Radial Solutions to an Elliptic Boundary Valued Problem

Ivan Ventura

Prof. Alfonso Castro, Advisor

Prof. Jon Jacobsen, Reader

May, 2007

HARVEY MUDD
COLLEGE
Department of Mathematics

Abstract

In this paper we prove that

$$\begin{cases} div(|\mathbf{x}|^{\beta}\nabla u) + |\mathbf{x}|^{\alpha} f(u) = 0, & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

has infinitely many solutions when f is *superlinear* and grows *subcritically* for $u \geq 0$ and up to *critically* for u less than 0 with

$$1 < p < \tfrac{N+2\alpha-\beta+2}{N+\beta-2}, \quad N+\alpha > 0, \quad 1 < q \leq \tfrac{N+2\alpha-\beta+2}{N+\beta-2}, \quad \alpha < \beta < \alpha+1, \quad N > 3$$

We make extensive use of *Pohozaev* identities and *phase plane* and *energy* arguments.

Acknowledgments

I would like to thank Professor Alfonso Castro for his guidance and support throughout the entire thesis process. I would also like to thank Professor Andrew Bernoff for giving me advice to better my presentations on the thesis and Professor Jon Jacobsen for being my second reader. Also I would like to thank my parents for their unconditional support. Additionally I would like to thank the Harvey Mudd math department for knowledge and resources they provided me throughout this process. Finally thank you Claire Conneley for making the class files and printing posters that made my thesis experience so much easier.

Contents

Chapter 1

Introduction

1.1 Motivation

In the field of partial differential equations (PDEs) there are many different methods to help find and characterize their solutions. When dealing with a PDE defined on a radially symmetric set it is often advantageous to first consider the solutions which are radially symmetric. By finding and characterizing the radial solutions one can often determine the regularity, concentrated compactness, or other properties of the more general solutions to the PDEs. In this paper, we expand on the results of Chou and Geng (1996). Throughout this paper we will use several abbreviations for different terms. BVP will stand for boundary value problem while IVP will be an initial value problem. PDE will be a partial differential equation while ODE will be an ordinary differential equation.

1.2 Elliptic PDEs and Radial Solutions

In this paper, we are going to look at radial solutions of a PDE. A solution is considered a *radial* solution if its value is constant on the surface of every ball centered at the origin.

We will also need the notion of a super linear differential equation. Consider a partial differential equation of the form

$$\mathcal{L}u + f(u) = 0, \tag{1.1}$$

where $\mathcal{L}$ is some linear operator. We say that equation (1.1) is superlinear if

$$\lim_{|u|\to\infty} \frac{f(u)}{u} = \infty.$$

Superlinearity is helpful when trying to bound the energy of the system.
One of the more important aspects of working with partial differential equations comes from having to classify the different types of equations. For the most part PDEs can be classified in three different categories: Parabolic, Elliptic, and Hyperbolic, where each of these have their own special properties.
Since we are focusing only on elliptic PDEs we will not define the others. First of all an $n \times n$ matrix A is said to be *positive definite* if there exists some fixed $\alpha > 0$ such that $\langle Ax, x\rangle > \alpha|x|^2$ for all $x \in \mathbb{R}^N$. Now let us consider a PDE of the form

$$F(\mathbf{x}, u, u_{x_1}, u_{x_2} \ldots, u_{x_i x_j} \ldots) = 0 \quad 1 \leq i, j \leq N$$

We say such a PDE is elliptic if the matrix

$$\left(\frac{\partial F}{\partial u_{x_i x_j}}\right),$$

with $1 \leq i, j \leq N$, is positive definite. For example a generic two dimensional PDE $au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$, with a through g being functions of x and y, is said to be elliptic if $b^2 - ac = 0$ (example from p 57-58 of Garabedian (1964)).

1.3 Sobolev-Hardy Inequality

Throughout this paper there will be many references to the Sobolev-Hardy inequality. Sobolev inequalities usually refer to inequalities relating the norms or seminorms of a function in different spaces. A general Sobolev inequality looks like $\|u\|_{L^q(U)} \leq \|u\|_{W^{k,p}(U)}$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{N}$ and $U \subset \mathbb{R}^N$. This guarantees that a function will be in L^q if it is in $W^{k,p}(U)$. These inequalities help to determine when there are solutions to differential equations. A Sobolev inequality known as the Sobolev-Hardy inequality plays a large role in many of the problems this paper will discuss. Here is an example of a Sobolev-Hardy inequality:

$$\left(\int_{\mathbb{R}^N} |\mathbf{x}|^\alpha |u|^p \mathrm{dx}\right)^{1/p} \leq C \left(\int_{\mathbb{R}^N} |\mathbf{x}|^\beta |\nabla u|^2 \mathrm{dx}\right)^{1/2}.$$

If our value of p is small enough, this inequality will hold. We say p is equal to the *critical value* if it causes the Sobolev inequality to be an equality. More over the critical value is unique in the our equations. Similarly we will say that the value of p is *subcritical* (respectively *supercritical*) if it is less (respectively greater) than the critical value.

Chapter 2

Background

2.1 History of the Problem

Consider the ordianary differential equation

$$\begin{cases} u'' + f(u) = 0 \\ u(0) = u(\pi) = 0 \end{cases} \tag{2.1}$$

It is a well known classical result that if $f(u)$ is superlinear then there exists a $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ (2.1) has two solutions with exactly k zeroes on $[0, \pi]$. Though this result seems peculiar, it is easy to show why the k_0 is necessary (that is why there are not two solutions for every positive integer).

Example 2.1 *Consider when $f(u) = u^3 + 14u$. We know that $f'(u) = 3u^2 + 14 \geq 14$. This means that $\frac{f(u)}{u} \geq 14$ on $[0, \pi]$. With a little rearranging* (2.1) *becomes $u'' + \frac{f(u)}{u}u = 0$. But $u'' + \frac{f(u)}{u}u \geq u'' + 14u$. Notice that the solution to this ODE oscillates faster than the solutions to $u'' + 9u = 0$. Thus any solution to* (2.1) *with our chosen f must have at least 3 zeroes. Thus our $k_0 \geq 3$.*

A similar problem was visited by Struwe in Struwe (1981) where he discussed the solutions to

$$\begin{cases} \Delta u + f(u) = 0 & \text{on } \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{2.2}$$

where $f(u) \sim |u|^{p-1}u$ and $1 < p < \frac{N+2}{N-2}$. He found that this equation has infinitely many solutions. It is important to note that the exponent $\frac{N+2}{N-2}$ is the critical exponent for this equation. Later in Castro and Kurepa (1987) the same result was found for a much stronger $f(u)$, namely

$$f(u) = \begin{cases} |u|^{p-1}u & 1 < p < \frac{N}{N-1} \\ |u|^{q-1}u & q > \frac{N+2}{N-2} \end{cases}$$

Then in 2006 the results were expanded upon in Castro et al.. Here they found the solution for the same equation when

$$f(u) = \begin{cases} |u|^{p-1}u & 1 < p < \frac{N+2}{N-2} \\ |u|^{q-1}u & q > \frac{N+2}{N-2} \end{cases}$$

The result of Castro et al. is simply increasing the size of the positive exponent in Castro and Kurepa (1987). These two results imply the interesting fact, that the critical exponent seems to have a greater influence when $u > 0$ than when $u < 0$, since both papers choose $q > \frac{N+2}{N-2}$.

2.2 Positive solutions

Often times when working with the radial solutions it often helps to first try and find positive solutions. If $u(r) > 0$ for all r the energy is much easier to bound making many arguments easier to do. Let us consider an equation very similar to (2.2). It is known for $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) smooth and star shaped that

$$\begin{cases} \Delta u + u^{(N+2)/(N-2)} = 0, \quad u > 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has no solution. It is clear that the exponent of u plays a big role in the existence of solutions. The exponent, $\frac{N+2}{N-2}$ is once again the critical Sobolev-Hardy exponent so u^a where $a < \frac{N+2}{N-2}$ has infinitely many solutions, while $a > \frac{N+2}{N-2}$ has none. Interestingly it was shown in Brezis and Nirenberg (1983) that by simply adding λu, the equation

$$\begin{cases} \Delta u + u^{(N+2)/(N-2)} + \lambda u = 0, \quad u > 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

has a solution when $N \geq 4$ and $\lambda \in (0, \lambda_1)$ where λ_1 is the first eigenvalue of the Laplacian operator. They continued to show that a similar agrement works for different criterion if $N = 3$.

2.3 A Small Change in the Laplacian

Thus far we have discussed existence of solutions and results relating to equations like (2.2). Though their solutions form a very important class of functions, much of the work on them has already been done. Notice that we can think of $\Delta u = div(\nabla u)$. So we can easily weight the gradient of u with a function $a(x)$ to generalize the operator, $div(a(x)\nabla u)$. Since we are considering radial solutions it is best if $a(x)$ is itself a radial function, the simplest example being $|x|^\beta$. That is exactly what Chou and Geng (1996) and Catrina ended up doing. With a couple more radial weights to make the problem rounder they ended up studying the solutions to:

$$\begin{cases} \partial_i(|\mathbf{x}|^\beta \partial_i u) + |\mathbf{x}|^\alpha u^p + \lambda |x|^\sigma u = 0, & u > 0, \quad \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

with

$$p > 1, \quad N + \alpha > 0, \quad \frac{N+\alpha}{p+1} + 1 = \frac{N+\beta}{2}, \quad \frac{\beta}{2} \geq \frac{\alpha}{p+1}, \quad \sigma > \beta - 2.$$

There are several things to be noted about this equation. First of all in the discussion of these two papers we will be using the notation of Chou and Geng (1996) with two minor exceptions: N will be the dimension not n and p will be the critical exponent not $p-1$. Also note that they are working specifically with positive solutions, much like Brezis and Nirenberg (1983). Finally we must note that p is now the Sobolev critical exponent, but in the case of this problem that exponent is

$$p - 1 = \frac{2(N+\alpha)}{N+\beta-2} - 1 = \frac{N + 2\alpha - \beta + 2}{N + \beta - 2}.$$

which satisfies the Sobolev-Hardy inequality for this problem:

$$\left(\int_{\mathbb{R}^N} |\mathbf{x}|^\alpha |u|^p \mathrm{dx} \right)^{1/p} \leq C \left(\int_{\mathbb{R}^N} |\mathbf{x}|^\beta |\nabla u|^2 \mathrm{dx} \right)^{1/2} \tag{2.6}$$

Chou and Geng used this setup to prove that if λ_1 is the first eigenvalue of

$$\begin{cases} -\partial_i(|\mathbf{x}|^\beta \partial_i u) = \lambda |\mathbf{x}|^\sigma u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases}$$

then the following is true:

Theorem 2.1 *If the above holds with $\beta \leq 0$ then*

1. *If $N \geq 4 - 2\beta + \sigma$ there exists a solution for* (2.5) *for every $\lambda \in (0, \lambda_1)$.*
2. *If $N < 4 - 2\beta + \sigma$ then there exists a solution for* (2.5) *for every $\lambda \in (\mu, \lambda_1)$ where*

$$\mu = \lambda_1 - (C(\alpha, \beta)^2 \int_\Omega |\mathbf{x}|^\sigma \psi^p dx)^{-1}.$$

The $C(\alpha, \beta)$ is the best constant in equation (2.6)*.*

After proving the existence of radial solutions they extend their results to nonradial solutions of the same equation and prove that there exits non-radial positive solutions to the equation.

2.4 Methodology

For the class of problems involving the Laplacian, there are very specific tools that one can use in order to get the desired results. These tools though easy to use are often hard to invent from scratch. So as it turns out that the value in much of the literature is not in the result but in the process that is used to reach the result.
In the papers by Castro et al. and Castro and Kurepa (1987)they discussed radial solutions to the equation

$$\begin{cases} \Delta u + g(u) = q(x) & x \in \mathbb{R}^N \quad ||x|| \leq 1 \\ u(x) = 0 & for \quad ||x|| = 1 \end{cases} \tag{2.7}$$

Castro et al. considered the case where $q(x) = 0$ while Castro and Kurepa (1987) did a more general $q(x)$ but with weaker $f(u)$.
Though we will go into it in greater depth in Chapter 4, in Pohozaev (1965) the author discovers and demonstrates the derivation of a very useful identity that will from now on be refereed to as the Pohozaev identity. This identity is useful because it is positive when the exponent is subcritical, negative when super critical, and identically zero when its critical. This gives us a quantity with nice properties (sometimes referred to as the Pohozaev Energy). To begin Castro et al. found the Pohozaev identity for their equation. This allowed them to do an energy analysis to determine if the solution will have enough energy to make it to $u(1) = 0$. This Pohozaev-type identity was critical in bounding values of r for which the function u

could take different values. Like that paper we will utilize the same method to prove a variety of inequalities. In their paper, Castro and Kurepa (1987) and Castro and Lazer (1981) use similar methods to characterize solutions to their equation. The most important of these methods is their phase plane analysis, which is used to study the path parameterized by (u, u'), where u is a solution to their differential equation, in the u, u' plane. By cleverly defining the angle function, it is easy to reduce the problem to showing the angle function approaches infinity with the starting conditions. This idea was critical in allowing them to find the necessary and sufficient conditions for their equation to have a solution.

Chapter 3

Statement and Setup of the Problem

3.1 The Equation

In our work we will be considering the equation

$$\begin{cases} div(|\mathbf{x}|^{\beta}\nabla u) + |\mathbf{x}|^{\alpha} f(u) = 0, & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{3.1}$$

Where B is the unit ball as a subset of $\mathbb{R}^N$ and also

$$f(u) = \begin{cases} u^p & u \geq 0 \\ |u|^{q-1}u & u < 0 \end{cases} \tag{3.2}$$

where p and q satisfy the following conditions

$$1 < p < \tfrac{N+2\alpha-\beta+2}{N+\beta-2}, \quad N+\beta > 2, \quad 1 < q \leq \tfrac{N+2\alpha-\beta+2}{N+\beta-2}, \quad \alpha < \beta < \alpha+1, \quad N \geq 3 \tag{3.3}$$

Note that the Sobolev critical value is $\frac{N+2\alpha-\beta+2}{N+\beta-2}$. In this paper we will prove that there exists infinitely many sign changing solutions to this equation. Though at first glance it seems like equation (3.1) is a special case of equation (2.5), there is a subtle difference which is that (3.1) is not restricted to only positive solutions. Also, in (3.1), we are choosing the exponent to be subcritical if u is positive and up to critical if it is negative. This will make bounding the energy of a solution more difficult, but will provide a greater insight into the behavior of radial solutions of this PDE. Also since we already know when positive solutions exist, we can concentrate specifically on strictly sign changing solutions.

3.2 Conversion to Radial

Converting a PDE in $\mathbb{R}^N$ to radial coordinates can be delicate, I will outline the process for equation (3.1). Note first that the second term $|\mathbf{x}|^\alpha f(u) = r^\alpha f(u(r))$. So the complicated part will be the first term. There are a couple identities to note before we continue. I will use $u' = \frac{\partial u}{\partial r}$ for the remainder of the paper. In the same spirit u'' will be $\frac{\partial^2 u}{\partial r^2}$. So now notice that $r = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$ by definition. A simple derivative gives us that

$$\frac{\partial r}{\partial x_i} = \frac{2x_i}{2\sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}} = \frac{x_i}{r}. \tag{3.4}$$

This implies that

$$\frac{\partial u}{\partial x_i} = \frac{x_i}{r} u' \tag{3.5}$$

by the chain rule. This means that $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}) = (\frac{x_1}{r} u', \frac{x_2}{r} u', \ldots, \frac{x_N}{r} u')$. It follows that $|\mathbf{x}|^\beta \nabla u = (x_1 r^{\beta-1} u', \ldots, x_N r^{\beta-1} u')$. Thus we get

$$div(|\mathbf{x}|^\beta \nabla u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(x_i r^{\beta-1} u' \right).$$

If we simplify term by term we get that $\frac{\partial}{\partial x_i} \left(x_i r^{\beta-1} u' \right) = (\beta - 1) r^{\beta-2} \frac{\partial r}{\partial x_i} x_i u' + r^{\beta-1} u' + r^{\beta-1} x_i \frac{\partial u'}{\partial x_i}$. Using equations (3.4) and the u' form of (3.5) we get that $\frac{\partial}{\partial x_i} \left(x_i r^{\beta-1} u' \right) = (\beta - 1) r^{\beta-3} x_i^2 u' + r^{\beta-1} u' + r^{\beta-2} x_i^2 u''$. Taking these sums with $i = 1$ to N leaves us with $div(|\mathbf{x}|^\beta \nabla u) = (\beta - 1) r^{\beta-1} u' + N r^{\beta-1} u' + r^\beta u''$. Combining this with our previous result (3.1) becomes

$$\begin{cases} u'' + \frac{N+\beta-1}{r} u' + r^{\alpha-\beta} f(u) = 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{3.6}$$

Thus we have made our partial differential equation into a ordinary differential equation. Now the only issue with which we need to concern ourselves is how to address the boundary condition. Notice at this point the equation is not even a BVP since there is no value for u at 0. The conventional way of dealing with this problem is to arbitrarily give it behavior at 0 making it into an a full BVP. Then we make our new ODE from a BVP into an IVP. We will assume that our solution does not oscillate quickly (like $\sin(\frac{1}{x})$) near 0. Since we know that it is radial we know there are two possibilities for u at the origin. Either $u(0) = d$ for some d or $u \to \infty$ as $r \to 0$. Similarly u' can either be 0 or approach ∞ at 0. For this particular problem we will analyze what happens when $u(0) = d > 0$ and $u'(0) = 0$. Thus (3.6) becomes

$$\begin{cases} u'' + \frac{N+\beta-1}{r}u' + r^{\alpha-\beta}f(u) = 0 & r \in [0,1] \\ u(0) = d & u'(0) = 0 \end{cases} \tag{3.7}$$

There are many reasons to prefer an IVP to a BVP, the biggest being the existence of solutions. As the author proves in Hurewicz (1958), any system of the form $y' = f(x,y)$ where $f(x,y)$ is continuous and Lipzschitzian in y has a unique solution with $u(0) = d$. Since our 2nd order ODE can be written as a system of first order ODEs with the condition met (since $p, q > 1$), we know that there exists a unique solution to our IVP. Also, we chose the simplest two starting conditions since they are the only pair that assume the initial energy is finite. Notice that the energy of the system is

$$E(r) = \frac{(u'(r))^2}{2} + r^\alpha F(u(r)), \tag{3.8}$$

where $F(u) = \int_0^u f(s)ds$. So if either u' or u are infinite so is $E(0)$.

Chapter 4

Pohozaev Identities

4.1 Bounding u'

We will now use calculus along with a few clever tricks to obtain a bound for u' on an interval $[0,r]$, where $u(s) > 0$ for all $s \in [0,r]$. If we multiply equation (3.7) by $r^{N-1+\beta}$ we get that $r^{N-1+\beta}u'' + (N-1+\beta)r^{N-2+\beta}u' + r^{\alpha+N-1}f(u) = (r^{N-1+\beta}u')' + r^{\alpha+N-1}f(u) = 0$. This gives us that $-(r^{N-1+\beta}u')' = r^{\alpha+N-1}f(u)$. Integrating the left side from 0 to r we get that

$$\begin{aligned} -\int_0^r (s^{N-1+\beta}u'(s))'\mathrm{ds} &= -(r^{N-1+\beta}u'(r) - 0^{N-1+\beta}u'(0)) \\ &= -r^{N-1+\beta}u'(r) \end{aligned}$$

since in (3.7) $u'(0) = 0$. This gives us that

$$-r^{N-1+\beta}u'(r) = \int_0^r s^{N+\alpha-1}f(u(s))\mathrm{ds} \tag{4.1}$$

We know there is an interval $[0,R] \subset [0,1]$ such that $u(r) > 0$ for all $r \in [0,R]$ since $u(0) > 0$ and u is continuous. Thus since $r, u(r)$ are positive for $r \in [0,R]$, (4.1) gives us that $-u'(r) > 0$. Hence $u'(r) < 0$ for all $r \in [0,R]$. Therefore 0 is a local maximum for u. This means that $0 < u(r) \leq d = u(0)$ implying that $f(u(r)) = u^p \leq d^p$ for $r \in [0,R]$. Thus we get that (4.1) implies the inequality

$$-r^{N-1+\beta}u'(r) \leq d^p \int_0^r s^{N+\alpha-1}\mathrm{ds} = \frac{d^p r^{N+\alpha}}{\alpha+N},$$

which becomes

$$-u'(r) \leq \frac{d^p r^{\alpha+1-\beta}}{\alpha+N}. \tag{4.2}$$

Since $u' < 0$ on $(0, R]$, we get that $u(r+\epsilon) < u(r)$ for small $\epsilon > 0$. Also note that $-u'(r) = |u'(r)|$ (it is positive). Since $u' < 0$ in this region and $f(u) < d^p$, we know that $u'' < 0$ from (3.7). Hence we can find $r_0 \in [0,1]$ such that $u(r_0) = \frac{d}{2}$. We can repeat the process that took (4.1) to (4.2), using $u(s) \geq \frac{d}{2}$ on $[0, r_0]$. This will yield

$$-u'(r) \geq \frac{d^p r^{\alpha+1-\beta}}{2^p(\alpha+N)}. \tag{4.3}$$

Notice that (4.3) tells us that $|u'| \to \infty$ as $d \to \infty$. So the function will fall faster as you increase d.
We can use the above equations to bound r_0. Consider integrating both sides of (4.2) from 0 to r_0 with respect to r. The left side becomes $u(0) - u(r_0) = \frac{d}{2}$, while the right becomes $\frac{d^p r_0^{\alpha+2-\beta}}{(N+\alpha)(\alpha+2-\beta)}$. When rearranged we get that

$$r_0^{\alpha+2-\beta} \geq \frac{d^{1-p}(N+\alpha)(\alpha+2-\beta)}{2}. \tag{4.4}$$

Using doing the same for (4.3) yields that

$$r_0^{\alpha+2-\beta} \leq \frac{d^{1-p}(N+\alpha)(\alpha+2-\beta)}{2^{1-p}}. \tag{4.5}$$

Notice that since $p > 1$ we know that $1 - p < 0$. This along with (4.5) implies that $r_0 \to 0$ as $d \to 0$.

4.2 Finding A Pohozaev Identity

Here we will use a process similar to Pohozaev (1965) to find a Pohozaev Identity (also eqreferred to as a Pohozaev energy). To do this we will perform some simple calculus on (3.7). First we will multiply (3.7) by $r^{N+\beta}u'$ to obtain:

$$r^{N+\beta}u'u'' + (N+\beta-1)r^{N-1+\beta}(u')^2 + r^{N+\alpha}f(u)u' = 0$$

Thus

$$r^{N+\beta}\left(\frac{(u')^2}{2}\right)' + (N+\beta-1)r^{N-1+\beta}(u')^2 + r^{N+\alpha}f(u)u' = 0.$$

Let F be a function such that $F' = f$ and $F(0) = 0$. If we integrate both sides on $[0, r]$ and use integration by parts we get that

$$\begin{gathered}\tfrac{r^{N+\beta}(u')^2}{2} - \int_0^r \left[(N+\beta)s^{N+\beta-1}\tfrac{(u')^2}{2} - (N-1+\beta)s^{N-1+\beta}(u')^2 \right] \mathrm{ds} \\ + r^{N+\alpha}F(u) - \int_0^r (N+\alpha)s^{N+\alpha-1}F(u)\mathrm{ds} \\ = 0\end{gathered}$$

$$\begin{aligned}\frac{r^{N+\beta}(u')^2}{2} - \int_0^r \left[N-1+\beta - \frac{N+\beta}{2} \right] s^{N+\beta-1}(u')^2 \mathrm{ds} \\ + r^{N+\alpha}F(u) - \int_0^r (N+\alpha)s^{N+\alpha-1}F(u)\mathrm{ds} = 0. \end{aligned} \tag{4.6}$$

Now we multiply (3.7) by $r^{N-1+\beta}u$ and integrate on $[0, r]$. This will yield a another equation

$$r^{N+\beta-1}uu' - \int_0^r s^{N+\beta-1}(u')^2\mathrm{ds} + \int_0^r s^{N+\alpha-1}uf(u)\mathrm{ds} = 0 \tag{4.7}$$

If we combine (4.6) and (4.7) by cancelling out their $\int_0^r s^{N+\beta-1}(u')^2\mathrm{ds}$ terms we are left with a Pohozaev type identity. Specifically, the identity is

$$\begin{aligned} r^{N+\beta}\tfrac{(u')^2}{2} + r^{N+\alpha}F(u) + \left((N+\beta-1) - \tfrac{N+\beta}{2} \right) r^{N+\beta-1}uu' \\ = \int_0^r s^{N+\alpha-1} \left[(N+\alpha)F(u(s)) - (N+\beta-1-\tfrac{N+\beta}{2})uf(u(s)) \right] \mathrm{ds} \end{aligned} \tag{4.8}$$

This identity is in general for any $f(u)$ that we choose in the original equation. For our specific equation we know that $f(u)$ is given by equation (3.2). If we apply this to (4.8), the right side becomes

$$\int_0^r s^{N+\alpha-1} \left[\frac{N+\alpha}{p+1} - \left(N+\beta-1-\frac{N+\beta}{2} \right) \right] u^{p+1}\mathrm{ds} \tag{4.9}$$

Consider the coefficient of this equation. Notice that if $p < \frac{N+2\alpha-\beta+2}{N+\beta-2}$ then we get that $\frac{N+\alpha}{p+1} - \left(N+\beta-1-\frac{N+\beta}{2} \right) > 0$. This means that the right side of (4.8) is always positive. Thus the left side must always be positive.
So the Pohozaev identity with our $f(u)$ substituted in becomes

$$\begin{aligned} H(r) = r^{N+\beta}\tfrac{(u')^2}{2} + \tfrac{r^{N+\alpha}}{p+1}u^{p+1} + \left(\tfrac{N+\beta}{2} - 1 \right) r^{N+\beta-1}uu' \\ = \int_0^r s^{N+\alpha-1} \left[\tfrac{N+\alpha}{p+1} - \left(\tfrac{N+\beta}{2} - 1 \right) \right] u^{p+1}\mathrm{ds} \end{aligned} \tag{4.10}$$

Since u can be thought of as a function of both r and d (i.e. $u(r, d)$) we will sometimes write $H(r, d)$ to emphasize this point.

4.3 Applying the bounds

We will now use sections 4.1 and 4.2 together to prove the following theorem

Theorem 4.1 *Given a real number M then exists $d_0(M)$ such that if $d \geq d_0(M)$ then*
$$H(r,d) \geq M$$
for all $r \geq r_0(d)$ (recall $u(r_0(d),d) = \frac{d}{2}$).

Proof: Thus we are proving that

$$\lim_{d\to\infty} H(r,d) = \infty \tag{4.11}$$

uniformly. It is sufficient to prove that

$$\lim_{d\to\infty} H(r_0,d) = \infty$$

is equivalent to equation (4.11). This is clear since

$$\begin{aligned} H(r,d) &= \int_0^r s^{N+\alpha-1}\left[\frac{N+\alpha}{p+1} - \left(\frac{N+\beta}{2} - 1\right)\right] u^{p+1} ds \\ &= H(r_0,d) + \int_{r_0}^r s^{N+\alpha-1}\left[\frac{N+\alpha}{p+1} - \left(\frac{N+\beta}{2} - 1\right)\right] u^{p+1} ds \end{aligned}$$

This is where the bounds are used. We know that on the interval $[0, r_0]$ that $u \geq \frac{d}{2}$. So this means $H(r)$ on $r \in [0, r_0]$

$$\begin{aligned} H(r_0,d) &\geq \left(\frac{d}{2}\right)^{p+1} \int_0^{r_0} s^{N+\alpha-1}\left[\frac{N+\alpha}{p+1} - \left(\frac{N+\beta}{2} - 1\right)\right] ds. \\ &= \left(\frac{d}{2}\right)^{p+1} \left[\frac{N+\alpha}{p+1} - \left(\frac{N+\beta}{2} - 1\right)\right] \frac{r_0^{N+\alpha}}{N+\alpha} \\ &\geq K d^{p+1} d^{\frac{(1-p)(N+\alpha)}{\alpha+2-\beta}} \end{aligned}$$

where $K > 0$ is some constant not dependent on d. Notice that the exponent of d is

$$\begin{aligned} p+1+\frac{(1-p)(N+\alpha)}{\alpha+2-\beta} &= \frac{(1-p)(N+\alpha)+(\alpha+2-\beta)(p+1)}{\alpha+2-\beta} \\ &= \frac{(2-\beta-N)p+N+2\alpha+2-\beta}{\alpha+2-\beta} \end{aligned} \tag{4.12}$$

The denominator is clearly positive. Also we know that $2-\beta-N<0$ (the choice of p had both numerator and denominator positive) so we get by substituting the limits of p found in (3.3)

$$\begin{aligned} N+2\alpha+2-\beta-(\beta+N-2)p &> N+2\alpha+2-\beta-(\beta+N-2)\left(\frac{N+2\alpha-\beta+2}{N+\beta-2}\right) \\ &= N+2\alpha+2-\beta-(N+2\alpha+2-\beta) \\ &= 0 \end{aligned} \tag{4.13}$$

This implies the numerator of (4.12) is positive. Thus as $d\to\infty$ we know that $H(r_0,d)\to\infty$. Thus the proof is complete.

Corollary 4.2 *There exists d_1 such that if $d>d_1$ for any $\hat{r}\geq 0$, if $u(\hat{r},d)=0$ then $u'(\hat{r},d)\neq 0$. In other words u and u' cannot be zero simultaneously be 0.*

Proof: Clearly $u(r)>0$ for $r\in[0,r_0]$. So the corollary holds on this interval. Consider $d_0(1)$ from theorem 4.2. Choose $d>d_0(1)$. We know that for all $\hat{r}\in[r_0,1]$, $H(\hat{R})>1$. From (4) we know that if $H(\hat{r},d)\neq 0$ then either $u(\hat{r},d)\neq 0$ or $u'(\hat{r},0)\neq 0$. QED.

Corollary 4.2 will prove to be quite important in chapter 5 since this means the curve carved out by u and u' on the phase plane will never hit the origin! With this and some more clever approximations we are close to proving that there are infinitely many sign-changing radial solutions to (3.1).

Chapter 5

Phase Plane Analysis

5.1 The Pruffer Transformation

Since by corollary 4.2 we have that u and u' will never both be 0 we can now begin the phase plane analysis. Here we shall consider the u, u' plane. We know that our solution traces out a path parametrized by r as $(u(r,d), u'(r,d))$. Following a similar method to Castro and Kurepa (1987) (which is better outlined in Castro and Lazer (1981)) we use a Pruffer transformation to reparameterize our curve into polar coordinates as

$$u(r,d) = \rho(r,d)\cos(\theta(r,d)) \quad \text{and} \quad u'(r,d) = -\rho(r,d)\sin(\theta(r,d)). \quad (5.1)$$

Contrary to the usual convention of polar coordinates, in a Pruffer transformation we will count the angle to be positive in the clockwise direction as can be noted by the negative in the expression for u' in (5.1). Though we are moving backwards through the angles we will still refer to the quadrants in the same fashion (i.e. the $u > 0$ $u' > 0$ quadrant is still the first quadrant and the fourth is when $u > 0$ and $u' < 0$). This leaves us with

$$(\rho(r,d))^2 = (u(r,d))^2 + (u'(r,d))^2. \quad (5.2)$$

For simplicity we will not include (r,d) (i.e. $u = u(r,d)$). Notice that $\rho(0,d) > 0$. So $\rho > 0$ for all r and d. It is hard to explicitly express θ as a function of u and u' since inverse trigonometric functions have limits in their range. We could say $\theta = -\arctan(\frac{u'}{u})$, but this would cause a problem if the curve ever crossed the u'-axis. Using the first part of (5.1) w get that

$$u' = \rho' \cos(\theta) - \rho \sin(\theta)\theta' \quad (5.3)$$

Using (5.2) we get $\rho' = \frac{uu'+u'u''}{\rho}$. As a result (5.3) becomes

$$\theta' = 1 + \frac{(uu' + u'u'')\cos(\theta)}{\rho^2 \sin(\theta)} \tag{5.4}$$

(5.4) works as a definition for θ' when $\sin(\theta) \neq 0$. Using the second part of (5.1) we get a similar result for θ' that does not hold when $\cos(\theta) = 0$. Surprisingly these two expressions agree whenever they are both defined. It is much simpler to say that have $\theta' = (-\arctan(\frac{u'}{u}))'$. This leaves us with

$$\theta' = -\frac{1}{1+\left(\frac{u'}{u}\right)^2}\left(\frac{u''u-(u')^2}{u^2}\right) = \frac{(u')^2 - u''u}{u^2+(u')^2} = \frac{(u')^2-u''u}{\rho^2} \tag{5.5}$$

We know from (3.7) that

$$u'' + \frac{N+\beta-1}{r}u' + r^{\alpha-\beta}f(u) = 0$$

Hence

$$-u'' = \frac{N+\beta-1}{r}u' + r^{\alpha-\beta}f(u)$$

substituting this into (5.5) yields that

$$\theta' = \frac{(u')^2 + (r^{\alpha-\beta}f(u) + \frac{N+\beta-1}{r}u')u}{u^2+(u')^2}. \tag{5.6}$$

Substituting (5.1) into (5.6) we get that

$$\theta' = \sin^2(\theta) + \frac{N+\beta-1}{r}\sin(\theta)\cos(\theta) + \frac{r^{\alpha-\beta}}{\rho^2}uf(u) \tag{5.7}$$

Since u and $f(u)$ are always of the same sign we know by (5.7) we know that $\theta' \geq \frac{N+\beta-1}{r}\sin(\theta)\cos(\theta)$. Notice also that $\theta'(0,d) > 0$ and that when we are in the fourth quadrant both $\sin(\theta) > 0$ and $\cos(\theta) > 0$. Thus we can conclude that $\theta' > 0$ in the fourth quadrant, which proves that $\theta(r,d) > 0$. From (5.6) we obtain two conclusions:

1. If u=0 then $\theta' = 1$
2. If $u' = 0$ then $\theta' = \frac{f(u)}{u}r^{\alpha-\beta}$.

As a consequence of these two conclusions we obtain that

Lemma 5.1 *$\theta' > 0$ on both the u and u' axis, (i.e. the function θ will not "go backwards" across quadrants). More importantly $\theta(r,d) > 0$ for all $r \in [0,1]$.*

5.2 The Theorem

We can now bound θ' near both the u and u' axes since $f(u)$ is superlinear and a large d causes $\frac{f(u)}{u}$ to be large.

Lemma 5.2 *Given a real number M there exists $d_0(M)$ such that if $d \geq d_1(M)$ then*

$$\rho(r,d) \geq M$$

for all $r \in [0,1]$.

Proof: Let M be given. We break it into two cases.
Case 1: $r \in [0, r_0]$ then we choose $d_1 \geq \max\{2M, 2\}$. If $d > d_1$ we know that $u(r,d) \geq \frac{d_1}{2} > M$. This gives us that $\rho(r,d) = u^2 + (u')^2 \geq u^2 > M^2 \geq M$ if $M \geq 1$. If $M < 1$ then $\rho(r,d) \geq u^2 > 1 > M$. **Case 2:** If $r \in [r_0, 1]$. Let $d_0(M)$ be as in theorem 4.1 and let $d > d_0(M)$. We know that one term of $H(r,d)$ has to be greater than $\frac{M}{3}$. We will show the proof for one specific term as the other two are similar. Let us assume that it is the third term, yielding

$$\left(\frac{N+\beta-1}{2}\right) r^{N+\beta-1} u u' \geq \frac{M}{3} \tag{5.8}$$

Now we break this into two cases: First, if $N+\beta-1 \geq 0$ then $r^{N+\beta-1} \leq 1$; and second, if $N+\beta-1 < 0$ then $r^{N+\beta-1} \leq r_0^{N+\beta-1}$. Either way $r^{N+\beta-1}$ is bounded above by a positive constant, lets just call it k. Thus (5.8) becomes

$$\begin{aligned} \left(\frac{N+\beta-1}{2}\right) k u u' &\geq \frac{M}{3} \\ u u' &\geq \frac{2M}{3k(N+\beta-1)} \end{aligned}$$

Since $\frac{1}{2}\rho = \frac{1}{2}u^2 + \frac{1}{2}(u')^2 \geq uu'$, Case 2 is done. So let $d_2(M) > \max\{d_1(M), d_0(M)\}$ which completes the proof. In particular $\theta(r,d)$ is defined for all $r \in [0,1]$.
So finally we have

Theorem 5.3 *There are infinitely many sign-changing radial solutions to* (3.1).

Proof: It is sufficient to prove that

$$\lim_{d \to \infty} \theta(1,d) = \infty.$$

In order to do so, we show that, given any $\epsilon > 0$ there exists d_0 such that if $d \geq d_0$ and if $\theta(r_2, d) = \theta(r_1, d) + 2\pi$ then $|r_2 - r_1| < \epsilon$. If $x_0 > 0$ and $m(x_0) := \min\{\frac{f(x)}{x} : |x| \geq x_0\}$. Then by the superlinearity of f we know

$$m(x_0) \to \infty \quad \text{as} \quad x \to \infty \tag{5.9}$$

Let

$$\delta \in \left(0, \min\left\{\cos^{-1}(\sqrt{.95}), \frac{\sin^{-1}(.05)}{4(N+\beta-1)}, \frac{(.9)\epsilon}{8}\right\}\right) \tag{5.10}$$

Let $Q = \min\{p, q\}$ (p, q from (3.2)). We will now assume that $r \geq \frac{1}{4}$. We know that $\theta(\frac{1}{4}, d) > 0$ from lemma 5.1

Case 1: Assume that $\delta \in [\frac{k\pi}{2} + \delta, \frac{(k+2)\pi}{2} - \delta]$ for some $k \in \mathbb{Z}$ and odd. Using $r \geq \frac{1}{4}$, $\sin(\theta)\cos(\theta) \geq -\frac{1}{2}$, (5.7) and that $\sin(\delta)$ is a lower bound for $\sin(\theta)$ in such a region, we obtain that

$$\theta' \geq \sin^2(\theta) - 2(N+\beta-1) + r^{\alpha-\beta}\frac{uf(u)}{\rho^2}$$

since $r^{\alpha-\beta} \geq 1$ for $r \in [0,1]$, because $-1 < \alpha - \beta < 0$, we get

$$\theta' \geq -2(N+\beta-1) + \rho^{Q-1}\sin^{Q+1}(\delta) \tag{5.11}$$

$$\geq 0 \tag{5.12}$$

We know that $-2(N+\beta-1)$ is just a constant, and by lemma 5.2 we can choose d large enough so that (5.12) is true. But more importantly is that in a region such as $[\frac{k\pi}{2} + \delta, \frac{(k+2)\pi}{2} - \delta]$ the expression in (5.11) goes to ∞ as $d \to \infty$. Let r_a and r_b be such that $\theta(r_a, d) = \frac{k\pi}{2} + \delta$ and $\theta(r_b, d) = \frac{(k+2)\pi}{2} - \delta$. Thus there exists d_0 such that if $d > d_0$ then $|r_b - r_a| \leq \frac{\pi - 2 + \delta}{-2(N+\beta-1)+\rho^Q \sin^{Q+1}(\delta)} < \frac{\epsilon}{4}$ by lemma 5.2.

Case 2: Now assume that $\theta \in [\frac{k\pi}{2} - \delta, \frac{k\pi}{2} + \delta]$ where $k \in \mathbb{Z}$ and odd. By using (5.7) and $r \geq \frac{1}{4}$ we get that

$$\theta' \geq \sin^2\theta - 4(N+\beta-1)|\sin(\theta)||\cos(\theta)| \tag{5.13}$$

Because of our choice of δ we know that $\sin^2(\theta) \geq .95$ and $4(N+\beta-1)|\cos(\theta)| \leq .05$. This gives us that in this region $\theta' \geq .9$. Thus if $\theta(r_c, d) = \frac{k\pi}{2} - \delta$ and $\theta(r_d, d) = \frac{k\pi}{2} + \delta$, then $|r_d - r_c| \leq \frac{2\delta}{.9} < \frac{\epsilon}{4}$.

Now consider an r_1 and r_2 such that $\theta(r_2, d) = \theta(r_1, d) + 2\pi)$. We know that if $d > d_0$

$$|r_2 - r_1| = 2|r_a - r_b| + 2|r_c - r_d|$$

$$< \epsilon \tag{5.14}$$

Thus

$$\lim_{d \to \infty} \theta(1, d) = \infty$$

and the theorem is proved.

5.3 What's Next?

There are many different problems we can explore as a follow up to our results. For instance consider whether the same problem with $q > \frac{N+2\alpha-\beta+2}{N+\beta-2}$ has sign-changing (or any) radial solutions when the exponent is super critical when negative. The same general body of literature would help provide the tools to deal with this problem. Another possibility would be to continue in the same direction as Chou and Geng (1996) and add back in the $\lambda|x|^{\sigma}u$ term that was previously left out and try to once again find sign-changing solutions. This would be slightly more complicated since we would have to work with the eigenvalues (and functions) of equation. Finally, we could determine if there are any nonradial solutions to (3.1). This would require a different bag of tricks since these solutions are the most difficult to find and understand. Though these are some possibilities for further study, they are not the only ones, since combinations of these problems are also a possibility.

Bibliography

H Brezis and L Nirenberg. Positive solutions of nonlinear elliptic equations involving critical sobolev exponents. *Communs Pure Applied Math*, 36: 437477, 1983.

Alfonso Castro and Alexandra Kurepa. Infinitely many radial solutions to a superlinear dirichlet problem in a ball. *Proc. of the AMS*, 101:57, 1987.

Alfonso Castro and Alan C. Lazer. On periodic solutions of weakly coupled systems of differential equations. *Boll. Un. Mat. Ital.*, 18, 1981.

Alfonso Castro, John Kwon, and Chee Meng Tan. Infinitely many radial solutions for a sub-super critical dirichlet boundary value problem in a ball.

Florin Catrina. On a brezis-nirenberg type problem.

Kai-Seng Chou and Di Geng. On the critical dimension of a semilinear degenerate elliptic equation involving critical sobolev-hardy exponent. *Nonlinear Analysis, Theory, Methods & Applications*, 126:1965–1984, 1996.

Paul R. Garabedian. *Partial Differential Equations*. John Wiley and Sons inc., 1964.

Witold Hurewicz. *Lectures on Ordinary Differential Equations*. MIT Press, Cambridge, 1958.

S. I. Pohozaev. On the eigenfunctions of the equation $\delta u + f(u) = 0$. *Dokl. Akad. Nauk, SSSR*, 165:36, 1965.

Michael Struwe. Infinitely many solutions of superlinear boundary value problems with rotational symmetry. *Arch. Math. (Basel)*, 36:360, 1981.