Math113 Problems: Homework 4 (Due Wednesday, July 18)

(1) Remember that the "dihedral group of order $2n$", D_n, is the group with generators σ and τ, with relations $\sigma^n = e$, $\tau^2 = e$, and $\sigma\tau = \tau\sigma^{n-1} = \tau\sigma^{-1}$. In other words, the elements of D_n are:

$$\text{elements: } \left\{ \begin{array}{ccccc} e & \sigma & \sigma^2 & \dots & \sigma^{n-1} \\ \tau & \tau\sigma & \tau\sigma^2 & \dots & \tau\sigma^{n-1} \end{array} \right\} \quad \text{relations: } \begin{array}{l} \sigma^n = e, \quad \tau^2 = e \\ \sigma\tau = \tau\sigma^{n-1} \end{array}$$

  (a) Explain in a few sentences what it means to say that "D_n is the symmetry group of the regular n-gon".
  (b) Show that $D_3 \simeq S_3$. Is $D_4 \simeq S_4$?
  (c) Show that for any n, we have $\langle \sigma \rangle \lhd D_n$
  (d) What is the center of D_n? [Hint: consider odd and even n separately.]
  (e) Find a subgroup of D_6 that is isomorphic to D_3, and interpret this isomorphism geometrically (in terms of symmetries).

(2) (a) Write down all the elements of S_4.
  (b) In S_4, how many elements are there of order 2? of order 3? of order 4?
  (c) It is a fact that the symmetry group of the cube is (isomorphic to) S_4. Find and describe "enough" symmetries of each order in this group.

(3) Suppose we are given an action of a group G on a set S. Recall that the stabilizer of an element $s \in S$ is defined as $Stab(s) = \{g \in G \,|gs = s\}$. Show that $Stab(s)$ is a subgroup of G.

(4) Let G be a group, and consider the map $G \times G \to G$ given by $(g, x) \mapsto gxg^{-1}$. Show that this maps defines an action of G on itself. We describe this scenario by saying "G acts on itself by conjugation". What is this action in the case where G is abelian?

(5) Let S_4 act on itself by conjugation. Find the orbits of the action. You may find one of the results from HW1 useful.

(6) You have three different colors of beads, and wish to make a (circular) necklace of 12 (evenly spaced) beads. How many distinct such necklaces are there?