MATH113 PROBLEMS: HOMEWORK 3 (DUE MONDAY, JULY 9)

(1) **§2.4 #32** Let $G$ be a finite group, and $H$ a subgroup of $G$. Fix an element $a \in G$. Let $f(a)$ be the smallest positive integer such that $a^{f(a)} \in H$. Show that $f(a)|o(a)$.

(2) **§2.5 #1** Determine in each of the parts if the given mapping is a homomorphism. If so, identify its kernel and whether or not the mapping is 1-1 or onto.
   (a) $G = \mathbb{Z}, +$, $G' = \mathbb{Z}/n\mathbb{Z}, +_n$, and $\phi(a) = [a]_n$, for $a \in \mathbb{Z}$
   (b) $G$ a group, $\phi : G \to G$ defined by $\phi(a) = a^{-1}$ for $a \in G$.
   (c) Same as in (b), but $G$ is an abelian group.
   (d) $G = \mathbb{R}^\times$, and $G' = \{1, -1\}$, with the operations of multiplication, and $\phi(r) = r/|r|$ for $r \in \mathbb{R}^\times$

(3) For each of the following pairs of groups, $G$ and $G'$, either write down an isomorphism $G \to G'$, or explain why none exist (by finding a 'structural difference' between them).
   (a) $G = \mathbb{Z}$ and $G' = \mathbb{Z}/n\mathbb{Z}$
   (b) $G = \mathbb{Z}$ and $G' = \mathbb{Q}$
   (c) $G = U_{14}$ and $G' = \mathbb{Z}/6\mathbb{Z}$
   (d) $G = S_3$ and $G' = \mathbb{Z}/6\mathbb{Z}$
   (e) $G = \mathbb{Q}$ and $G' = \mathbb{Q}^\times$
   (f) $G = \mathbb{R}$ and $G' = \mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$ (with the operation of multiplication)

(4) **§2.5 #7** Let $\phi : G \to G'$ be a homomorphism of groups. Prove that $\phi$ is an injection if and only if $\ker(\phi) = e$.

(5) **§2.5 #15** Let $G$ be a group, $N \lhd G$, and $\phi : G \to G'$ a homomorphism of $G$ *onto* $G'$. Prove that the image, $\phi(N)$, of $N$ is a normal subgroup of $G'$.

(6) **§2.5 #23** Let $G$ be a group in which all subgroups of $G$ are normal in $G$. If $a, b \in G$, show that $ba = a^j b$ for some $j \in \mathbb{Z}$.

(7) For a group $G$, we define the *center of* $G$, $Z(G) = \{g \in G \mid ga = ag\,, \forall a \in G\}$.
   (a) **§2.5 # 12** Show that for any group $G$, $Z(G) \lhd G$.
   (b) **§2.6 #11** Show that if $G/Z(G)$ is cyclic, then $G$ is abelian.

(8) Let $G = GL_n(\mathbb{R})$ (recall, this is the group of $n \times n$ matrices with non-zero determinant, and the operation of matrix multiplication). Let $H = \{A \in G \mid \det(A) = 1\}$.
   (a) Show that $H \lhd G$
   (b) Show that $G/H \simeq \mathbb{R}^\times$

(9) **§2.7 #5** Let $G$ be a group, $H$ a subgroup of $G$, and $N \lhd G$.
   Define $HN = \{hn \mid h \in H, n \in N\}$. Prove that:
   (a) $H \cap N \lhd H$
   (b) $HN$ is a subgroup of $G$
   (c) $N \subset HN$ and $N \lhd HN$
   (d) $(HN/N) \simeq H/(H \cap N)$