A simple model of the integrable Hamiltonian equation[a)]

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A method of analysis of the infinite-dimensional Hamiltonian equations which avoids the introduction of the Bäcklund transformation or the use of the Lax equation is suggested. This analysis is based on the possibility of connecting in several ways the conservation laws of special Hamiltonian equations with their symmetries by using symplectic operators. It leads to a simple and sufficiently general model of integrable Hamiltonian equation, of which the Korteweg–de Vries equation, the modified Korteweg–de Vries equation, the nonlinear Schrödinger equation and the so-called Harry Dym equation turn out to be particular examples.

INTRODUCTION

The aim of this paper is to suggest a constructive approach to the infinite-dimensional integrable Hamiltonian equations, i.e., to the evolution equations possessing an infinite sequence of independent integrals which are in involution. The present analysis is based on the study of the connection between the symmetries and the conservation laws of the evolution equations. The main result is in showing a simple model of integrable Hamiltonian equation, of which the Korteweg–de Vries equation, the modified Korteweg–de Vries equation, the nonlinear Schrödinger equation, and the so-called Harry Dym equation turn out to be particular examples.

The analysis proceeds as follows. In Sec. 1 it is shown that any conservation law of an infinite-dimensional Hamiltonian equation is connected with a symmetry transformation. The study of the connection between the symmetries and the conservation laws of a given evolution equation is thus reduced to the study of its Hamiltonian structures. In Sec. 2 it is shown by an example that a given evolution equation may be endowed with different Hamiltonian structures. Each of them provides a way of connecting the conservation laws with the symmetries. Let us then consider an equation endowed with two of such connections, and let us use the former to associate the conservation laws with the symmetries and the latter to conversely associate the symmetries with the conservation laws. One is thus able to obtain a new conservation law from a given one. In Sec. 3 it is shown that highly ordered chains of integrals which are in involution can be constructed in this way for special twofold Hamiltonian equations. Such equations provide a simple model of integrable Hamiltonian equation. The examples of Sec. 5 seem to suggest that this model is not only conceptually simple but also effective in the applications.

1. SYMMETRIES AND CONSERVATION LAWS OF HAMILTONIAN EVOLUTION EQUATIONS

In this section an operator approach to the symmetries and to the conservation laws of any system of evolution equations

$$\partial_t u^A(\underline{x},t)=k^A(u^B,u^B_j,u^B_{jl},\cdots),\tag{1.1}$$

is suggested. The field functions $u^A(\underline{x},t)$ are supposed to be defined, at any instant of time, in a fixed region Ω of $\mathbb{R}^3$ and to vanish on the boundary of this region; the subscripts denote the partial derivatives of these functions with respect to the space coordinates x^j.

We set up the study of Eq. (1.1) into the linear space U of the field functions regarded as functions of the space coordinates only. Consequently, any n-tuple $u^A(\underline{x},t_0)$ will be simply denoted by $u(t_0)$ and will be referred to as a point of this space. The given evolution equations will be synthetized into the single operator equation

$$\partial_t u=\mathbf{K}(u),\tag{1.2}$$

where $\mathbf{K}$ is the formal differential operator defined by the functions $k^A(u^B,u^B_j,\cdots)$. The space U will be called the configuration space associated with the abstract evolution equation (1.2). The purpose of this operator approach is to suggest a simple way of extending to infinite-dimensional systems the geometric analysis developed for the classical Hamiltonian mechanics in the phase space.[1]

A. Symmetries

The object of the theory of the symmetries is the study of the manifold of the solutions of Eq. (1.2) in the configuration space U. We shall limit ourselves to a local study of such a manifold and so we only consider the infinitesimal symmetry transformations. They are the infinitesimal point transformations

$$\bar{u}=u+\epsilon\mathbf{S}(u),\tag{1.3}$$

of the configuration space into itself which map every solution again into a solution.[2] The operator $\mathbf{S}$ is called the generator of the symmetry mapping and is regarded as defining a "contravariant" vector field on the space U. The lines of this vector field are the orbits of the symmetry mapping.

The symmetry condition is readily obtained if one observes that for any solution $u(t)$ it is

$$\partial_t\bar{u}-\mathbf{K}(\bar{u})$$

$$\overset{(A2)}{=}\cancel{\partial_t u}+\epsilon\partial_t\mathbf{S}(u)-\mathbf{K}(\cancel{u})-\epsilon\mathbf{K}'_u\mathbf{S}(u)$$

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$$\overset{(A6)}{=} \epsilon[\mathbf{S}'_u \partial_t u - \mathbf{K}'_u \mathbf{S}(u)] \tag{1.4}$$

$$\overset{(1.2)}{=} \epsilon[\mathbf{S}'_u \mathbf{K}(u) - \mathbf{K}'_u \mathbf{S}(u)],$$

where $\mathbf{S}'_u$ is the Gateaux derivative of the operator $\mathbf{S}$, which is supposed to not explicitly depend on the time (see Appendix A). Hence, the *symmetry condition* is

$$\mathbf{S}'_u \mathbf{K}(u) - \mathbf{K}'_u \mathbf{S}(u) \doteq 0, \tag{1.5}$$

where the symbol $\doteq$ means that the equality is required to hold only for the solutions. For simplicity, however, in this paper we shall only consider the symmetry generators for which condition (1.5) is identically verified (the condition being *a fortiori* verified on the manifold of the solutions).

Equation (1.5) expresses the structural relation which connects the given equation to its symmetries, independently of the specific form either of the equation or of the symmetry mappings. It is a commutation relation, the left side being the commutator of the two nonlinear operators $\mathbf{S}$ and $\mathbf{K}$.[3] Therefore, the set of the generators of the symmetry mappings constitutes a Lie algebra. This means that if two of such generators $\mathbf{S}_j$ and $\mathbf{S}_k$ are composed according to the formula

$$[\mathbf{S}_j, \mathbf{S}_k](u) \equiv \mathbf{S}'_{ju}\mathbf{S}_k(u) - \mathbf{S}'_{ku}\mathbf{S}_j(u), \tag{1.6}$$

a third generator is obtained again.

B. Conservation laws

The study of the manifold of the solutions is the object also of the theory of the conservation laws, but the standpoint is different and, so to speak, dual to that of the theory of the symmetries.

Besides the configuration space U, one considers a second space V, put in duality with U by a convenient bilinear form $\langle v, u\rangle$,[4] and then one considers the operators $\mathbf{Q}: U \rightarrow V$ (see Fig. 1). Such operators may be regarded as defining the "covariant" vector fields on U. For such fields it is possible to introduce the concept of elementary circulation

$$\delta C \equiv \langle \mathbf{Q}(u), \delta u\rangle, \tag{1.7}$$

and so it is possible to consider the *conservative* covariant vector fields, for which the circulation does not depend on the line but only on the endpoints. As is known,[4] in order that the field be conservative it is necessary and sufficient that

$$\langle \mathbf{Q}'_u du, \delta u\rangle = \langle \mathbf{Q}'_u \delta u, du\rangle, \tag{1.8}$$

for any pair of variations du and δu of the field functions. The operators $\mathbf{Q}$ verifying this condition are called *potential operators*.[4] For such operators the circulation from a fixed point u_0 to any point u defines a functional $F[u]$, so that the elementary circulation is given by

$$\delta F[u] = \langle \mathbf{Q}(u), \delta u\rangle. \tag{1.9}$$

For this reason, the operator $\mathbf{Q}$ is also called the *gradient* of the functional F.

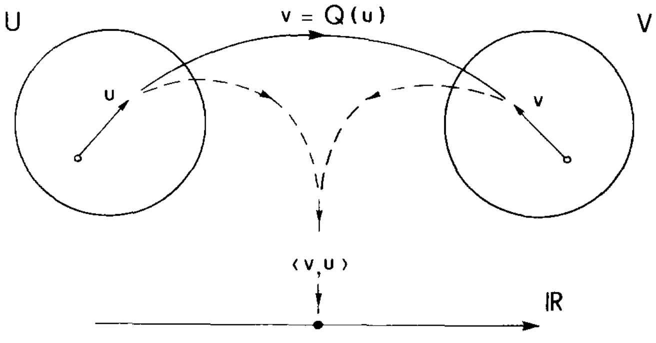

FIG. 1.

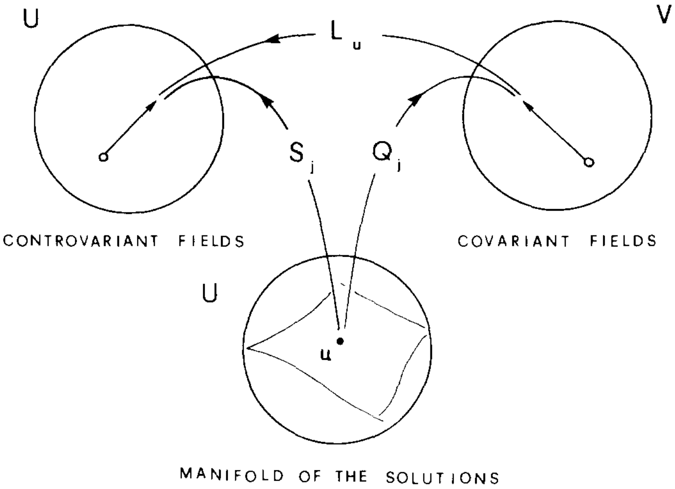

FIG. 2.

The theory of the conservation laws associates a special set of conservative covariant vector fields with the given equation by the requirement that the corresponding functionals keep their value $F[u(t)]$ independent of t for any solution $u(t)$. These functionals are called *integrals*[5] of the given equation and the corresponding potential operators may be called *"integrating"* operators.[6] Therefore, the theory of the symmetries studies the manifold of the solutions by using contravariant vector fields while the theory of the conservation laws studies the same manifold by using covariant vector fields. In this sense the two formalisms are dual.

The following condition

$$\langle \mathbf{Q}(u), \mathbf{K}(u)\rangle \doteq 0, \tag{1.10}$$

on the integrating operators $\mathbf{Q}$ is readily obtained if one observes that it is

$$\partial_t F[u(t)] \overset{(1.9)}{=} \langle \mathbf{Q}(u), \partial_t u\rangle$$

$$\overset{(1.2)}{=} \langle \mathbf{Q}(u), \mathbf{K}(u)\rangle, \tag{1.11}$$

for any solution $u(t)$. As in the case of the symmetry generators, however, we shall only consider the integrating operators for which the condition (1.10) is identically verified.

C. Connecting the conservation laws with the symmetries

The problem of connecting the conservation laws with the symmetries requires the introduction of a metric operator, which associates the covariant with the contravariant vector fields as is usual in Riemannian geometry. By a *metric operator* it is meant a linear operator $\mathbf{L}_u : V \to U$ (which may nonlinearly depend on the point u) mapping the covariant vector fields $\mathbf{Q}_j$ into the contravariant vector fields $\mathbf{S}_j$ according to the relation

$$\mathbf{S}_j(u) = \mathbf{L}_u \mathbf{Q}_j(u) \tag{1.12}$$

(examples will be given in Secs. 2 and 5, see Fig. 2).

It is the purpose of this subsection to study the special class of the metric operators verifying the following two conditions:

$$\langle dv, \mathbf{L}_u \delta v\rangle = -\langle \delta v, \mathbf{L}_u dv\rangle, \tag{1.13}$$

$$\langle dv, \mathbf{L}'_u(\delta v; \mathbf{L}_u \Delta v)\rangle + \langle \delta v, \mathbf{L}'_u(\Delta v; \mathbf{L}_u dv)\rangle + \langle \Delta v, \mathbf{L}'_u(dv; \mathbf{L}_u \delta v)\rangle = 0, \tag{1.14}$$

where $\mathbf{L}'_u$ is the Gateaux derivative of $\mathbf{L}_u$ with respect to u (see Appendix B), and to show that they allow to connect the integrating operators with the symmetry generators of the evolution equations. Such metric operators will be called *symplectic operators* with respect to the prefixed bilinear form $\langle v, u\rangle$.

To this end, consider any operator $\mathbf{S}_j$ associated with a *potential* operator $\mathbf{Q}_j$ by means of a symplectic operator $\mathbf{L}_u$. It is called a *Hamiltonian operator* and it verifies the following condition,

$$\langle dv, \mathbf{S}'_{ju}\mathbf{L}_u \delta v\rangle - \langle \delta v, \mathbf{S}'_{ju}\mathbf{L}_u dv\rangle = \langle dv, \mathbf{L}'_u(\delta v; \mathbf{S}_j(u))\rangle, \tag{1.15}$$

(see Appendix B). This condition implies that the commutator $[\mathbf{S}_j, \mathbf{S}_k]$ of any pair of Hamiltonian operators verifies the relation

$$\begin{aligned}
\langle dv, [\mathbf{S}_j, \mathbf{S}_k](u)\rangle
&\equiv \langle dv, \mathbf{S}'_{ju}\mathbf{S}_k(u) - \mathbf{S}'_{ku}\mathbf{S}_j(u)\rangle \\
&\overset{(B4)}{=} \langle dv, \mathbf{L}_u\mathbf{Q}'_{ju}\mathbf{S}_k(u) + \mathbf{L}'_u(\mathbf{Q}_j(u); \mathbf{S}_k(u)) - \mathbf{S}'_{ku}\mathbf{L}_u\mathbf{Q}_j(u)\rangle \\
&\overset{(1.15)}{=} \langle dv, \mathbf{L}_u\mathbf{Q}'_{ju}\mathbf{S}_k(u)\rangle + \langle dv, \mathbf{L}'_u(\mathbf{Q}_j(u); \mathbf{S}_k(u))\rangle \\
&\qquad - \langle \mathbf{Q}_j(u), \mathbf{S}'_{ku}\mathbf{L}_u dv\rangle - \langle dv, \mathbf{L}'_u(\mathbf{Q}_j(u); \mathbf{S}_k(u))\rangle \\
&\overset{(1.13)}{=} \langle dv, \mathbf{L}_u(\mathbf{Q}'_{ju}\mathbf{S}_k(u) + \tilde{\mathbf{S}}'_{ku}\mathbf{Q}_j(u))\rangle,
\end{aligned} \tag{1.16}$$

which shows that

$$[\mathbf{S}_j, \mathbf{S}_k](u) = \mathbf{L}_u(\mathbf{Q}'_{ju}\mathbf{S}_k(u) + \tilde{\mathbf{S}}'_{ku}\mathbf{Q}_j(u)), \tag{1.17}$$

where $\tilde{\mathbf{S}}'_{ku}$ is the adjoint of $\mathbf{S}'_{ku}$ with respect to the prefixed bilinear form $\langle v, u\rangle$ (see Appendix A).

$$\begin{tikzcd}
F_j \arrow[r] \arrow[d] & F_{jk} \arrow[d, "\text{POISSON BRACKET}"'] & F_k \arrow[l] \arrow[d] \\
Q_j \arrow[r] \arrow[d] & Q_{jk} \arrow[d] & Q_k \arrow[l] \arrow[d] \\
S_j \arrow[r] & {[S_j, S_k]} & S_k \arrow[l] \\
 & \text{COMMUTATOR} &
\end{tikzcd}$$

FIG. 3

Since the operator

$$\mathbf{Q}_{jk}(u) \equiv \mathbf{Q}'_{ju}\mathbf{S}_k(u) + \tilde{\mathbf{S}}'_{ku}\mathbf{Q}_j(u), \tag{1.18}$$

is the gradient of the functional

$$F_{jk}[u] \equiv \langle \mathbf{Q}_j(u), \mathbf{S}_k(u)\rangle, \tag{1.19}$$

as is proved by

$$\begin{aligned}
\delta F_{jk}[u]
&= \langle \mathbf{Q}'_{ju}\delta u, \mathbf{S}_k(u)\rangle + \langle \mathbf{Q}_j(u), \mathbf{S}'_{ku}\delta u\rangle \\
&\overset{(1.8)}{=} \langle \mathbf{Q}'_{ju}\mathbf{S}_k(u), \delta u\rangle + \langle \tilde{\mathbf{S}}'_{ku}\mathbf{Q}_j(u), \delta u\rangle \\
&= \langle \mathbf{Q}'_{ju}\mathbf{S}_k(u) + \mathbf{S}'_{ku}\mathbf{Q}_j(u), \delta u\rangle,
\end{aligned} \tag{1.20}$$

relation (1.17) shows that $[\mathbf{S}_j, \mathbf{S}_k]$ is again a Hamiltonian operator relative to $\mathbf{L}_u$. Therefore, the operators $\mathbf{S}_j$ make a Lie algebra, and this Lie algebra structure induces a corresponding structure on the operators $\mathbf{Q}_j$ and on the functionals F_j according to the scheme of Fig. 3. The functional F_{jk} is the *Poisson bracket* of the functionals F_j and F_k associated with the Hamiltonian operators $\mathbf{S}_j$ and $\mathbf{S}_k$ (see Ref. 3, Sec. 5).

A simple property of this algebraic structure is that the condition

$$F_{jk}[u] \equiv \langle \mathbf{Q}_j(u), \mathbf{S}_k(u)\rangle = 0 \tag{1.21}$$

implies

$$[\mathbf{S}_j, \mathbf{S}_k] = 0. \tag{1.22}$$

From the point of view of the theory of the symmetries and of the conservation laws, conditions (1.21) and (1.22) mean that $\mathbf{Q}_j$ and $\mathbf{S}_j$ are respectively an integrating operator and a symmetry generator of the evolution equation

$$\partial_t u = \mathbf{S}_k(u) \equiv \mathbf{L}_u \mathbf{Q}_k(u). \tag{1.23}$$

Therefore, the symplectic operator $\mathbf{L}_u$, associating $\mathbf{Q}_j$ with $\mathbf{S}_j$ according to (1.12), connects the integrating operators of Eq. (1.23) with its symmetry generators. This property explains the importance of the symplectic operators in the study of the evolution equations.

The problem of connecting the conservation laws with the symmetries of the given equation (1.2) is thus reduced to that of recasting this equation into the Hamiltonian form (1.23), by decomposing the operator $\mathbf{K}$ as follows,

$$\mathbf{K}(u) = \mathbf{L}_u \mathbf{Q}(u), \tag{1.24}$$

where $\mathbf{L}_u$ is a suitable symplectic operator and $\mathbf{Q}$ is a potential operator. To find such operators, if any, it is useful to observe that, according to (1.15), $\mathbf{L}_u$ must be coupled to $\mathbf{K}$ by the condition

$$\langle dv, \mathbf{K}'_u\mathbf{L}_u\delta v\rangle - \langle \delta v, \mathbf{K}'_u\mathbf{L}_u\, dv\rangle = \langle dv, \mathbf{L}'_u(\delta v; \mathbf{K}(u))\rangle, \tag{1.25}$$

and that $\mathbf{Q}$ must be an integrating operator of the given equation, as is proved by

$$\langle \mathbf{Q}(u), \mathbf{K}(u)\rangle = \langle \mathbf{Q}(u), \mathbf{L}_u\mathbf{Q}(u)\rangle \overset{(1.13)}{=} 0 \tag{1.26}$$

(examples will be given in Sec. 5). When condition (1.25) is fulfilled, one says that the symplectic operator $\mathbf{L}_u$ makes the given equation Hamiltonian. It can thus be stated that *every* symplectic operator $\mathbf{L}_u$ making the given equation Hamiltonian maps its integrating operators $\mathbf{Q}_j$ into its symmetry generators $\mathbf{S}_j$ according to relation (1.12). This is the main result on which the following analysis of the integrable Hamiltonian equations rests upon.

2. AN EXAMPLE: THE KORTEWEG-de VRIES EQUATION

It is the purpose of this section to show by an example how the integrable Hamiltonian equations may be analyzed by using *only* the connection between the symmetries and the integrating operators previously pointed out.

Consider the KdV equation

$$u_t + auu_x + u_{xxx} = 0 \tag{2.1}$$

and observe that it admits the following *two* Hamiltonian decompositions:

$$u_t + \partial_x\left(\frac{a}{2}u^2 + u_{xx}\right) = 0, \tag{2.2}$$

$$u_t + \left(\partial_{xxx} + \tfrac{2}{3}au\partial_x + \frac{a}{3}u_x I\right)u = 0, \tag{2.3}$$

where the operators

$$\mathbf{L}_u\varphi \equiv \varphi_x, \tag{2.4a}$$

$$\mathbf{M}_u\varphi \equiv \varphi_{xxx} + \tfrac{2}{3}au\varphi_x + \frac{a}{3}u_x\varphi, \tag{2.4b}$$

and

$$\mathbf{Q}_1(u) \equiv u,$$

$$\mathbf{Q}_2(u) \equiv \frac{a}{2}u^2 + u_{xx}, \tag{2.5}$$

are respectively two symplectic operators and two potential operators with respect to the bilinear form

$$\langle v, u\rangle \equiv \int_\Omega v(x,t)u(x,t)\,dx. \tag{2.6}$$

The former decomposition is well known[7]; the latter seems not to have been previously reported.

It follows that two symplectic operators are at our disposal to pass from the integrating operators of the KdV equation to its symmetry generators. They can be used to recover the infinite sequence of conservation laws associated with this equation, as follows. Consider the integrating operator $\mathbf{Q}_1$ and associate with it the symmetry generator

$$\mathbf{S}_2(u) \equiv \mathbf{M}_u\mathbf{Q}_1(u) = auu_x + u_{xxx} = \partial_x\mathbf{Q}_2(u), \tag{2.7}$$

by means of the higher-order symplectic operator $\mathbf{M}_u$. The inverse operator of the second symplectic operator ∂_x then allows to obtain from $\mathbf{S}_2$ the new integrating operator $\mathbf{Q}_2$. By iterating this process, first obtain from $\mathbf{Q}_2$

$$\begin{aligned}\mathbf{S}_3(u) &\equiv \mathbf{M}_u\mathbf{Q}_2(u)\\ &= \partial_x(\tfrac{5}{18}a^2u^3 + \tfrac{5}{3}auu_{xx} + \tfrac{5}{6}au_x^2 + u_{xxxx}),\end{aligned} \tag{2.8}$$

and then

$$\mathbf{Q}_3(u) = \tfrac{5}{18}a^2u^3 + \tfrac{5}{3}auu_{xx} + \tfrac{5}{6}au_x^2 + u_{xxxx}, \tag{2.9}$$

and so on. In this way an infinite sequence of independent integrating operators of the KdV equation can be constructed according to the recursion formula

$$\partial_x\mathbf{Q}_{j+1}(u) = \left(\partial_{xxx} + \tfrac{2}{3}au\partial_x + \frac{a}{3}u_x I\right)\mathbf{Q}_j(u). \tag{2.10}$$

The functionals F_j associated with the potential operators $\mathbf{Q}_j$ constitute the infinite sequence of integrals of the KdV equation.

The previous result, of course, is well known[8]; but the present analysis, which is based only on the study of the given equation and which avoids the introduction of the Bäcklund transformation or the use of the Lax equation,[9] provides a different point of view. Above all, this analysis emphasizes the role of the pairs of suitably coupled symplectic operators in the study of the integrable Hamiltonian equations. This point of view will be systematically developed in the next section.

3. INTEGRABLE HAMILTONIAN EQUATIONS

The example of the KdV equation suggests the study of the *twofold* Hamiltonian equations. This requires that we have to first discuss under which conditions two symplectic operators $\mathbf{L}_u$ and $\mathbf{M}_u$ have Hamiltonian operators in common. In this section we prove that if $\mathbf{L}_u$ and $\mathbf{M}_u$ have at least *one* Hamiltonian operator in common and they verify the *coupling condition*[10]

$$\begin{aligned}&\langle dv, \mathbf{L}'_u(\delta v; \mathbf{M}_u\Delta v)\rangle + \langle \delta v, \mathbf{L}'_u(\Delta v; \mathbf{M}_u\, dv)\rangle + \langle \Delta v, \mathbf{L}'_u(dv; \mathbf{M}_u\delta v)\rangle\\ &\quad = -\,[\langle dv, \mathbf{M}'_u(\delta v; \mathbf{L}_u\Delta v)\rangle + \langle \delta v, \mathbf{M}'_u(\Delta v; \mathbf{L}_u\, dv)\rangle\\ &\qquad + \langle \Delta v; \mathbf{M}'_u(dv; \mathbf{L}_u\delta v)\rangle],\end{aligned} \tag{3.1}$$

and if for one of them, say $\mathbf{L}_u$, condition (1.15) on the Hamiltonian operators is sufficient as well,[11] then they have a possibly infinite sequence of commuting Hamiltonian operators in common.

Assume that

$$\mathbf{S}_j(u) \equiv \mathbf{L}_u\mathbf{Q}_j(u) \tag{3.2}$$

is any Hamiltonian operator common both to $\mathbf{L}_u$ and $\mathbf{M}_u$ and construct

$$\mathbf{S}_{j+1}(u) \equiv \mathbf{M}_u\mathbf{Q}_j(u) \tag{3.3}$$

(see Fig. 4). Our aim is to show that $\mathbf{S}_{j+1}$ is a common Hamiltonian operator again. For this purpose, it suffices to prove that $\mathbf{M}_u\mathbf{Q}_j$ satisfies condition (1.15) on the operators which are Hamiltonian with respect to $\mathbf{L}_u$. On account of (B4), this condition is explicitly given by

$$\begin{aligned}&\langle dv,\ \mathbf{M}_u\mathbf{Q}'_{ju}\mathbf{L}_u\delta v + \mathbf{M}'_u(\mathbf{Q}_j(u); \mathbf{L}_u\delta v)\rangle\\ &\quad -\langle \delta v,\ \mathbf{M}_u\mathbf{Q}'_{ju}\mathbf{L}_u\, dv + \mathbf{M}'_u(\mathbf{Q}_j(u); \mathbf{L}_u\, dv)\rangle\\ &\quad = \langle dv,\ \mathbf{L}'_u(\delta v; \mathbf{M}_u\mathbf{Q}_j(u))\rangle.\end{aligned} \tag{3.4}$$

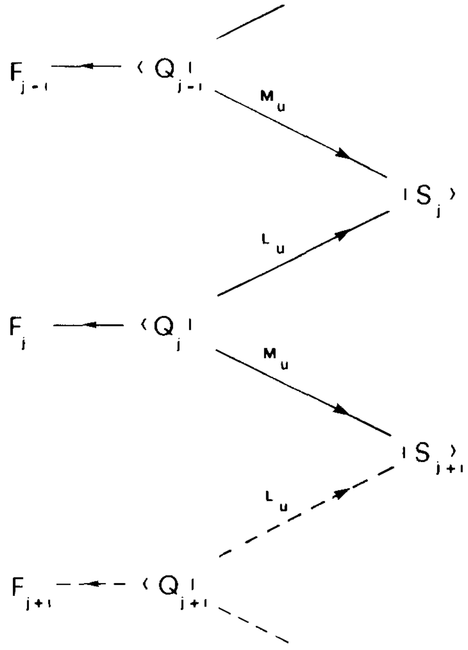

FIG. 4.

Now, this condition readily follows from the coupling condition and from the fact that $\mathbf{S}_j$ is a common Hamiltonian operator. In fact, the assumption on $\mathbf{S}_j$ means that $\mathbf{L}_u\mathbf{Q}_j$ obeys the following condition,

$$\begin{aligned}&\langle dv,\ \mathbf{L}_u\mathbf{Q}'_{ju}\mathbf{M}_u\delta v+\mathbf{L}'_u(\mathbf{Q}_j(u);\mathbf{M}_u\delta v)\rangle\\&\quad-\langle \delta v,\ \mathbf{L}_u\mathbf{Q}'_{ju}\mathbf{M}_u\, dv+\mathbf{L}'_u(\mathbf{Q}_j(u);\mathbf{M}_u\, dv)\rangle\\&\quad=\langle dv,\ \mathbf{M}'_u(\delta v;\mathbf{L}_u\mathbf{Q}_j(u))\rangle,\end{aligned}\tag{3.5}$$

and the coupling condition (3.1) implies that (3.4) and (3.5) coincide for any $\mathbf{Q}_j$.[12] This proves the statement.

It follows that, once we know one common Hamiltonian operator, we are able to construct successively a possibly infinite sequence of such operators according to the recursion formula

$$\mathbf{L}_u\mathbf{Q}_{j+1}(u)=\mathbf{M}_u\mathbf{Q}_j(u).\tag{3.6}$$

All the properties of this sequence stem from relation (3.6). So, for example, consider the Poisson bracket of any pair of functionals associated with the sequence (see Fig. 4). The following recursion formula is induced by (3.6),

$$\begin{aligned}F_{jk}[u]&\overset{(1.19)}{=}\langle \mathbf{Q}_j(u),\ \mathbf{L}_u\mathbf{Q}_k(u)\rangle\\&\overset{(3.6)}{=}-\langle \mathbf{Q}_j(u),\ \mathbf{M}_u\mathbf{Q}_{k-1}(u)\rangle\\&\overset{(1.13)}{=}-\langle \mathbf{Q}_{k-1}(u),\ \mathbf{M}_u\mathbf{Q}_j(u)\rangle\\&\overset{(3.6)}{=}\langle \mathbf{Q}_{k-1}(u),\ \mathbf{L}_u\mathbf{Q}_{j+1}(u)\rangle\\&\overset{(1.13)}{=}\langle \mathbf{Q}_{j+1}(u),\ \mathbf{L}_u\mathbf{Q}_{k-1}(u)\rangle\\&=F_{j+1,k-1}[u].\end{aligned}\tag{3.7}$$

By iteration, one finds (by assuming $j<k$)

$$\begin{aligned}F_{jk}[u]&=F_{j+1,k-1}[u]=F_{j+2,k-2}[u]\\&=\cdots=F_{kj}[u],\end{aligned}\tag{3.8}$$

and then

$$F_{jk}[u]\equiv\langle \mathbf{Q}_j(u),\mathbf{S}_k(u)\rangle=0,\tag{3.9}$$

on account of the skew symmetry of the Poisson bracket. *All the functionals F_j are thus in involution.* From this property the corresponding relation

$$[\mathbf{S}_j,\mathbf{S}_k]=0,\tag{3.10}$$

readily follows (recall Fig. 3), showing that *the operators $\mathbf{S}_j$ constitute a set of commuting operators.*

Hence we have constructed a sequence of twofold Hamiltonian equations

$$\partial_t u=\mathbf{S}_k(u),\tag{3.11}$$

which have remarkable properties. The conditions (3.9) and (3.10) mean that each of such equations has a possibly infinite sequence of symmetry generators $\mathbf{S}_j$, of integrating operators $\mathbf{Q}_j$, and of integrals F_j which are in involution. Often this sequence is indeed infinite (see the examples of Sec. 5). In this case, *any Eq.* (3.11) *is an infinite-dimensional integrable Hamiltonian equation.*

4. SUMMARY

The present paper has mainly dealt with the following two results: The symmetry generators and the integrating operators of any Hamiltonian equation are connected in pairs, and such pairs may be connected into a highly ordered chain for special twofold Hamiltonian equations. A characteristic property of this chain is that the symmetries are automatically in involution, so that the chain may actually define a whole hierarchy of integrable Hamiltonian equations.

These results suggest either a sufficiently general procedure of constructing the integrable Hamiltonian equations or a systematic way of analyzing the Hamiltonian structure of a given evolution equation. Dealing with the first problem, we consider a suitable pair of symplectic operators coupled according to (3.1), and we look for their Hamiltonian operators by solving condition (1.25), where $\mathbf{K}$ is regarded as the unknown operator. If we find *one* Hamiltonian operator common both to $\mathbf{L}_u$ and $\mathbf{M}_u$, we can construct an infinite hierarchy of integrable Hamiltonian equations. Dealing with the second problem, we look for the symplectic operators which make the given equation Hamiltonian by solving condition (1.25) with respect to the unknown operator $\mathbf{L}_u$. If we find *two* solutions of this condition which are coupled according to (3.1) and for which the sequence defined by the recursion formula (3.6) is infinite, the given equation turns out to be an infinite-dimensional integrable Hamiltonian equation. At the same time, the recursion formula (3.6) directly defines the sequence of conservation laws associated with this equation. Examples of this procedure will be given in the next section.

5. APPLICATIONS

As a first example, consider the so-called *Harry Dym equation*[13]

$$\partial_t u=\partial_{xxx}(u^{-1/2}).\tag{5.1}$$

It is manifestly a Hamiltonian equation, since the operators

$$\mathbf{M}_u\varphi \equiv \varphi_{xxx}, \tag{5.2}$$

and

$$\mathbf{Q}_1(u) \equiv u^{-1/2}, \tag{5.3}$$

are respectively a symplectic and a potential operator with respect to the bilinear form (2.6).

According to the method of analysis previously worked out, we look for a second symplectic operator making Eq. (5.1) Hamiltonian. To this end, let us consider the operators

$$\mathbf{L}_u\varphi \equiv 2a(u)\varphi_x + \mathring{a}(u)u_x\varphi, \tag{5.4}$$

which are symplectic with respect to (2.6) for any choice of the function $a(u)$ ($\mathring{a} = da/du$). By trying to fulfill condition (1.25) by means of (5.4), the function

$$a(u) = u, \tag{5.5}$$

is obtained. Since the two symplectic operators (5.2) and (5.4) [with $a(u)$ given by (5.5)] verify the coupling condition (3.1), it turns out that the Harry Dym equation is an integrable Hamiltonian equation, and that the infinite sequence of its conservation laws is defined by the recursion formula

$$(2u\partial_x + u_x I)\mathbf{Q}_{j+1}(u) = \partial_{xxx}\mathbf{Q}_j(u), \tag{5.6}$$

$\mathbf{Q}_1$ being given by (5.3). The search for the symplectic operators making the Harry Dym equation Hamiltonian has thus led to a simple analysis of this equation.

The previous procedure seems to require some preliminary guess of the form of the second symplectic operator in order to be effective. This difficulty can be bypassed as follows. Consider, as a second example, the *nonlinear Schrödinger equation*

$$i\psi_t + \psi_{xx} + 2\psi^2\bar{\psi} = 0, \tag{5.7}$$

and write it in the form

$$\psi_t = i(\psi_{xx} + 2\psi^2\bar{\psi}). \tag{5.8}$$

Since the operators

$$\mathbf{L}\varphi = i\varphi, \tag{5.9}$$

and

$$\mathbf{Q}(\psi) = \psi_{xx} + 2\psi^2\bar{\psi}, \tag{5.10}$$

are respectively a symplectic and a potential operator with respect to the bilinear form

$$\langle\psi, \varphi\rangle = \int_\Omega (\psi\bar{\varphi} + \bar{\psi}\varphi)\,dx, \tag{5.11}$$

(5.8) is a first (well known) Hamiltonian decomposition of the nonlinear Schrödinger equation.

To find a second symplectic operator $\mathbf{M}_\psi$ making the Eq. (5.7) Hamiltonian, let us look for the simplest integrating operators of this equation. The following four integrating operators:

$$\begin{aligned} &\mathbf{Q}_1(\psi) = \psi, && \mathbf{S}_1(\psi) = i\psi; \\ &\mathbf{Q}_2(\psi) = -i\psi_x, && \mathbf{S}_2(\psi) = \psi_x; \\ &\mathbf{Q}_3(\psi) = -(\psi_{xx} + 2\psi^2\bar{\psi}), && \mathbf{S}_3(\psi) = -i(\psi_{xx} + 2\psi^2\bar{\psi}); \\ &\mathbf{Q}_4(\psi) = i(\psi_{xxx} + 6\psi\bar{\psi}\psi_x), && \mathbf{S}_4(\psi) = -(\psi_{xxx} + 6\psi\bar{\psi}\psi_x) \end{aligned} \tag{5.12}$$

can be readily obtained by using the condition (1.10). The operators $\mathbf{S}_j$ are the symmetry generators associated with them by the symplectic operator (5.9). According to (3.3), our problem is to find a symplectic operator $\mathbf{M}_\psi$ fulfilling the relations ($j = 1, 2, 3$)

$$\mathbf{S}_{j+1}(\psi) = \mathbf{M}_\psi\mathbf{Q}_j(\psi). \tag{5.13}$$

The inspection of these relations leads to the following integrodifferential operator

$$\mathbf{M}_\psi\varphi = \varphi_x + 2\psi\int_a^x (\bar{\psi}\varphi - \psi\bar{\varphi})\,d\xi. \tag{5.14}$$

This operator, however, is not skew symmetric with respect to the bilinear form (4.11), because of the integral. Let us then write it in the equivalent form

$$\begin{aligned} \mathbf{M}_\psi\varphi = \varphi_x &+ \psi\int_a^x (\bar{\psi}\varphi - \psi\bar{\varphi})\,d\xi \\ &+ \psi\int_b^x (\bar{\psi}\varphi - \psi\bar{\varphi})\,d\xi, \end{aligned} \tag{5.15}$$

where a and b denote the endpoints of the interval Ω of definition of the field functions. The operator (4.15) obeys the conditions on the symplectic operators with respect to the bilinear form (4.11) and the coupling condition with (4.9). Therefore, the results obtained in Sec. 3 can be applied to the nonlinear Schrödinger equation. In particular, the following recursion formula for its conservation laws,

$$\begin{aligned} i\mathbf{Q}_{j+1}(\psi) = \partial_x\mathbf{Q}_j &+ \psi\int_a^x [\bar{\psi}\mathbf{Q}_j(\psi) - \psi\bar{\mathbf{Q}}_j(\psi)]\,d\xi \\ &+ \psi\int_b^x [\bar{\psi}\mathbf{Q}_j(\psi) - \psi\bar{\mathbf{Q}}_j(\psi)]\,d\xi, \end{aligned} \tag{5.16}$$

is obtained.

The same analysis, finally, can be repeated for the *modified KdV equation*

$$u_t + au^2u_x + u_{xxx} = 0. \tag{5.17}$$

One obtains the following two Hamiltonian decompositions:

$$u_t + \partial_x\left(\frac{a}{3}u^3 + u_{xx}\right) = 0, \tag{5.18a}$$

$$\begin{aligned} u_t + \Big(&\partial_{xxx} + \tfrac{2}{3}av^2\partial_x + \tfrac{2}{3}vv_xI \\ &-\frac{a}{3}v_x\int_a^x v_\xi I\,d\xi - \frac{a}{3}v_x\int_b^x v_\xi I\,d\xi\Big)v = 0, \end{aligned} \tag{5.18b}$$

whose symplectic operators again verify the coupling condition (3.1) with respect to the bilinear form (2.6). The modified KdV equation is thus another example of the special twofold Hamiltonian equations considered in this paper. These examples show that the method of analysis based on the search for the symplectic operators making the given equation Hamiltonian is not only conceptually simple but also effective in the applications. They point out, moreover, that some of the more interesting evolution equations considered in the literature have a *common structure*, which is well described by the model of integrable Hamiltonian equation developed in Sec. 3.

APPENDIX A

The Gateaux derivative of an operator $\mathbf{S}: U \to U$ may be denoted by $\mathbf{S}'_u$ and is defined by

$$\mathbf{S}'_u\varphi \equiv \frac{d}{d\epsilon}\mathbf{S}(u+\epsilon\varphi)\big|_{\epsilon=0}, \tag{A1}$$

so that to the first order in ϵ it is

$$\mathbf{S}(u+\epsilon\varphi)=\mathbf{S}(u)+\epsilon\mathbf{S}'_u\varphi. \tag{A2}$$

Its adjoint operator $\tilde{\mathbf{S}}'_u$, relative to the prefixed bilinear form $\langle v, u\rangle$, is defined by

$$\langle dv, \mathbf{S}'_u du\rangle = \langle \tilde{\mathbf{S}}'_u dv, du\rangle. \tag{A3}$$

It is a linear mapping of the dual space V into itself.

If $\mathbf{S}$ is given by

$$\mathbf{S}(u)=s(u, u_x, u_{xx}, \cdots), \tag{A4}$$

it is

$$\mathbf{S}'_u\varphi = \frac{\partial s}{\partial u}\varphi + \frac{\partial s}{\partial u_x}\varphi_x + \cdots, \tag{A5}$$

and so the following identity

$$\begin{aligned}\partial_t\mathbf{S}(u) &= \frac{\partial s}{\partial u}\partial_t u + \frac{\partial s}{\partial u_x}\partial_t u_x + \cdots \\ &= \frac{\partial s}{\partial u}u_t + \frac{\partial s}{\partial u_x}\partial_x u_t + \cdots \\ &= \mathbf{S}'_u\,\partial_t u,\end{aligned} \tag{A6}$$

can be readily verified.

APPENDIX B

The Gateaux derivative of the metric operator $\mathbf{L}_u$ is defined by

$$\mathbf{L}'_u(\varphi;\psi) \equiv \frac{d}{d\epsilon}\mathbf{L}_{u+\epsilon\psi}\varphi\big|_{\epsilon=0}. \tag{B1}$$

Consequently, to the first order in ϵ it is

$$\mathbf{L}_{u+\epsilon\psi}\varphi = \mathbf{L}_u\varphi + \epsilon\mathbf{L}'_u(\varphi;\psi), \tag{B2}$$

and the Gateaux derivative of the operator

$$\mathbf{S}(u) \equiv \mathbf{L}_u\mathbf{Q}(u) \tag{B3}$$

is given by

$$\begin{aligned}\mathbf{S}'_u\psi &= \frac{d}{d\epsilon}\mathbf{L}_{u+\epsilon\psi}\mathbf{Q}(u+\epsilon\psi)\big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}[\mathbf{L}_u\mathbf{Q}(u) + \epsilon\mathbf{L}_u\mathbf{Q}'_u\psi \\ &\quad + \epsilon\mathbf{L}'_u(\mathbf{Q}(u);\psi) + \cdots]_{\epsilon=0} \\ &= L_u\mathbf{Q}'_u\psi + \mathbf{L}'_u(\mathbf{Q}(u);\psi).\end{aligned} \tag{B4}$$

To prove condition (1.15), observe that every symplectic operator satisfies the relation

$$\langle \delta v, \mathbf{L}'_u(dv;du)\rangle = -\langle dv, \mathbf{L}'_u(\delta v;du)\rangle, \tag{B5}$$

which is obtained by differentiating (1.13) with respect to u. Therefore,

$$\begin{aligned}&\langle dv, \mathbf{S}'_u\mathbf{L}_u\delta v\rangle - \langle \delta v, \mathbf{S}'_u\mathbf{L}_u dv\rangle \\ &\overset{(\mathrm{B4})}{=} \langle dv, \mathbf{L}_u\mathbf{Q}'_u\mathbf{L}_u\,\delta v + \mathbf{L}'_u(\mathbf{Q}(u);\mathbf{L}_u\delta v)\rangle \\ &\qquad - \langle \delta v, \mathbf{L}_u\mathbf{Q}'_u\mathbf{L}_u\,dv + \mathbf{L}'_u(\mathbf{Q}(u);\mathbf{L}_u dv)\rangle \\ &\overset{(\mathrm{B5})}{=} -\langle \mathbf{Q}(u), \mathbf{L}'_u(dv;\mathbf{L}_u\delta v)\rangle - \langle \delta v, \mathbf{L}'_u(\mathbf{Q}(u);\mathbf{L}_u dv)\rangle \\ &\overset{(1.14)}{=} \langle dv, \mathbf{L}'_u(\delta v;\mathbf{L}_u\mathbf{Q}(u))\rangle \\ &= \langle dv, \mathbf{L}'_u(\delta v;\mathbf{S}(u))\rangle.\end{aligned} \tag{B6}$$

This proves that condition (1.15) is necessary. As regards the problem if this condition is sufficient as well, we can only remark that there exist symplectic operators for which this is true. This can be verified, for example, for the symplectic operators (2.4a), (5.4), (5.9) and (5.18a) considered in this paper. This fact justifies us to assume, in Sec. 3, that for the symplectic operator $\mathbf{L}_u$, condition (1.15) is sufficient as well.

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[6] The integrating operators may be regarded as an extension to general evolution equations of the well-known integrating factors of the ordinary differential equations of the first order.

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[10] The coupling condition (3.1) has a simple algebraic meaning, namely that the sum of the two symplectic operators $\mathbf{L}_u$ and $\mathbf{M}_u$ is itself a symplectic operator. I thank Professor C. Cercignani for this observation.

[11] See the remark at the end of Appendix B.

[12] Take into account property (B5), which is verified by every symplectic operator.

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