hH = eH$. Even if we don't know the action, we can solve the problem. Since H is a Sylow- p -subgroup, $|H| = p^n$, and so $|G/H| = m$ is not divisible by p . But the set G/H is partitioned by the orbits of the action, so we can write

$$m = |G/H| = \sum_s |\text{Orb}(s)| = \sum_s |H/\text{Stab}(s)|$$

Since $\text{Stab}(s)$ is always a subgroup of the group H that's acting, the terms in the sum are all divisors of $|H|$, namely they are powers of p . But if they are all higher powers of p , then the right hand side is divisible by p , which cannot be since p doesn't divide m . So at least one of the terms in the right hand sum must be $p^0 = 1$. Thus there is an orbit of size 1.

4. Consider the ring $R = \mathbb{Z} \times \mathbb{Z}$.

(a) (7 points) Let H be the cyclic subgroup generated by the element $(1, 3) \in R$. Show that there is an isomorphism of groups $R/H \cong \mathbb{Z}$.

Consider the map $\phi : R \rightarrow \mathbb{Z}$ given by $\phi((a, b)) = 3a - b$. This is a homomorphism of groups, since $\phi((a, b) + (c, d)) = \phi((a+c, b+d)) = 3(a+c) - (b+d) = 3a - b + 3c - d = \phi((a, b)) + \phi((c, d))$. The kernel of ϕ is $\{(a, b) \in \mathbb{Z} \times \mathbb{Z} | b = 3a\} = \{(a, 3a) \in \mathbb{Z} \times \mathbb{Z} | a \in \mathbb{Z}\} = H$. ϕ is onto since for any $k \in \mathbb{Z}$ we have $\phi((k, 2k)) = 3(k) - (2k) = k$, so by the first homomorphism theorem for groups, we get an isomorphism $R/H \cong \mathbb{Z}$.

(b) (5 points) Is the isomorphism from the first part an isomorphism of rings? No. H is not an ideal of R , since, for example $(1, 3) \in H$ and $(2, 1) \in R$, but $(2, 1)(1, 3) = (2, 3) \notin H$. So there's no natural ring structure on the quotient, and it doesn't make sense to ask about ring homomorphisms.

(c) (8 points) Now let I be the ideal generated by the element $(1, 3) \in R$. Show that I is a maximal ideal.

Define a map $\phi : R \rightarrow \mathbb{Z}/3\mathbb{Z}$ by $\phi((a, b)) = [b]_3$. It is easy to check that this map is a ring homomorphism onto $\mathbb{Z}/3\mathbb{Z}$, and that the kernel of the map is exactly $\{(a, 3b) | a, b \in \mathbb{Z}\} = \{(a, b)(1, 3) | (a, b) \in \mathbb{Z} \times \mathbb{Z}\} = \langle (1, 3) \rangle$. So by the first homomorphism theorem for rings, we get an isomorphism $(\mathbb{Z} \times \mathbb{Z})/I \cong \mathbb{Z}/3\mathbb{Z}$. Since the factor ring R/I is a field (since 3 is prime), I is a maximal ideal of R .

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